Loop group schemes and Abhyankar's lemma

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LOOP GROUP SCHEMES AND ABHYANKAR'S LEMMA

SCHÉMAS EN GROUPES DE LACETS ET LEMME D'ABHYANKAR

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Abstract. We define the notion of loop reductive group schemes defined over the localization of a regular henselian ring A at a strict normal crossing divisor D. We provide a criterion for the existence of parabolic subgroups of a given type.

Résumé. On définit la notion de schémas en groupes réductifs de lacets au-dessus du localisé d'un anneau hensélien A en un diviseur à croisements normaux stricts D. On établit un critère pour qu'un tel schéma en groupes admette un sous–schéma en groupes paraboliques d'un type donné.

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Soit A un anneau local hensélien régulier muni d'un système de paramètres $f_1, \dots, f_r$. On note k le corps résiduel de A et D le diviseur $D = \mathrm{div}(f_1) + \cdots + \mathrm{div}(f_r)$, il est à croisements normaux stricts. On pose $X = \mathrm{Spec}(A)$ et $U = X \setminus D = \mathrm{Spec}(A_D)$. La théorie d'Abhyankar décrit les revêtements finis étales connexes de $U = X \setminus D = \mathrm{Spec}(A_D)$ qui sont modérément ramifiés le long de D [10, XIII.2]. Un tel objet est dominé par un revêtement galoisien de la forme

$$B_n = B[T_1^{\pm 1}, \dots, T_r^{\pm 1}]/(T_1^n - f_1, \dots, T_r^n - f_r)$$

où n désigne un entier ≥ 1 premier à la caractéristique de k et B est une A–algèbre galoisienne contenant une racine primitive n–ième l'unité. Le groupe de Galois $\mathrm{Gal}(B_n/A_D)$ est le produit semi-direct $\mu_n(B)^r \rtimes \mathrm{Gal}(B/A)$ où $\mu_n(B)^r$ agit par multiplication sur les $T_1, \dots, T_r$. Si G est un A-schéma en groupes localement de présentation finie, un 1-cocycle $z : \mathrm{Gal}(B_n/A_D) \to G(B_n)$ est dit de *lacets* (loop en anglais) s'il est à valeurs dans $G(B) \subset G(B_n)$. Cette terminologie est inspirée par l'analogie avec le cas des polynômes de Laurent [7, ch. 3].

On note $\widehat{X}$ l'éclaté de $X = \mathrm{Spec}(A)$ en son point fermé, c'est un schéma régulier [9, §8.1, th. 1.19] et le diviseur exceptionnel $E \subset \widehat{X}$ est un diviseur de Cartier isomorphe à $\mathbb{P}_k^{r-1}$. On note alors $R = \mathcal{O}_{\widehat{X},\eta}$ l'anneau local au point générique η de E. Cet anneau R est de valuation discrète de corps des fractions K et de corps

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résiduel $F = k(E) = k(t_1, \dots, t_{r-1})$ où t_i désigne l'image de $\frac{f_i}{f_r} \in R$ par l'application de spécialisation $R \to F$. On note alors $v : K^\times \to \mathbb{Z}$ la valuation discrète associée à R et K_v le complété de K. Le résultat principal de cette note est le suivant.

Théorème (extrait du th. 3.1). *On suppose que G agit sur un A–schéma propre et lisse Z. Soit ϕ un 1–cocycle de lacets pour G. On note ${}_\phi Z/U$ le tordu par ϕ de $Z \times_X U$ Alors les assertions suivantes sont équivalentes:*

(a) $({}_\phi Z)(U) \neq \emptyset$;

(b) $({}_\phi Z)(K_v) \neq \emptyset$. big

C'est assez proche d'un résultat sur les polynômes de Laurent [7, §, thm. 7.1]. L'application principale concerne le cas d'un schéma en groupes réductifs de lacets. Par définition, un U-schéma en groupes réductifs G est *de lacets* si il est isomorphe à un tordu de sa forme déployée G_0 par un 1–cocycle de lacets à valeurs dans le schéma en groupes des autmorphismes $\mathrm{Aut}(G_0)$. On applique alors le résultat ci-dessus à des A–schémas de sous groupes paraboliques de G_0 d'un type donné (th. 4.1). On en déduit par exemple que si G est un U-schéma en groupes réductifs *de lacets*, alors G admet un U–schéma en groupes de Borel si et seulement si le K_v–schéma en groupes G_{K_v} est quasi-déployé. Plus généralement l'isotropie de G est contrôlée par l'indice de Tits de G_{K_v}.

