# CONVERGENCE OF IPDG FOR COUPLED TIME-DEPENDENT NAVIER-STOKES AND DARCY EQUATIONS

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NABIL CHAABANE, VIVETTE GIRAULT, CHARLES PUELZ, BEATRICE RIVIERE

Abstract. A numerical method is proposed and analyzed for the coupled time-dependent Navier-Stokes equations and Darcy equations. Existence and uniqueness of the solution is obtained under a small data condition. A priori error estimates are derived. Numerical examples confirm the theoretical convergence rates.

**Keywords:** multiphysics, error analysis, Beavers-Joseph-Saffman

## 1. Introduction

The coupling of free flow and porous media is an important model problem arising from several fields, such as energy, environment and bio-medicine. While the case of coupled Stokes and Darcy equations has been extensively studied mathematically and numerically (see a non-exhaustive list in the introduction of [6]), the case of coupled Navier-Stokes and Darcy equations has been the subject of very few publications. We refer to [15] for the steady-state coupling; both existence and uniqueness of a weak solution are proved and a priori error estimates for a numerical error are derived. Other related works are [13, 14, 2, 9, 10].

This article is dedicated to the numerical analysis for a discretization of the time-dependent Navier-Stokes equations coupled with Darcy equations. The interface conditions are the Beavers-Joseph-Saffman conditions. To our knowledge, the only previous publications on the time-dependent coupling are [7, 8]; however the interface conditions in those two papers include inertial effects, which makes the analysis easier both for the weak solution and the numerical solution. In a more recent work, we showed existence and uniqueness of a weak solution to the coupled time-dependent Navier-Stokes and Darcy problem [6]. The present work is a continuation of this analysis, in the sense that we propose a discrete solution and we show optimal a priori error estimates between the weak solution and the numerical solution. We choose to discretize the equations by locally

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mass conservative interior penalty discontinuous Galerkin methods, of arbitrary order. We remark that the proof of existence and uniqueness of the discrete solution, which requires a small data condition, is not straightforward and follows a technical and convoluted argument, which can be traced back to [18]. The proof for the error estimates is also non-standard in the sense that one has to carefully bound terms with the appropriate norms. To our knowledge, the current paper is the first one to analyze convergence of a scheme for the time-dependent coupled case with the Beavers-Joseph-Saffman equations. While the obtained theoretical results are what we would expect, one important contribution of this paper is on the analysis itself. Our analysis is valid in 2D and 3D, and allows for rough (non-smooth) interface.

An outline of the paper follows. Section 2 introduces the equations and the discrete spaces. The following section describes the numerical method, and is followed by a section on useful inequalities. Section 5 contains the proof for existence and uniqueness of the numerical solution. A priori error estimates are derived next. They are confirmed by numerical tests in Section 7. Concluding remarks follow.

## 2. Model problem

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded connected domain with a Lipschitz–continuous boundary, partitioned into two disjoint subdomains, $\Omega_1$ and $\Omega_2$, so that $\Omega = \Omega_1 \cup \Omega_2$. The free flow region and the porous medium are $\Omega_1$ and $\Omega_2$ respectively. We assume that each subdomain $\Omega_i$ also has a Lipschitz–continuous boundary. Let $\Gamma_{12}$ denote the interface between $\Omega_1$ and $\Omega_2$. The interface $\Gamma_{12}$ may be a polygonal curve or surface and does not have to be smooth. Let $\Gamma_i$ be the exterior boundary of $\Omega_i$, $i = 1, 2$. The boundary $\Gamma_2$ is partitioned into two disjoint open sets, $\Gamma_2 = \Gamma_{2D} \cup \Gamma_{2N}$, and we assume that $|\Gamma_1| > 0$ and $|\Gamma_{2D}| > 0$, where $|\cdot|$ denotes the measure.

We denote by $\mathbf{n}_{\Omega_i}$ the exterior unit vector normal to $\Gamma_i$ and by $\mathbf{n}_{12}$ the unit normal vector to $\Gamma_{12}$ pointing from $\Omega_1$ to $\Omega_2$. In the case $d = 3$, we choose a pair of orthonormal tangent vectors, $\boldsymbol{\tau}_{12}^j$, $j = 1, 2$, on the tangent plane to $\Gamma_{12}$. For two-dimensional domains, the vector $\boldsymbol{\tau}_{12}^1$ is the unit tangent vector to $\Gamma_{12}$.

We are interested in studying a fully discrete discontinuous Galerkin approximation to the following equations posed in each subdomain (the equalities below hold almost everywhere in the domains or on the boundaries, according to the context):

$$\frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot (2\mu \mathbf{D}(\mathbf{u}) - p_1 \mathbf{I}) + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{f}_1, \quad \text{in} \quad \Omega_1 \times (0,T), \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in} \quad \Omega_1 \times (0,T), \tag{2}$$

$$-\nabla \cdot \mathbf{K} \nabla p_2 = f_2, \quad \text{in} \quad \Omega_2 \times (0,T), \tag{3}$$

with the symmetric gradient $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$. Equations (1), (2) represent the incompressible Navier-Stokes equations, where $\mathbf{u}$ and $p_1$ are the fluid velocity and pressure respectively in $\Omega_1$. The fluid viscosity is denoted by $\mu > 0$. Equation (3) represents the single phase flow equation in a porous medium, where $p_2$ is the fluid pressure in $\Omega_2$. The matrix $\mathbf{K}$ is the ratio of the permeability matrix to the fluid viscosity. It is assumed to be symmetric positive definite, with eigenvalues uniformly bounded above and bounded below away from zero. This system is complemented by the boundary and the interface conditions below. First we prescribe standard conditions on $\Gamma_i$:

$$\mathbf{u} = \mathbf{0}, \quad \text{on} \quad \Gamma_1 \times (0,T), \tag{4}$$

$$p_2 = 0, \quad \text{on} \quad \Gamma_{2D} \times (0,T), \tag{5}$$

$$\mathbf{K} \nabla p_2 \cdot \mathbf{n}_{\Omega_2} = 0, \quad \text{on} \quad \Gamma_{2N} \times (0,T). \tag{6}$$

On the interface $\Gamma_{12}$, we prescribe the following interface conditions:

$$\mathbf{u} \cdot \mathbf{n}_{12} = -\mathbf{K} \nabla p_2 \cdot \mathbf{n}_{12}, \text{ on } \Gamma_{12} \times (0,T), \tag{7}$$

$$\big((-2\mu \mathbf{D}(\mathbf{u}) + p_1 \mathbf{I}) \mathbf{n}_{12}\big) \cdot \mathbf{n}_{12} = p_2, \text{ on } \Gamma_{12} \times (0,T), \tag{8}$$

$$\mathbf{u} \cdot \boldsymbol{\tau}_{12}^j = -2\mu G^j \big(\mathbf{D}(\mathbf{u}) \mathbf{n}_{12}\big) \cdot \boldsymbol{\tau}_{12}^j, 1 \leq j \leq d-1, \text{ on } \Gamma_{12} \times (0,T), \tag{9}$$

where

$$G^j = \frac{\mu \alpha}{(\mathbf{K} \boldsymbol{\tau}_{12}^j, \boldsymbol{\tau}_{12}^j)^{1/2}}.$$

The interface conditions (7)–(9) have been discussed extensively in the literature for the steady-state coupling of porous media and free flows [3, 21, 10, 12, 15]. The condition (9) is the Beavers-Joseph-Saffman condition. We note that $\alpha > 0$ is a given constant, usually obtained from experimental data.