1. Introduction

In the reference [7], we investigated a theory of loop reductive group schemes over the ring of Laurent polynomials $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Using Bruhat-Tits' theory, this permitted to relate the study of those group schemes to that of reductive algebraic groups over the field of iterated Laurent series $k((t_1))\dots((t_n))$. The main issue of this note is to start a similar approach for reductive group schemes defined over the localization A_D of a regular henselian ring A at a strict normal crossing divisor D and to relate with algebraic groups defined over a natural field associated to A and D, namely the completion K_v of the fraction field K with respect to the valuation arising from the blow-up of $\mathrm{Spec}(A)$ at its maximal ideal. The example which connects the two viewpoints is $k[[t_1, \dots, t_n]][\frac{1}{t_1}, \dots \frac{1}{t_n}]$ where $K_v \cong k\big(\frac{t_1}{t_n}, \dots, \frac{t_1}{t_{n-1}}\big)((t_n))$.

After defining the notion of loop reductive group schemes in this setting, we show that for this class of group schemes, the existence of parabolic subgroups over the localization A_D is controlled by the parabolic subgroups over K_v (Theorem 4.1).

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2. Tame fundamental group

2.1. **Abhyankar's lemma.** Let $X = \mathrm{Spec}(A)$ be a regular local scheme (not assumed henselian at this stage). Let k be the residue field of A and $p \geq 0$ be its characteristic. We put $\widehat{\mathbb{Z}}' = \prod_{l \neq p} \mathbb{Z}_l$. Let K be the fraction field of A, and let K_s be a separable closure of K. It determines a base point $\xi : \mathrm{Spec}(K) \to X$ so that we can deal with the Grothendieck fundamental group $\Pi_1(X, \xi)$ [10].

Let $(f_1, \dots, f_r)$ be a regular sequence of A and consider the divisor $D = \sum D_i = \sum \mathrm{div}(f_i)$, it has strict normal crossing. We put $U = X \setminus D = \mathrm{Spec}(A_D)$.

We recall that a finite étale cover $V \to U$ is *tamely ramified* with respect to D if the associated étale K–algebra $L = L_1 \times \cdots \times L_a$ is tamely ramified at the D_i's, that is, for each i, there exists j_i such that for the Galois closure $\widetilde{L}_{j_i}/K$ of L_{j_i}/K, the inertia group associated to v_{D_i} has order prime to p [10, XIII.2.0].

Grothendieck and Murre defined the tame (*modéré* in French) fundamental group $\Pi_1^D(U, \xi)$ with respect to $U \subset X$ as defined in [10, XIII.2.1.3] and [8, §2]. This is a profinite quotient of $\Pi_1(U, \xi)$ whose quotients by open subgroups provides finite Galois tame cover of U.

We are given a finite étale tame cover $V \to U$. In this case Abhyankar's lemma states that there exists a flat Kummer cover $X' = \mathrm{Spec}(A') \to X$ where

$$A' = A[T_1, \dots, T_r]/(T_1^{n_1} - f_1, \dots, T_r^{n_r} - f_r)$$

and the n_i's are coprime to p such that $V' = V \times_X X' \to X'$ extends uniquely to a finite étale cover $Y' \to X'$ [10, XIII.5.2].

Lemma 2.1. *Let $V \to U$ be a finite étale cover which is tame. Then* $\mathrm{Pic}(V) = 0$.

Proof. We use the some notation as above. We know that X' is regular [10, XIII.5.1] so a fortiori locally factorial. It follows that the restriction maps $\mathrm{Pic}(X') \to \mathrm{Pic}(V') \to \mathrm{Pic}(V)$ are surjective [5, 21.6.11]. Since A' is finite over the local ring A, it is semilocal so that $\mathrm{Pic}(A') = \mathrm{Pic}(X') = 0$. Thus $\mathrm{Pic}(V) = 0$ as desired. □

From now on we assume that A is henselian. According to [5, 18.5.10], the finite A–ring A' is a finite product of henselian local rings. We observe that $A' \otimes_A k = k[T_1, \dots, T_r]/(T_1^{n_1}, \dots, T_r^{n_r})$ is a local Artinian algebra so that A' is connected. It follows that A' is a henselian local ring. Its maximal ideal is $\mathfrak{m}' = \mathfrak{m} \otimes_A A' + \langle T_1, \dots, T_r \rangle$ so that $A'/\mathfrak{m}' = k$. Since there is an equivalence of categories between finite étale covers of A (resp. A') and étale k–algebras [5, 18.5.15], the base change from A to A' provides an equivalence of categories between the category of finite étale covers of A and that of A'.