Finally, to simplify the discussion, we prescribe a zero initial condition:

$$\mathbf{u} = \mathbf{0}, \quad \text{in} \quad \Omega_1 \times \{0\}. \tag{10}$$

The relevant spaces for the exact solution $(\mathbf{u}, p_1, p_2)$ are

$$\mathbf{X} = \{\mathbf{v} \in H^1(\Omega_1)^d\,;\, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}, \tag{11}$$

$$M_1 = L^2(\Omega_1), \tag{12}$$

$$M_2 = \{q \in H^1(\Omega_2)\,;\, q = 0 \text{ on } \Gamma_{2D}\}. \tag{13}$$

For the discretization, we assume that the domain has a polygonal or polyhedral boundary, according to the dimension, so that it can be entirely triangulated. This simplifies substantially the numerical analysis. Let $\mathcal{E}_i^h$ be a regular family of conforming triangulations of $\Omega_i$, $i = 1, 2$, with maximum mesh size $h$, made of triangles or tetrahedra, according to the dimension, regular in the sense of Ciarlet [11], i.e., there exists a constant $\eta > 0$, independent of $h$, such that

$$\forall E \in \mathcal{E}_h^i, \quad \frac{h_E}{\varrho_E} \leq \eta, \tag{14}$$

where $h_E$ is the diameter of $E$ and $\varrho_E$ is the diameter of ball inscribed in $E$. For the sake of brevity, we only consider the case when the traces of $\mathcal{E}_i^h$ on the interface $\Gamma_{12}$ coincide; but the analysis below extends easily to the case of non matching grids. We denote by $\Gamma_i^h$ the set of faces of $\mathcal{E}_i^h$ interior to $\Omega_i$, for $i = 1, 2$. For each interior face $e$ we associate a fixed unit normal vector $\mathbf{n}_e$. For a boundary face $e \in \Gamma_1 \cup \Gamma_2$ the unit vector $\mathbf{n}_e$ coincides with the outward normal vector. For a face in $\Gamma_{12}$, we set $\mathbf{n}_e = \mathbf{n}_{12}$, so that $\mathbf{n}_e$ is the outward normal to $\Omega_1$. If a normal vector $\mathbf{n}_e$ points from the element $E^1$ to the element $E^2$, we define the jump and average of discontinuous functions as

$$[\varphi] = \varphi|_{E^1} - \varphi|_{E^2}, \quad \{\varphi\} = \frac{1}{2}\left(\varphi|_{E^1} + \varphi|_{E^2}\right).$$

By convention, for a face on the boundary $\Gamma_1 \cup \Gamma_2$, the jump and average are defined to be equal to the trace of the function on that face.

To define the finite element spaces, take two integers, $k_1 \geq 1$ and $k_2 \geq 1$, and set

$$\mathbf{X}^h = \big\{\mathbf{v} \in L^2(\Omega_1)^d; \quad \forall E \in \mathcal{E}_1^h, \quad \mathbf{v}|_E \in \mathbb{P}_{k_1}(E)^d\big\}, \tag{15}$$

$$M_1^h = \big\{q \in L^2(\Omega_1); \quad \forall E \in \mathcal{E}_1^h, \quad q|_E \in \mathbb{P}_{k_1-1}(E)\big\}, \tag{16}$$

$$M_2^h = \big\{q \in L^2(\Omega_2); \quad \forall E \in \mathcal{E}_2^h, \quad q|_E \in \mathbb{P}_{k_2}(E)\big\}. \tag{17}$$

The discrete DG spaces $\mathbf{X}^h$ and $M_2^h$ are equipped with mesh dependent norms denoted respectively by $\|\cdot\|_X$ and $\|\cdot\|_{M_2}$,

$$\forall \mathbf{v} \in \mathbf{X}^h, \quad \|\mathbf{v}\|_X = \left( 2 \sum_{E \in \mathcal{E}_1^h} \|\mathbf{D}(\mathbf{v})\|^2_{L^2(E)} + \sum_{e \in \Gamma_1^h \cup \Gamma_1} \frac{\sigma_e}{h_e} \|[\mathbf{v}]\|^2_{L^2(e)} \right)^{1/2}, \tag{18}$$

$$\forall q \in M_2^h, \quad \|q\|_{M_2} = \left( \sum_{E \in \mathcal{E}_2^h} \|\mathbf{K}^{1/2} \nabla q\|^2_{L^2(E)} + \sum_{e \in \Gamma_2^h \cup \Gamma_{2D}} \frac{\sigma_e}{h_e} \|[q]\|^2_{L^2(e)} \right)^{1/2}, \tag{19}$$

where $h_e$ denotes the diameter of $e$ and $\sigma_e$ denotes a positive parameter, possibly depending on $e$, but independent of $h$, and bounded below and above,

$$0 < \sigma_{\min} \leq \sigma_e \leq \sigma_{\max}.$$

The norm on $M_1^h$ is the $L^2$ norm,

$$\forall q \in M_1^h, \quad \|q\|_{M_1} = \|q\|_{L^2(\Omega_1)}. \tag{20}$$

We end this section by introducing additional notation. We define the inner-product of two functions $f, g$ on a domain $\mathcal{O} \subset \mathbb{R}^d$, $d = 2, 3$, as

$$(f, g)_{\mathcal{O}} = \int_{\mathcal{O}} fg.$$

This notation is extended to vector functions. The classical Sobolev spaces are denoted by $W^{k,m}(\mathcal{O})$ and by $H^k(\mathcal{O})$ if $m = 2$. Finally the usual Sobolev norms and semi-norms are denoted by $\|f\|_{H^k(\mathcal{O})}$ and $|f|_{H^k(\mathcal{O})}$ respectively.

## 3. Numerical scheme

The discretization of problem (1)–(10) makes use of the following discrete bilinear forms:

$$\begin{aligned} a_S(\mathbf{u}, \mathbf{v}) =& 2\mu \sum_{E \in \mathcal{E}_1^h} (\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_E - 2\mu \sum_{e \in \Gamma_1^h \cup \Gamma_1} (\{\mathbf{D}(\mathbf{u})\mathbf{n}_e\}, [\mathbf{v}])_e \\ &+ 2\epsilon_1 \mu \sum_{e \in \Gamma_1^h \cup \Gamma_1} (\{\mathbf{D}(\mathbf{v})\mathbf{n}_e\}, [\mathbf{u}])_2 + \mu \sum_{e \in \Gamma_1^h \cup \Gamma_1} \frac{\sigma_e}{h_e} ([\mathbf{u}], [\mathbf{v}])_e, \end{aligned} \tag{21}$$

$$b_S(\mathbf{v}, q) = -\sum_{E \in \mathcal{E}_1^h} (q, \nabla \cdot \mathbf{v})_E + \sum_{e \in \Gamma_1^h \cup \Gamma_1} (\{q\}, [\mathbf{v}] \cdot \mathbf{n}_e)_e, \tag{22}$$

$$\begin{aligned} a_D(p,q) = &\sum_{E\in\mathcal{E}_2^h} (\mathbf{K}\nabla p, \nabla q)_E - \sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} (\{\mathbf{K}\nabla p\cdot\mathbf{n}_e\},[q]) \\ &+ \epsilon_2 \sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} (\{\mathbf{K}\nabla q\cdot\mathbf{n}_e\},[p])_e + \sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} \frac{\sigma_e}{h_e}([p],[q])_e, \end{aligned} \tag{23}$$

where the parameters $\epsilon_i$, $i = 1, 2$, take the value 1, 0, or $-1$, according to the choice of DG method: non symmetric, incomplete, or symmetric. We use an upwind discretization of the nonlinear convection term,

$$c_{NS}(\mathbf{z},\mathbf{u};\mathbf{v},\mathbf{w}) = \ell_h(\mathbf{u};\mathbf{v},\mathbf{w}) + n_h(\mathbf{z},\mathbf{u};\mathbf{v},\mathbf{w}), \tag{24}$$

where

$$\ell_h(\mathbf{u};\mathbf{v},\mathbf{w}) = \sum_{E\in\mathcal{E}_1^h} (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w})_E + \frac{1}{2}\sum_{E\in\mathcal{E}_1^h} (\nabla\cdot\mathbf{u},\mathbf{v}\cdot\mathbf{w})_E - \frac{1}{2}\sum_{e\in\Gamma_1^h\cup\Gamma_1} ([\mathbf{u}]\cdot\mathbf{n}_e,\{\mathbf{v}\cdot\mathbf{w}\})_e, \tag{25}$$

$$n_h(\mathbf{z},\mathbf{u};\mathbf{v},\mathbf{w}) = \sum_{E\in\mathcal{E}_1^h} (|\{\mathbf{u}\}\cdot\mathbf{n}_E|(\mathbf{v}^{\text{int}}-\mathbf{v}^{\text{ext}}),\mathbf{w}^{\text{int}})_{\partial E_{-}(\mathbf{z})\setminus\Gamma_{12}}, \tag{26}$$

and $\partial E_{-}(\mathbf{z})$ denotes the portion of $\partial E$ where flow enters $E$,

$$\partial E_{-}(\mathbf{z}) = \{\mathbf{x}\in\partial E\,;\,\{\mathbf{z}\}\cdot\mathbf{n}_E < 0\}, \tag{27}$$

and the superscript int (respectively, ext) refers to the interior value to $E$ (respectively, exterior value to $E$) at any point $x$ of $\partial E$). Finally, the time derivative is approximated by backward Euler in time and the nonlinear term is linearized by time–lagging. Denote the time step by $\Delta t > 0$, and let $t_i = i\Delta t$, with $0 \le i \le N_T$, such that $t_{N_T} = T$.

The numerical scheme is the following: Starting from $\mathbf{u}_h^0 = \mathbf{0}$, find $(\mathbf{u}_h^{n+1}, p_{1h}^{n+1}, p_{2h}^{n+1}) \in \mathbf{X}^h \times M_1^h \times M_2^h$, for all $0 \le n \le N_T$ such that

$$\begin{aligned} (\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t},\mathbf{v})_{\Omega_1} &+ a_S(\mathbf{u}_h^{n+1},\mathbf{v}) + b_S(\mathbf{v},p_{1h}^{n+1}) + a_D(p_{2h}^{n+1},q) + c_{NS}(\mathbf{u}_h^n,\mathbf{u}_h^n;\mathbf{u}_h^{n+1},\mathbf{v}) \\ &+(p_{2h}^{n+1},\mathbf{v}\cdot\mathbf{n}_{12})_{\Gamma_{12}} - (\mathbf{u}_h^{n+1}\cdot\mathbf{n}_{12},q)_{\Gamma_{12}} + \sum_{j=1}^{d-1}\big(\frac{1}{G^j}(\mathbf{u}_h^{n+1}\cdot\boldsymbol{\tau}_{12}^j,\mathbf{v}\cdot\boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} \\ &= (\mathbf{f}_1^{n+1},\mathbf{v})_{\Omega_1} + (f_2^{n+1},q)_{\Omega_2},\quad \forall(\mathbf{v},q)\in\mathbf{X}^h\times M_2^h, \end{aligned} \tag{28}$$

$$b_S(\mathbf{u}_h^{n+1},q) = 0.\quad \forall q\in M_1^h. \tag{29}$$

The discretization of the spatial operators in (1)–(3) by the DG forms $a_S$, $b_S$ and $a_D$ has been described in previous works on the steady-state coupling. Readers can refer for instance to [15]. The discretization of the time derivative by a first order finite difference is trivial.

In view of (29), it is convenient to introduce the kernel

$$\mathbf{V}^h = \big\{\mathbf{v} \in \mathbf{X}^h\,; \quad \forall q \in M_1^h, \quad b_S(\mathbf{v}, q) = 0\big\}. \tag{30}$$

## 4. Useful inequalities

In this section, we recall inequalities that will be used in the subsequent estimates. All constants are independent of $h$ and the functions. To simplify, the range of exponents related to Sobolev's inequalities are given in the case $d = 3$. First, we have the equivalence of norms,

$$\forall \mathbf{v}_h \in \mathbf{X}^h, \quad \|\mathbf{v}_h\|_X \leq C \left( \sum_{E \in \mathcal{E}_1^h} \|\nabla \mathbf{v}_h\|^2_{L^2(E)} + \sum_{e \in \Gamma_1^h \cup \Gamma_1} \frac{\sigma_e}{h_e} \|[\mathbf{v}_h]\|^2_{L^2(e)} \right)^{1/2}, \tag{31}$$

$$\forall \mathbf{v}_h \in \mathbf{X}^h, \quad \big( \sum_{E \in \mathcal{E}_1^h} \|\nabla \mathbf{v}_h\|^2_{L^2(E)} \big)^{1/2} \leq C \|\mathbf{v}_h\|_X. \tag{32}$$