It follows that $Y' \to X'$ descends uniquely to a finite étale cover $\widetilde{f} : \widetilde{Y} \to X$. From now on, we assume that V is furthermore connected, it implies that

$$H^0(V, \mathcal{O}_V) = B[T_1, \dots, T_r]/(T_1^n - f_1, \dots, T_r^n - f_r)$$

where B is finite connected étale cover of A. If follows that $V \to U$ is a quotient of a Galois cover of the shape

$$B_n = B[T_1^{\pm 1}, \dots, T_r^{\pm 1}]/(T_1^n - f_1, \dots, T_r^n - f_r)$$

where B is Galois cover of A containing a primitive n–root of unity. We record that B_n is the localization at $T_1 \dots T_r$ of $B'_n = B[T_1, \dots, T_r]/(T_1^n - f_1, \dots, T_r^n - f_r)$. We have

$$\mathrm{Gal}(B_n/A_D) = \big(\prod_{i=1}^{r} \mu_n(B)\big) \rtimes \mathrm{Gal}(B/A).$$

Passing to the limit we obtain an isomorphism

$$\pi_1^t(U, \xi) \cong \big(\prod_{i=1}^{r} \widehat{\mathbb{Z}}'(1)\big) \rtimes \pi_1(X, \xi).$$

We denote by $f : U^{sc,t} \to U$ the profinite étale cover associated to the quotient $\pi_1^t(U, \xi)$ of $\pi_1(U, \xi)$. According to [8, thm. 2.4.2], it is the universal tamely ramified cover of U. It is a localization of the inductive limit $\widetilde{B}'$ of the B'_n. On the other hand we consider the inductive limit $\widetilde{B}$ of the B's and observe that $\widetilde{B}'$ is a $\widetilde{B}$-ring.

2.2. **Blow-up.** We follow a blowing-up construction arising from [5, lemma 15.1.1.6]. We denote by $\widehat{X}$ the blow-up of $X = \mathrm{Spec}(A)$ at his closed point, this is a regular scheme [9, §8.1, th. 1.19] and the exceptional divisor $E \subset \widehat{X}$ is a Cartier divisor isomorphic to $\mathbb{P}_k^{r-1}$. We denote by $R = \mathcal{O}_{\widehat{X},\eta}$ the local ring at the generic point η of E. The ring R is a DVR of fraction field K and of residue field $F = k(E) = k(t_1, \dots, t_{r-1})$ where t_i is the image of $\frac{f_i}{f_r} \in R$ by the specialization map. We denote by $v : K^\times \to \mathbb{Z}$ the discrete valuation associated to R.

We deal now with a Galois extension B_n of A_D as above. Since B is a connected finite étale cover of A, B is regular and local; it is furthermore henselian [5, 18.5.10]. We denote by L the field of fraction of B and by L_n that of B_n. We have $[L_n : L] = n^r$. We want to extend the valuation v to L and to L_n.

We denote by $l = B/\mathfrak{m}_B$ the residue field of B, this is a finite Galois field extension of k. Also $(t_1, \dots, t_r)$ is a system of parameters for B. We denote by $w : L^\times \to \mathbb{Z}$ the discrete valuation associated to the exceptional divisor of the blow-up of $\mathrm{Spec}(B)$ at its closed point. Then w extends v and L_w/K_v is an unramified extension of degree $[L : K]$ and of residual extension $F_l = l(t_1, \dots, t_{r-1})/k(t_1, \dots, t_{r-1})$.