This last inequality comes from Proposition 4.7 in [15], see also [4, 5]. Next, we have the trace inequalities,

$$\text{for } 2 \leq r \leq 4, \quad \forall \mathbf{v}_h \in \mathbf{X}^h, \quad \|\mathbf{v}_h\|_{L^r(\Gamma_{12})} \leq C \left( \sum_{E \in \mathcal{E}_1^h} \|\nabla \mathbf{v}_h\|^2_{L^2(E)} + \sum_{e \in \Gamma_1^h \cup \Gamma_1} \frac{\sigma_e}{h_e} \|[\mathbf{v}_h]\|^2_{L^2(e)} \right)^{1/2}, \tag{33}$$

$$\text{for } 2 \leq r \leq 4, \quad \forall q_h \in M_2^h, \quad \|q_h\|_{L^r(\Gamma_{12})} \leq C \|q_h\|_{M_2}. \tag{34}$$

Similarly, we have Sobolev's imbeddings [1, 17],

$$\text{for } 2 \leq r \leq 6, \quad \forall \mathbf{v}_h \in \mathbf{X}^h, \quad \|\mathbf{v}_h\|_{L^r(\Omega_1)} \leq C \left( \sum_{E \in \mathcal{E}_1^h} \|\nabla \mathbf{v}_h\|^2_{L^2(E)} + \sum_{e \in \Gamma_1^h \cup \Gamma_1} \frac{\sigma_e}{h_e} \|[\mathbf{v}_h]\|^2_{L^2(e)} \right)^{1/2}, \tag{35}$$

$$\text{for } 2 \leq r \leq 6, \quad \forall q_h \in M_2^h, \quad \|q_h\|_{L^r(\Omega_2)} \leq C \|q_h\|_{M_2}. \tag{36}$$

When $r = 2$, the constant in (36) will be denoted by $C_1$. By combining (32) with (35) or (33), we also obtain,

$$\text{for } 2 \leq r \leq 6, \quad \forall \mathbf{v}_h \in \mathbf{X}^h, \quad \|\mathbf{v}_h\|_{L^r(\Omega_1)} \leq C \|\mathbf{v}_h\|_X, \tag{37}$$

$$\text{for } 2 \leq r \leq 4, \quad \forall \mathbf{v}_h \in \mathbf{X}^h, \quad \|\mathbf{v}_h\|_{L^r(\Gamma_{12})} \leq C \|\mathbf{v}_h\|_X. \tag{38}$$

When $r = 2$, the constant in (37) will be denoted by $C_2$.

Lastly, here are some inequalities satisfied by the discrete forms. The bilinear forms have the following ellipticity properties [20],

$$\forall \mathbf{v}_h \in \mathbf{X}^h, \quad C_3\mu\|\mathbf{v}_h\|_X^2 \le a_S(\mathbf{v}_h, \mathbf{v}_h), \tag{39}$$

$$\forall q_h \in M_2^h, \quad C_4\|q_h\|_{M_2}^2 \le a_D(q_h, q_h), \tag{40}$$

provided the coefficients $\sigma_e$ are chosen appropriately when $\epsilon_i = 0$ or $-1$. Moreover, whatever the coefficients $\sigma_e$, the forms $a_D$ and $a_S$ are continuous in the discrete spaces,

$$\forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{X}^h, \quad a_S(\mathbf{v}_h, \mathbf{w}_h) \le C_5\mu\|\mathbf{v}_h\|_X\|\mathbf{w}_h\|_X, \tag{41}$$

$$\forall p_h, q_h \in M_2^h, \quad a_D(p_h, q_h) \le C_6\|p_h\|_{M_2}\|q_h\|_{M_2}. \tag{42}$$

For the nonlinear form, instead of ellipticity, we have (see Lemma 4.11 in [15])

$$\forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{X}^h, \quad c_{NS}(\mathbf{u}_h, \mathbf{u}_h; \mathbf{v}_h, \mathbf{v}_h) \ge \frac{1}{2}(\mathbf{u}_h \cdot \mathbf{n}_{12}, \mathbf{v}_h \cdot \mathbf{v}_h)_{\Gamma_{12}}. \tag{43}$$

Using (38), we easily obtain, for a constant $C_7$:

$$\forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{X}^h, \quad \frac{1}{2}(\mathbf{v}_h \cdot \mathbf{n}_{12}, \mathbf{w}_h \cdot \mathbf{w}_h)_{\Gamma_{12}} \le C_7\|\mathbf{v}_h\|_X\|\mathbf{w}_h\|_X^2. \tag{44}$$

We also have the following continuity with respect to the last three variables (see (4.23) in [15]):

$$\forall \mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h, \mathbf{z}_h \in \mathbf{X}^h + \mathbf{X}, \quad c_{NS}(\mathbf{z}_h, \mathbf{u}_h; \mathbf{v}_h, \mathbf{w}_h) \le C_8\|\mathbf{u}_h\|_X\|\mathbf{v}_h\|_X\|\mathbf{w}_h\|_X. \tag{45}$$

## 5. Analysis of the discrete problem

5.1. **Preliminary results.** Problem (28)–(29) may be reduced by eliminating $p_{1h}^{n+1}$ as follows: Starting from $\mathbf{u}_h = \mathbf{0}$, find $(\mathbf{u}_h^{n+1}, p_{2h}^{n+1}) \in \mathbf{V}^h \times M_2^h$ such that

$$\begin{aligned}
&(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v})_{\Omega_1} + a_S(\mathbf{u}_h^{n+1}, \mathbf{v}) + a_D(p_{2h}^{n+1}, q) + c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n; \mathbf{u}_h^{n+1}, \mathbf{v}) \\
&+(p_{2h}^{n+1}, \mathbf{v}\cdot\mathbf{n}_{12})_{\Gamma_{12}} - (\mathbf{u}_h^{n+1}\cdot\mathbf{n}_{12}, q)_{\Gamma_{12}} + \sum_{j=1}^{d-1}(\frac{1}{G^j}(\mathbf{u}_h^{n+1}\cdot\boldsymbol{\tau}_{12}^j, \mathbf{v}\cdot\boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} \\
&\qquad = (\mathbf{f}_1^{n+1}, \mathbf{v})_{\Omega_1} + (f_2^{n+1}, q)_{\Omega_2}, \quad \forall(\mathbf{v}, q) \in \mathbf{V}^h \times M_2^h.
\end{aligned} \tag{46}$$

In the above we used the notation:

$$\mathbf{f}_1^{n+1}(\cdot) = \mathbf{f}_1(t^{n+1}, \cdot), \quad f_2^{n+1}(\cdot) = f_2(t^{n+1}, \cdot).$$

We start with a useful property of the nonlinear term.

**Lemma 5.1.** *For all $\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in \mathbf{X}^h$, we have*

$$\begin{aligned} c_{NS}(\mathbf{u}_h, \mathbf{u}_h; \mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h) - c_{NS}(\mathbf{v}_h, \mathbf{v}_h; \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) \\ \geq \frac{1}{2}(\mathbf{v}_h \cdot \mathbf{n}_{12}, (\mathbf{u}_h - \mathbf{v}_h) \cdot (\mathbf{u}_h - \mathbf{v}_h))_{\Gamma_{12}} - C_9 \|\mathbf{u}_h\|_X \|\mathbf{u}_h - \mathbf{v}_h\|_X^2, \end{aligned} \tag{47}$$

$$|c_{NS}(\mathbf{u}_h, \mathbf{u}_h; \mathbf{u}_h, \mathbf{w}_h) - c_{NS}(\mathbf{v}_h, \mathbf{v}_h; \mathbf{u}_h, \mathbf{w}_h)| \leq C_{10} \|\mathbf{u}_h - \mathbf{v}_h\|_X \|\mathbf{w}_h\|_X \|\mathbf{u}_h\|_X, \tag{48}$$

*with constants $C_9$ and $C_{10}$ independent of $h$.*

*Proof.* In order to obtain (47), we rewrite, using the notation $\mathbf{w}_h = \mathbf{u}_h - \mathbf{v}_h$

$$\begin{aligned} c_{NS}(\mathbf{u}_h, \mathbf{u}_h; \mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h) - c_{NS}(\mathbf{v}_h, \mathbf{v}_h; \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) = c_{NS}(\mathbf{v}_h, \mathbf{v}_h; \mathbf{w}_h, \mathbf{w}_h) + c_{NS}(\mathbf{u}_h, \mathbf{w}_h; \mathbf{u}_h, \mathbf{w}_h) \\ + n_h(\mathbf{u}_h, \mathbf{v}_h; \mathbf{u}_h, \mathbf{w}_h) - n_h(\mathbf{v}_h, \mathbf{v}_h; \mathbf{u}_h, \mathbf{w}_h). \end{aligned}$$

Using (43), (45), and Proposition 4.10 in [15], we obtain (47).

In order to obtain (48), we rewrite the difference as follows:

$$\begin{aligned} c_{NS}(\mathbf{u}, \mathbf{u}; \mathbf{u}, \mathbf{w}) - c_{NS}(\mathbf{v}, \mathbf{v}; \mathbf{u}, \mathbf{w}) &= \ell_h(\mathbf{u}; \mathbf{u}, \mathbf{w}) - \ell_h(\mathbf{v}; \mathbf{u}, \mathbf{w}) + n_h(\mathbf{u}, \mathbf{u} - \mathbf{v}; \mathbf{u}, \mathbf{w}) \\ &+ n_h(\mathbf{u}, \mathbf{v}; \mathbf{u}, \mathbf{w}) - n_h(\mathbf{v}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \\ &= \ell_h(\mathbf{u} - \mathbf{v}; \mathbf{u}, \mathbf{w}) + n_h(\mathbf{u}, \mathbf{u} - \mathbf{v}; \mathbf{u}, \mathbf{w}) + n_h(\mathbf{u}, \mathbf{v}; \mathbf{u}, \mathbf{w}) - n_h(\mathbf{v}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \\ &= c_{NS}(\mathbf{u}, \mathbf{u} - \mathbf{v}; \mathbf{u}, \mathbf{w}) + n_h(\mathbf{u}, \mathbf{v}; \mathbf{u}, \mathbf{w}) - n_h(\mathbf{v}, \mathbf{v}; \mathbf{u}, \mathbf{w}). \end{aligned}$$

The proof follows from the continuity of $c_{NS}$ (see (45)) and Proposition 4.10 in [15]. □

The next theorem, which is the most technical part of this work, proves an important bound for the interaction term on the interface. It will be used in the existence proof of the discrete solution for controlling the discrete time derivative after one time step, see Proposition 5.8.

**Theorem 5.2.** *There is a constant $C_{11}$, independent of $h$ such that for all $\mathbf{u}_h \in \mathbf{V}^h$ and for all $q_{2h} \in M_2^h$,*

$$|(q_{2h}, \mathbf{u}_h \cdot \mathbf{n}_{12})_{\Gamma_{12}}| \leq C_{11} \|q_{2h}\|_{M_2} \|\mathbf{u}_h\|_{L^2(\Omega_1)}. \tag{49}$$

*Proof.* Observe that (49) follows immediately from Green's formula when $q_{2h}$ is in $H^1(\Omega_2)$ and $\mathbf{u}_h$ in $H_0(\text{div}, \Omega)$ with $\text{div}\, \mathbf{u}_h = 0$. Although, Green's formula cannot be applied here because the discretization is nonconforming, this observation suggests to regularize the discrete functions in order to use Green's formula.

First, we examine the local effect of Green's formula on functions of $M_1^h$. Let $q_{1h}$ be a function of $M_1^h$ and let $\mathbf{u}_h$ be a function of $\mathbf{X}^h$. Using the definition of $b_S$ and Green's formula in each $E \in \mathcal{E}_1^h$, we infer

$$(q_{1h}, \mathbf{u}_h \cdot \mathbf{n}_{12})_{\Gamma_{12}} = -b_S(\mathbf{u}_h, q_{1h}) + \sum_{E\in\mathcal{E}_1^h} (\nabla q_{1h}, \mathbf{u}_h)_E - \sum_{e\in\Gamma_1^h} ([q_{1h}], \{\mathbf{u}_h\} \cdot \mathbf{n}_e)_e.$$

If in addition $\mathbf{u}_h$ satisfies (29), then we have

$$(q_{1h}, \mathbf{u}_h \cdot \mathbf{n}_{12})_{\Gamma_{12}} = \sum_{E\in\mathcal{E}_1^h} (\nabla q_{1h}, \mathbf{u}_h)_E - \sum_{e\in\Gamma_1^h} ([q_{1h}], \{\mathbf{u}_h\} \cdot \mathbf{n}_e)_e.$$

Finally, by using an equivalence of norms, we easily obtain:

$$|(q_{1h}, \mathbf{u}_h \cdot \mathbf{n}_{12})_{\Gamma_{12}}| \le C\|\mathbf{u}_h\|_{L^2(\Omega_1)} \left( \sum_{E\in\mathcal{E}_1^h} \|\nabla q_{1h}\|^2_{L^2(E)} + \sum_{e\in\Gamma_h^1} \frac{1}{h_e} \|[q_{1h}]\|^2_{L^2(e)} \right)^{1/2}. \tag{50}$$

Therefore, if we able to relate functions of $M_2^h$ to suitable functions of $M_1^h$, we should be able to obtain (49). This is achieved in three steps: regularizing the functions of $M_2^h$, extending the regularized functions continuously to $H^1(\Omega_1)$, and discretizing the extended functions in $M_1^h$.