On the other hand we denote by $w_n : L_n^\times \to \mathbb{Z}$ the discrete valuation associated to the exceptional divisor of the blow-up of $\mathrm{Spec}(B_n)$ at its closed point. We put $l_n = B_n/\mathfrak{m}_{B_n}$, we have $l = l_n$. The valuation $\frac{w_n}{n}$ on L_n extends w and its residual extension is $F_{l,n} = l\Big(t_1^{1/n}, \dots, t_{r-1}^{1/n}\Big)/k\big(t_1, \dots, t_{r-1}\big)$ so that $[F_{l,n} : F_l] = n^{r-1}$. Furthermore the ramification index e_n of L_n/L is $\geq n$. Since $n^r \leq e_n\,[F_{l,n} : F_l] \leq [L_n : K] = n^r$(where the last inequality is [2, §VI.3, prop. 2]) it follows that $e_n = n$. The same statement shows that the map $L_w \otimes_L L_n \to L_{w_n}$ is an isomorphism. To summarize L_{w_n}/L_w is tamely ramified of ramification index n and of degree n^r. All together we have $L_{w_n} = L_w \otimes_K L_n$ so that L_{w_n} is Galois over K_v of group $\prod_i \mu_n(B) \rtimes \mathrm{Gal}(B/A) = \prod_i \mu_n(l) \rtimes \mathrm{Gal}(l/k)$.

We denote by $\Delta : \mu_n(l) \subset \prod_i \mu_n(l)$ the diagonal subgroup. We put $L_{w_n}^{\Delta} = L_n^{\Delta(\mu_n(B))}$. Since t_r is an uniformizing parameter of K_v and since $\Delta(\zeta)\,.\,t_r = \zeta.t_r$ for each $\zeta \in \mu_n(B)$, it follows that $(L_{w_n})^{\Delta}$ is the maximal unramified extension of L_{w_n}/K_v.

2.3. **Loop cocycles and loop torsors.** Let G be an X–group scheme locally of finite presentation. A loop cocycle is an element of $Z^1\big(\pi_1^t(U), G(\widetilde{B})\big)$ and it defines a Galois cocycle in $Z^1(\pi_1^t(U), G(U^{sc,t}))$. We denote by $Z^1_{loop}(\pi_1^t(U), G(U^{sc,t}))$ the image of the map $Z^1\big(\pi_1^t(U), G(\widetilde{B})\big) \to Z^1(\pi_1^t(U), G(U^{sc,t}))$ and by $H^1_{loop}(U,G)$ the image of the map

$$Z^1\big(\pi_1^t(U), G(\widetilde{B})\big) \to H^1(\pi_1^t(U), G(U^{sc,t})) \to H^1(U,G).$$

We say that a G-torsor E over U (resp. a fppf sheaf G-torsor) is a loop torsor if its class belongs to $H^1_{loop}(U,G) \subset H^1(U,G)$.

A given class $\gamma \in H^1_{loop}(U,G)$ is represented by a 1–cocycle $\phi : \mathrm{Gal}(B_n/A_D) \to G(B)$ for some cover B_n/A as above. Its restriction $\phi^{ar} : \mathrm{Gal}(B/A) \to G(B)$ to the subgroup $\mathrm{Gal}(B/A)$ of $\mathrm{Gal}(B_n/A_D)$ is called the "arithmetic part" and the other restriction $\phi^{geo} : \prod_i \mu_n(B) \to \mathfrak{G}(B)$ is called the geometric part. We observe that ϕ^{geo} is a B-group homomorphism.

Furthermore for $\sigma \in \mathrm{Gal}(B/A)$ and $\tau \in \prod_i \mu_n(B)$ the computation of [7, page 16] shows that $\phi^{geo}(\sigma\tau\sigma^{-1}) = \phi^{ar}(\sigma)\, {}^{\sigma}\phi(\tau)\, \phi^{ar}(\sigma)^{-1}$ so that ϕ^{geo} descends to a homomorphism of A-group schemes $\phi^{geo} : \mu_n^r \to {}_{\phi^{ar}}G$. This provides a parameterization of loop cocycles.