1) The functions $q_{2h}$ of $M_2^h$ are regularized by means of a Scott & Zhang interpolant [22], but since the functions $q_{2h}$ are smooth in each $E$, averages can be replaced by pointwise values, as follows. For a vertex $a$ that does not belong to the interface $\Gamma_{12}$, we associate an element $E_a \in \mathcal{E}_2^h$ such that $a$ is a vertex of $E_a$ and choose for nodal value at $a$ the value $q_{2h}|_{E_a}(a)$. For a vertex $a \in \Gamma_{12}$, we associate an element $E_a \in \mathcal{E}_2^h$ such that $E_a$ has a face on $\Gamma_{12}$ and $a$ is a vertex of this face and similarly choose for nodal value at $a$ the value $q_{2h}|_{E_a}(a)$. Then the interpolant $R_h(q_{2h}) \in H^1(\Omega_2)$ is the globally continuous function that is piecewise $\mathbb{P}_1$ in each element $E$ and takes the above nodal values at the vertices $a$ of $\mathcal{E}_2^h$. Note that its trace $R_h(q_{2h})|_{\Gamma_{12}}$ only depends on $q_{2h}|_{\Gamma_{12}}$. The operator $R_h$ is stable in the sense that there is a constant $C$ independent of $h$ such that

$$\forall q_{2h} \in M_2^h, \quad \|R_h(q_{2h})\|_{H^1(\Omega_2)} \le C\|q_{2h}\|_{M_2}. \tag{51}$$

This uniform stability of $R_h$ is established in Lemma 5.3.

2) As $R_h(q_{2h})$ belongs to $H^1(\Omega_2)$, we apply the standard extension operator $\mathcal{E} : H^1(\Omega_2) \to H^1(\Omega)$ to $R_h(q_{2h})$, so that $\mathcal{E}(R_h(q_{2h})) \in H^1(\Omega)$, coincides with $R_h(q_{2h})$ in $\Omega_2$,

$$\|\mathcal{E}(R_h(q_{2h}))\|_{H^1(\Omega)} \le C\|R_h(q_{2h})\|_{H^1(\Omega_2)}, \tag{52}$$

with a constant $C$ independent of $h$, and

$$\mathcal{E}(R_h(q_{2h})) = R_h(q_{2h}), \quad \text{on } \Gamma_{12}. \tag{53}$$

3) The extended function $\mathcal{E}(R_h(q_{2h}))$ is approximated in $\Omega_1$ by a function of $M_1^h$ defined as follows. For any $q$ in $H^1(\Omega_1)$, for each element $E \in \mathcal{E}_1^h$ that does not lie on the interface $\Gamma_{12}$, we choose

$$S_h(q)|_E = \frac{1}{|E|}\int_E q;$$

for an element $E \in \mathcal{E}_1^h$ that has a face $e$ on $\Gamma_{12}$, we choose

$$S_h(q)|_E = \frac{1}{|e|}\int_e q.$$

Note that in both cases, $S_h$ preserves the constant functions in each $E$, i.e. if $q|_E = C$, then $S_h(q)|_E = C$. Now, we write

$$|(q_{2h}, \mathbf{u}_h \cdot \mathbf{n}_{12})_{\Gamma_{12}}| \leq |(q_{2h} - S_h(q_{2h}), \mathbf{u}_h \cdot \mathbf{n}_{12})_{\Gamma_{12}}| + |(S_h(q_{2h}), \mathbf{u}_h \cdot \mathbf{n}_{12})_{\Gamma_{12}}|, \tag{54}$$

where, to simplify, $S_h(q_{2h})$ stands for $S_h(\mathcal{E}(R_h(q_{2h})))$. Let us start with the second term. As $S_h(q_{2h}) \in M_1^h$ is a piecewise constant, (50) reduces to

$$\begin{aligned}\big|(S_h(q_{2h}), \mathbf{u}_h \cdot \mathbf{n}_{12})_{\Gamma_{12}}\big| &\leq C\|\mathbf{u}_h\|_{L^2(\Omega_1)} \left(\sum_{e\in\Gamma_1^h} \frac{1}{h_e}\|[S_h(q_{2h})]\|^2_{L^2(e)}\right)^{1/2} \\ &\leq C\|\mathbf{u}_h\|_{L^2(\Omega_1)} \left(\sum_{e\in\Gamma_1^h} \frac{1}{h_e}\|[S_h(q_{2h}) - \mathcal{E}(R_h(q_{2h}))]\|^2_{L^2(e)}\right)^{1/2}.\end{aligned} \tag{55}$$

We have for any face $e \subset \Gamma_1^h$:

$$\|[S_h(q_{2h}){-}\mathcal{E}(R_h(q_{2h}))]\|_{L^2(e)} \leq \|(S_h(q_{2h}){-}\mathcal{E}(R_h(q_{2h})))|_{E_-}\|_{L^2(e)} {+} \|(S_h(q_{2h}){-}\mathcal{E}(R_h(q_{2h})))|_{E_+}\|_{L^2(e)},$$

where $E_-$ and $E_+$ are elements of $\mathcal{E}_1^h$ sharing the face $e$. Since $S_h$ preserves the piecewise constants in each $E$, we have for any $q \in H^1(E)$, when passing to the reference element $\hat{E}$

$$\|(S_h(q) - q)|_{E_-}\|_{L^2(e)} \leq C\,|e|^{1/2}|\hat{q}|_{H^1(\hat{E})} \leq C\big(\frac{|e|}{|E_-|}\big)^{1/2} h_{E_-}|q|_{H^1(E_-)}.$$

This yields

$$\frac{1}{h_e}\|(S_h(q_{2h}) - \mathcal{E}(R_h(q_{2h})))|_{E_-}\|^2_{L^2(e)} \leq C\|\nabla\mathcal{E}(R_h(q_{2h}))\|^2_{L^2(E_-)},$$

with a similar bound for the term in $E_+$, and thus we have

$$\Big(\sum_{e\in\Gamma_1^h}\frac{1}{h_e}\|[S_h(q_{2h})-\mathcal{E}(R_h(q_{2h}))]\|^2_{L^2(e)}\Big)^{1/2}\le C\|\nabla\mathcal{E}(R_h(q_{2h}))\|_{L^2(\Omega_1)}.$$

Then from (55) and the stability of $\mathcal{E}$ (see (52)) and $R_h$ (see (51)), we conclude that

$$\big|(S_h(q_{2h}),\mathbf{u}_h\cdot\mathbf{n}_{12})_{\Gamma_{12}}\big|\le C\|\mathbf{u}_h\|_{L^2(\Omega_1)}\|\nabla\mathcal{E}(R_h(q_{2h}))\|_{L^2(\Omega_1)}\le C\|\mathbf{u}_h\|_{L^2(\Omega_1)}\|q_{2h}\|_{M_2}. \tag{56}$$

Next we insert $R_h(q_{2h})$ in the first term of (54),

$$|(q_{2h}-S_h(q_{2h}),\mathbf{u}_h\cdot\mathbf{n}_{12})_{\Gamma_{12}}|\le|(q_{2h}-R_h(q_{2h}),\mathbf{u}_h\cdot\mathbf{n}_{12})_{\Gamma_{12}}|+|(R_h(q_{2h})-S_h(q_{2h}),\mathbf{u}_h\cdot\mathbf{n}_{12})_{\Gamma_{12}}|.$$

By construction, on any face $e\subset\Gamma_{12}$, since $S_h$ preserves the constant functions on $e$, we have

$$\|q-S_h(q)\|_{L^2(e)}\le C|e|^{1/2}|\hat{q}|_{H^1(\hat{e})},$$

for sufficiently smooth functions $q$. Here $q$ is the trace on $e$ of $R_h(q_{2h})\in\mathbb{P}_1(E_2)$, where $E_2$ is the element of $\mathcal{E}_2^h$ adjacent to $e$. Therefore,

$$|\hat{q}|_{H^1(\hat{e})}=|\widehat{R_h(q_{2h})}|_{H^1(\hat{e})}\le C|\widehat{R_h(q_{2h})}|_{H^1(\hat{E})}.$$

Hence, we have on one hand

$$\|R_h(q_{2h})-S_h(q_{2h})\|_{L^2(e)}\le C\big(\frac{|e|}{|E_2|}\big)^{1/2}h_{E_2}|R_h(q_{2h})|_{H^1(E_2)}.$$

On the other hand, denoting by $E_1$ the element of $\mathcal{E}_1^h$ adjacent to $e$, we have

$$\|\mathbf{u}_h\cdot\mathbf{n}_{12}\|_{L^2(e)}\le C\big(\frac{|e|}{|E_1|}\big)^{1/2}\|\mathbf{u}_h\|_{L^2(E_1)}.$$

By multiplying these two estimates, we obtain

$$\big|(R_h(q_{2h})-S_h(q_{2h}),\mathbf{u}_h\cdot\mathbf{n}_{12})_e\big|\le C|R_h(q_{2h})|_{H^1(E_2)}\|\mathbf{u}_h\|_{L^2(E_1)}.$$

Therefore, denoting by $D_i$ the layer of elements of $\mathcal{E}_i^h$ adjacent to $\Gamma_{12}$, $i=1,2$, and applying (59) below, we infer

$$\big|(R_h(q_{2h})-S_h(q_{2h}),\mathbf{u}_h\cdot\mathbf{n}_{12})_{\Gamma_{12}}\big|\le C|R_h(q_{2h})|_{H^1(D_2)}\|\mathbf{u}_h\|_{L^2(D_1)}\le C\|q_{2h}\|_{M_2}\|\mathbf{u}_h\|_{L^2(D_1)}. \tag{57}$$

It remains to evaluate $(q_{2h}-R_h(q_{2h}),\mathbf{u}_h\cdot\mathbf{n}_{12})_{\Gamma_{12}}$. For this, by proceeding exactly as in Lemma 5.3 below, we easily derive that

$$\big|(q_{2h}-R_h(q_{2h}),\mathbf{u}_h\cdot\mathbf{n}_{12})_{\Gamma_{12}}|\le C\|q_{2h}\|_{M_2}\|\mathbf{u}_h\|_{L^2(D_1)}. \tag{58}$$

Then (49) follows from (54), (56)–(58). ∎

**Lemma 5.3.** *The operator $R_h$, defined above, is stable in the following sense: There exists a constant $C$, independent of $h$, such that*

$$\forall q_{2h} \in M_2^h, \quad \|R_h(q_{2h})\|_{H^1(\Omega_2)} \le C\|q_{2h}\|_{M_2}. \tag{59}$$

*Proof.* In each $E$ of $\mathcal{E}_2^h$, we insert the standard Lagrange interpolant of $q_{2h}$, $L_h(q_{2h}) \in \mathbb{P}_1$,

$$R_h(q_{2h}) - q_{2h} = R_h(q_{2h}) - L_h(q_{2h}) + L_h(q_{2h}) - q_{2h}.$$

So

$$\begin{aligned}(\sum_{E\in\mathcal{E}_2^h} \|\nabla(R_h(q_{2h}) - q_{2h})\|^2_{L^2(E)})^{1/2} \le &(\sum_{E\in\mathcal{E}_2^h} \|\nabla(R_h(q_{2h}) - L_h(q_{2h}))\|^2_{L^2(E)})^{1/2}\\ &+ (\sum_{E\in\mathcal{E}_2^h} \|\nabla(L_h(q_{2h}) - q_{2h})\|^2_{L^2(E)})^{1/2}.\end{aligned}$$

The classical Lagrange interpolation results, the regularity of the family of triangulations, and a local inverse inequality yield

$$\|\nabla(L_h(q_{2h}) - q_{2h})\|_{L^2(E)} \le Ch_E|q_{2h}|_{H^2(E)} \le C\,|q_{2h}|_{H^1(E)}.$$

Similarly

$$\|L_h(q_{2h}) - q_{2h}\|_{L^2(E)} \le Ch_E|q_{2h}|_{H^1(E)}.$$

Hence

$$(\sum_{E\in\mathcal{E}_2^h} \|\nabla(L_h(q_{2h}) - q_{2h})\|^2_{L^2(E)})^{1/2} + \|L_h(q_{2h}) - q_{2h}\|_{L^2(\Omega_2)} \le C(\sum_{E\in\mathcal{E}_2^h} \|\nabla q_{2h}\|^2_{L^2(E)})^{1/2}. \tag{60}$$

To bound the term $\nabla(R_h(q_{2h}) - L_h(q_{2h}))$ in $E$, we write

$$(R_h(q_{2h}) - L_h(q_{2h}))|_E = \sum_{i=1}^{d+1}(R_h(q_{2h})(A_i) - L_h(q_{2h})(A_i))\varphi_i,$$

where $A_i$, $1 \le i \le d+1$, are the vertices of $E$ and $\varphi_i$ the corresponding nodal basis functions. Thus,

$$\nabla(R_h(q_{2h}) - L_h(q_{2h}))|_E = \sum_{i=1}^{d+1}(R_h(q_{2h})(A_i) - q_{2h}(A_i))\nabla\varphi_i.$$

Since

$$\|\nabla\varphi_i\|_{L^2(E)} \le C(i)\frac{|E|^{1/2}}{\varrho_E},$$

where $C(i)$ is a constant that depends only on the reference element $\hat{E}$, we conclude that

$$\|\nabla(R_h(q_{2h}) - L_h(q_{2h}))\|_{L^2(E)} \leq C\frac{|E|^{1/2}}{\varrho_E}\sum_{i=1}^{d+1}|R_h(q_{2h})(A_i) - q_{2h}(A_i)|. \quad (61)$$

For a given vertex $A_i$ of $E$, there is a sequence of elements, say $E_1, E_2, ..., E_m$, such that $E = E_1$, $A_i$ is a vertex of all $E_1, E_2, ..., E_m$, $R_h(q_{2h})(A_i) = q_{2h}|_{E_m}(A_i)$ and each consecutive pair of elements share an interior face (i.e. a face that belongs to $\Gamma_2^h$) denoted by $e_j$, $2 \leq j \leq m$. The regularity assumption on the family of meshes imply that the integer $m$ is bounded by a constant, say $M$, that is independent of $E$ and $h$. The difference $q_{2h} - R_h(q_{2h})$ can be re-rewritten as follows:

$$\begin{aligned} q_{2h}(A_i) - R_h(q_{2h})(A_i) &= q_{2h}|_{E_1}(A_i) - q_{2h}|_{E_m}(A_i) \\ &= q_{2h}|_{E_1}(A_i) - q_{2h}|_{E_2}(A_i) + \cdots + q_{2h}|_{E_{m-1}}(A_i) - q_{2h}|_{E_m}(A_i). \end{aligned}$$