Lemma 2.2. *(1) For B_n/A_D as above, the map $\phi \mapsto (\phi^{ar}, \phi^{geo})$ provides a bijection between $Z^1_{loop}\big(\mathrm{Gal}(B_n/A_D), G(B)\big)$ and the couples (z,η) where $z \in Z^1\big(\mathrm{Gal}(B/A), G(B)\big)$ and $\eta : \prod_i \mu_n \to {}_zG$ is an A–group homomorphism.*

(2) The map $\phi \mapsto (\phi^{ar}, \phi^{geo})$ provides a bijection between $Z^1_{loop}\big(\pi^1(U,\xi)^t, G(\widetilde{B})\big)$ and the couples (z,η) where $z \in Z^1\big(\pi^1(X,\xi), G(\widetilde{B})\big)$ and $\eta : \prod_{i=1}^r \widehat{\mathbb{Z}}' \to {}_zG$ is an A–group homomorphism.

Proof. This is similar with [7, lemma 3.7]. □

We examine more closely the case of a finite étale X–group scheme $\mathfrak{F}$ of constant degree d.

Lemma 2.3. *(1) $\mathfrak{F}(\widetilde{B}) = \mathfrak{F}(X^{sc}) = \mathfrak{F}(U^{sc,t})$.*

(2) We assume that d is prime to p. We have $H^1_{loop}(U,\mathfrak{F}) = H^1(U,\mathfrak{F})$.

(3) We assume that d is prime to p. Let $f : \mathfrak{F} \to \mathfrak{H}$ be a homomorphism of A–group schemes (locally of finite type). Then $f_*\Big(H^1(U,\mathfrak{F})\Big) \subset H^1_{loop}(U,\mathfrak{H})$.

Proof. (1) We are given a cover B_n/A_D as above such that $\mathfrak{F}_{B_n} \cong \Gamma_{B_n}$ is finite constant. as above. Since B and B_n are connected, the map $\mathfrak{F}(B) \to \mathfrak{F}(B_n)$ reads as the identity $\Gamma \cong \mathfrak{F}(B) \to \mathfrak{F}(B_n) \cong \Gamma$ so is bijective. By passing to the limit we get $\mathfrak{F}(\widetilde{B}) = \mathfrak{F}(U^{sc,t})$.

(2) Let $\mathfrak{E}$ be a $\mathfrak{F}$–torsor over U. This is a finite étale U–scheme. Since U is noetherian and connected, we have a decomposition $\mathfrak{E} = V_1 \times_U \cdots \times_U V_l$ where each V_i is a connected finite étale U–scheme of constant degree d_i. We have $d_1 + \cdots + d_l = d$ so that we can assume that d_1 is prime to p. We have then $\mathfrak{E}(V_1) \neq \emptyset$.

It follows that $f_1 : V_1 \to U$ is a finite étale cover so that there exists a factorization $U^{sc,t} \to V_1 \overset{h}{\to} U$ of f so that $\mathfrak{E}(U^{sc,t}) \neq \emptyset$. Therefore $[\mathfrak{E}]$ arises from $H^1(\pi_1^t(U,\xi), \mathfrak{F}(U^{sc,t})) \subset H^1(U, \mathfrak{F})$. It follows that $H^1(\pi_1^t(U,\xi), \mathfrak{F}(U^{sc,t})) \xrightarrow{\sim} H^1(U, \mathfrak{F})$. We use now (1) and obtain the desired bijection $H^1(\pi_1^t(U,\xi), \mathfrak{F}(B)) \xrightarrow{\sim} H^1(U, \mathfrak{F})$.

(3) This follows readily from (2). □

2.4. **Twisting by loop torsors.** We assume that the A–group scheme G acts on an A–scheme Z. Let $\phi : (\prod_i^r \mu_n)(B) \rtimes \mathrm{Gal}(B/A) \to G(B)$ be a loop cocycle. It gives rise to an A–action of μ_n^r on ${}_{\phi^{ar}}Z$. We denote by $({}_{\phi^{ar}}Z)^{\phi^{geo}}$ the fixed point locus for this action, it is representable by a closed A–subscheme of ${}_{\phi^{ar}}Z$ [4, A.8.10.(1)]. We have a closed embedding $({}_{\phi^{ar}}Z)^{\phi^{geo}} \times_X U \subset {}_{\phi}Z$ of U-schemes.

3. Fixed points method

Theorem 3.1. *Let $X = \mathrm{Spec}(A)$ be a henselian regular local scheme and $U = X \setminus D$ as above. We denote by $v : K^\times \to \mathbb{Z}$ the discrete valuation associated to the exceptional divisor E of the blow-up of X at its closed point.*

Let G be an affine A-group scheme of finite presentation acting on a proper smooth A–scheme Z. Let ϕ be a loop cocycle for G. Then $Y = \big({}_{\phi^{ar}}Z\big)^{\phi^{geo}}$ is a smooth proper A–scheme and the following are equivalent:

(i) $({}_{\phi}Z)(K_v) \neq \emptyset$;

(ii) $Y(k) \neq \emptyset$;

(iii) $Y(U) \neq \emptyset$;

(iv) $({}_{\phi}Z)(U) \neq \emptyset$.

This is quite similar with the fixed point theorem [7, §, thm. 7.1]. The following example makes the connection.

Example 3.2. We assume that $A = k[[t_1, \dots, t_r]]$ for a field k and $k[U] = k[[t_1, \dots, t_n]]\big[\frac{1}{t_1}, \dots, \frac{1}{t_r}\big]$. We are given an affine algebraic k–group G acting on a smooth proper k–scheme Z. In this case $K = k((t_1, \dots, t_r))$ and A embeds in $k\big(\frac{t_1}{t_r}, \dots, \frac{t_{r-1}}{t_r}\big)[[t_r]]$ so that K embeds in $k\big(\frac{t_1}{t_n}, \dots, \frac{t_{r-1}}{t_r}\big)((t_r))$ which is nothing but the complete field K_v. If Q is a loop G-torsor over U, the statement is then that ${}^{Q}Z(U) \neq \emptyset$ if and only if ${}^{Q}Z(K_v) \neq \emptyset$. Taking a cocycle $\phi \in Z^1(\pi_1(U)^t, G(k_s))$ for E, this rephrases by the equivalence between $({}_{\phi}Z)(U) \neq \emptyset$ and $({}_{\phi}Z)(K_v) \neq \emptyset$.

What we have from [7, thm. 7.1] (in characteristic zero but this extends to this tame setting) is the equivalence between $({}_\phi Z)(k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]) \neq \emptyset$ and $({}_\phi Z)\big(k((t_1))\dots((t_r))\big) \neq \emptyset$. Since $({}_\phi Z)(k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]) \subset ({}_\phi Z)(U)$ and $({}_\phi Z)\big(K_v\big) \subset ({}_\phi Z)\big(k((t_1))\dots((t_r))\big)$, it follows that this special case of Theorem 3.1 is a consequence of the fixed point result of [7].

We proceed to the proof of Theorem 3.1.

Proof. According to [4, A.8.10.(1)], $Y = \big({}_{\phi^{ar}}(Z^{\phi^{geo}})\big)$ is a closed A–scheme of ${}_{\phi^{ar}}Z$ so is proper. It is smooth over X according to point (2) of the same reference. Let $\phi : \mathrm{Gal}(B_n/A_D) \to G(B)$ be the loop 1-cocycle for some Galois cover B_n/A_D as above for some n prime to p. Up to replace G by ${}_{\phi^{ar}G}$ and G by ${}_{\phi^{ar}Z}$, we can assume that $\phi^{ar} = 1$ without lost of generality.

$(ii) \Longrightarrow (iii)$. Since Y_k is the special fiber of the smooth X–scheme Y, Hensel's lemma shows that $Y(A) \to Y(k)$ is onto. Since $Y(k)$ is not empty, it follows that $Y(A)$ is not empty and so is $Y(U)$.

$(iii) \Longrightarrow (iv)$. Since $Y(U) \subset {}_\phi Z(U)$, $Y(U) \neq \emptyset$ implies that ${}_\phi Z(U) \neq \emptyset$.

$(iv) \Longrightarrow (i)$. This is obvious.

$(i) \Longrightarrow (ii)$. We assume that $({}_\phi Z)(K_v) \neq \emptyset$. By definition we have

$$({}_\phi Z)(K_v) = \big\{ z \in Z(L_{w_n}) \mid \phi(\sigma).\sigma(z) = z \ \ \forall \sigma \in \mathrm{Gal}(L_n/K) \big\}$$

and our assumption is that this set is non-empty. Let O_{w_n} be the valuation ring of $Z(L_{w_n})$. Since Z is proper over X, we have a specialization map $Z(L_{w_n}) = Z(\mathcal{O}_{w_n}) \to Z_k(F_{l,n})$. We get that the set

$$\big\{ z \in Z_k(F_{l,n}), \mid \ \phi(\sigma).\sigma(z) = z \ \ \forall \sigma \in \mathrm{Gal}(L_{w_n}/K_v) \big\}$$

is not empty. Since we have an embedding

$$F_{l,n} = l\big(t_1^{1/n}, \dots, t_{r-1}\big) \ \hookrightarrow \ l\big(\big(t_1^{1/n}\big)\big) \dots \big(\big(t_{r-1}^{1/n}\big)\big)$$

in a higher field of Laurent series successive specializations, along the coordinates $t_1^{1/n}$, ..., $t_{r-1}^{1/n}$ show similarly that the set

$$\Big\{ z \in (Z_k)\big(l\big) \mid \phi(\sigma).\sigma(z) = z \ \ \forall \sigma \in \mathrm{Gal}(L_{w_n}/K_v) \Big\} \tag{3.1}$$

is not empty. Since $\eta^{ar} = 1$, this set is $(Z_k)^{\eta^{geo}}(k)$. Thus $Y(k) = (Z_k)^{\eta^{geo}}(k)$ is non empty. □

4. Parabolic subgroups of loop reductive group schemes

4.1. **Chevalley groups.** Let G_0 be Chevalley group defined over $\mathbb{Z}$. Let T_0 be a maximal split $\mathbb{Z}$-subtorus of G_0 together with a Borel subgroup B_0 containing it. We denote by Δ_0 the Dynkin diagram of (G_0, B_0, T_0). We denote by $G_{0,ad}$ the adjoint quotient of G_0 and by G_0^{sc} the simply connected covering of DG_0. We have a map $\mathrm{Aut}(G_0) \to \mathrm{Aut}(G_0^{sc}) \xrightarrow{\sim} \mathrm{Aut}(G_{0,ad})$ and a fundamental exact sequence

$$1 \to G_{0,ad} \to \mathrm{Aut}(G_{0,ad}) \to \mathrm{Out}(G_{0,ad}) \to 1$$

where $\mathrm{Out}(G_{0,ad}) \xrightarrow{\sim} \mathrm{Aut}(\Delta_0)$. We recall that there is a bijection $I \to P_{0,I}$ between the finite subsets of Δ_0 and the parabolic subgroups of G_0 containing B_0 [11, XXVI.3.8]; it is increasing for the inclusion order, in particular $B_0 = P_{0,\emptyset}$ and $G_0 = P_{0,\Delta_0}$. We consider the total scheme Par_{G_0} of parabolic subgroups of G_0, it is a projective smooth $\mathbb{Z}$–scheme equipped with a type map $\mathbf{t} : \mathrm{Par}_{G_0} \to \mathrm{Of}(\Delta_0)$ where $\mathrm{Of}(\Delta_0)$ stands for the finite constant scheme attached to the set of subsets of Δ_0 [11, XXVI.3]. The fiber at I is denoted by $\mathrm{Par}_{G_0,I}$, it has connected fibers and is the scheme of parabolic subgroups of G_0 of type I. We have a natural action of $\mathrm{Aut}(G_0)$ on Par_{G_0}. As in [6, §5.1], we denote by $\mathrm{Aut}_I(G_0)$ the stabilizer of I for this action. By construction $\mathrm{Aut}_I(G_0)$ acts on $\mathrm{Par}_{G_0,I}$.

4.2. **Definition.** Let G be a reductive U-group scheme in the sense of Demazure-Grothendieck [11, XIX]. Since U is connected and G is locally splittable [11, XXII.2.2] for the étale topology, G is an étale form of a Chevalley group G_0 as above defined over $\mathbb{Z}$.

We say that G is a *loop group scheme* if the $\mathrm{Aut}(G_0)$-torsor $Q = \mathrm{Isom}(G_0, G)$ (defined in [11, XXIV.1.9]) is a loop $\mathrm{Aut}(G_0)$-torsor. We denote by $G_{0,ad}$ the adjoint quotient of G_0 and ny G_0^{sc} the simply connected covering of DG_0. We have a map $\mathrm{Aut}(G_0) \to \mathrm{Aut}(G_0^{sc}) \xrightarrow{\sim} \mathrm{Aut}(G_{0,ad})$ which permits to see G_{ad} (resp. G^{sc}) as twisted forms of $G_{0,ad}$ (resp. G_0^{sc}) so that G_{ad} and G^{sc} are also loop reductive group schemes. We consider the map $\mathrm{Aut}(G_0) \to \mathrm{Aut}(G_{0,ad}) \to \mathrm{Out}(G_{0,ad}) \xrightarrow{\sim} \mathrm{Aut}(\Delta_0)$.

If $\phi : \mathrm{Gal}(B_n/A_D) \to \mathrm{Aut}(G_0)(B)$ is a loop cocycle, we get an action of $\mathrm{Gal}(B_n/A_D)$ on Δ_0 called the star action. If I is stable under the star action, we can twist $\mathrm{Par}_{G_0,I}$ by ϕ and deal with the scheme ${}_\phi\mathrm{Par}_{G_0,I}$ which is the scheme of parabolic subgroup schemes of G of type I.

4.3. **Parabolics.**

Theorem 4.1. *Assume that G is a loop reductive U-group scheme and let ϕ : $\mathrm{Gal}(B_n/A_D) \to \mathrm{Aut}(G_0)(B)$ be a loop cocycle such that $G \cong {}_\phi G_0$. Let $I \subset \Delta_0$ be a subset stable under the star action defined by ϕ. Then the following are equivalent:*

(i) G admits a U–parabolic subgroup of type I;

(ii) the k–morphism $\eta_k^{geo} : \mu_n^r \to \mathrm{Aut}({}_{\eta^{ar}}G_0)_k = \big({}_{\eta^{ar}}\mathrm{Aut}(G_0)\big)_k$ normalizes a parabolic k–subgroup of ${}_{\eta^{ar}}G_{0,k}$ of type I;

(iii) G_{K_v} admits a parabolic subgroup of type I.

Proof. Without loss of generality we can assume that G is adjoint. Our assumption on the star action rephrases by saying that ϕ takes values in $\mathrm{Aut}_I(G_0)$. We apply Theorem 3.1 to the action of $\mathrm{Aut}_I(G_0)$ on the proper A-scheme $\mathrm{Par}_{G_0,I}$. We consider the A-scheme $Y = ({}_{\phi^{ar}}\mathrm{Par}_{G_0,I})^{\phi^{geo}}$. Theorem 3.1 shows that the following statements are equivalent.

(i') $({}_{\phi}\mathrm{Par}_{G_0,I})(U) \neq \emptyset$;

(ii') $Y(k) \neq \emptyset$.

(iii') $({}_{\phi}\mathrm{Par}_{G_0,I})(K_v) \neq \emptyset$.

Clearly (i') is equivalent to condition (i) of the Theorem and similarly we have $(iii') \Longleftrightarrow (iii)$. It remains to establish the equivalence between (ii) and (ii').

Assume that $({}_{\phi^{ar}}\mathrm{Par}_{G_0,I})^{\phi^{geo}}(k)$ is not empty and pick a k–point z. Then the stabilizer $({}_{\phi^{ar}}G_0)_z$ is a k–parabolic subgroup of ${}_{\phi^{ar}}G_0$ of type I which is stabilized by the action ϕ^{geo}_k. In other words, ϕ^{geo}_k normalizes $({}_{\phi^{ar}}G)_z$. Conversely we assume that ${}_{\phi^{ar}}G$ admits a k–parabolic subgroup of type I normalized by ϕ^{geo}. It defines then a point $z \in ({}_{\phi^{ar}}\mathrm{Par}_{G_0,I})(k)$ which is fixed by ϕ^{geo}.

□

4.4. **An example.** Assume that the residue field k is not of characteristic two and consider the diagonal quadratic form of dimension 2^r

$$q = \sum_{I \subset \{1,\dots,r\}} u_I \, t^I (x_I)^2$$

where $t_I = \prod_{i \in I} t_i$ and $u_I \in A^\times$. Then $\mathrm{SO}(q)$ is a loop reductive group scheme over U. Since the projective quadric $\{q = 0\}$ is a scheme of parabolic subgroups of $\mathrm{SO}(q)$, Theorem 4.1 shows that q is isotropic over A_D if and only if q is isotropic over K_v. The two dimensional case is related with [3, proof of Theorem 3.1].

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