Each difference in the above sum is bounded by Lemma 5.4 below,

$$|q_{2h}(A_i) - R_h(q_{2h})(A_i)| \leq C\sum_{j=2}^{m}\frac{1}{|e_j|^{1/2}}\|[q_{2h}]\|_{L^2(e_j)}. \quad (62)$$

Therefore by substituting (62) into (61), using the regularity (14) of the family of triangulations, and combining with (60), we have

$$(\sum_{E\in\mathcal{E}_2^h}\|\nabla(R_h(q_{2h}) - q_{2h})\|^2_{L^2(E)})^{1/2} \leq C\|q_{2h}\|_{M_2}. \quad (63)$$

The same argument gives

$$\|R_h(q_{2h}) - q_{2h}\|_{L^2(\Omega_2)} \leq C\,h\|q_{2h}\|_{M_2}, \quad (64)$$

and by a triangle inequality, this yields (59). □

The next lemma is written in the setting of $M_2^h$, but it holds in a general situation.

**Lemma 5.4.** *Let $e$ be a face of $\Gamma_2^h$, $A$ a vertex of $e$ and let $E_+$ and $E_-$ be the two elements adjacent to $e$. There exists a constant $C$, independent of $h$ such that*

$$\forall q \in M_2^h, \quad |q(A)|_{E_+} - q(A)|_{E_-}| \leq C\frac{1}{|e|^{1/2}}\|[q]\|_{L^2(e)}.$$

*Proof.* Let us transform the union $E_+ \cup E_-$ by means of a continuous, piecewise affine mapping, into the union of two reference unit tetrahedra $\hat{E}_+ \cup \hat{E}_-$. As usual, the composition is denoted by a hat. We have

$$|q(A)|_{E_+} - q(A)|_{E_-}| = |\hat{q}|_{\hat{E}_+}(\hat{A}) - \hat{q}|_{\hat{E}_-}(\hat{A})| = |[\hat{q}](\hat{A})| \leq \|[\hat{q}]\|_{L^\infty(\hat{e})}.$$

Then equivalence of norms yields

$$\|[\hat{q}]\|_{L^\infty(\hat{e})} \le \hat{C}\|[\hat{q}]\|_{L^2(\hat{e})} \le C|e|^{-1/2}\|[q]\|_{L^2(e)}.$$

□

5.2. **Existence and uniqueness of the numerical solution.** Although, at each time step, (46) is a square linear system in finite dimension, proving the nonsingularity of its matrix without restricting the time step is not straightforward. This is due to the fact that the nonlinear term, not being antisymmetric (see (43) and (44)) cannot be eliminated on the interface. Let us start with a first sufficient condition for uniqueness. To simplify the discussion, all proofs are written in the case $d = 3$.

**Lemma 5.5.** *Let $1 \le n \le N_T - 1$. If at step $n$, $(\mathbf{u}_h^n, p_{2h}^n)$ is given satisfying*

$$\|\mathbf{u}_h^n\|_X \le \frac{\mu\, C_3}{C_7}, \tag{65}$$

*then, if the solution $(\mathbf{u}_h^{n+1}, p_{2h}^{n+1})$ of (46) exists, then it is unique.*

*Proof.* Assume that at step $n+1$, (46) has two solutions $(\mathbf{u}_h^{n+1}, p_{2h}^{n+1})$ and $(\tilde{\mathbf{u}}_h^{n+1}, \tilde{p}_{2h}^{n+1})$ for a given $(\mathbf{u}_h^n, p_{2h}^n)$; then their difference $(\mathbf{w}_h^{n+1}, z_{2h}^{n+1})$ solves

$$\frac{1}{\Delta t}(\mathbf{w}_h^{n+1}, \mathbf{v})_{\Omega_1} + a_S(\mathbf{w}_h^{n+1}, \mathbf{v}) + a_D(z_{2h}^{n+1}, q) + c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n; \mathbf{w}_h^{n+1}, \mathbf{v}) + (z_{2h}^{n+1}, \mathbf{v}\cdot\mathbf{n}_{12})_{\Gamma_{12}}$$
$$-(\mathbf{w}_h^{n+1}\cdot\mathbf{n}_{12}, q)_{\Gamma_{12}} + \sum_{j=1}^{d-1}\big(\frac{1}{G_j}\mathbf{w}_h^{n+1}\cdot\boldsymbol{\tau}_{12}^j, \mathbf{v}\cdot\boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} = 0, \quad \forall (\mathbf{v}, q) \in \mathbf{V}^h \times M_2^h. \tag{66}$$

By testing (66) with $(\mathbf{v}, q) = (\mathbf{w}_h^{n+1}, z_{2h}^{n+1})$ and using (39), (40), (43), and (44), we obtain

$$\frac{1}{\Delta t}\|\mathbf{w}_h^{n+1}\|^2_{L^2(\Omega_1)} + \mu\, C_3\|\mathbf{w}_h^{n+1}\|_X^2 + C_4\|z_{2h}^{n+1}\|^2_{M_2} + \sum_{j=1}^{d-1}\|\frac{1}{G_j^{1/2}}\mathbf{w}_h^{n+1}\cdot\boldsymbol{\tau}_{12}^j\|^2_{L^2(\Gamma_{12})}$$
$$\le C_7\|\mathbf{u}_h^n\|_X\|\mathbf{w}_h^{n+1}\|_X^2.$$

Then (65) implies $\mathbf{w}_h^{n+1} = \mathbf{0}$ and $z_{2h}^{n+1} = 0$. □

Checking (65) for arbitrary $n$ is not easy because it consists in a bound for the maximum in time in the norm $\|\cdot\|_X$, whereas the standard basic a priori estimate, written below, is a bound for the maximum in time in the $L^2$ norm. It is an a priori estimate because it assumes existence of a solution.

**Lemma 5.6.** *For $1 \leq n \leq N_T$, define*

$$\mathcal{H}_n = \frac{C_2^2}{\mu\, C_3} \Delta t \sum_{i=1}^n \|\mathbf{f}_1^i\|^2_{L^2(\Omega_1)} + \frac{C_1^2}{2\, C_4} \Delta t \sum_{i=1}^n \|f_2^i\|^2_{L^2(\Omega_2)}. \tag{67}$$

*Let $\mathbf{u}_h^0 = \mathbf{0}$ and let $1 \leq n \leq N_T - 1$. If for all $i$, $0 \leq i \leq n$, (46) has a solution $(\mathbf{u}_h^i, p_{2h}^i)$ satisfying*

$$\|\mathbf{u}_h^i\|_X \leq \frac{\mu\, C_3}{2\, C_7}, \tag{68}$$

*then for all $i$, $1 \leq i \leq n+1$,*

$$\|\mathbf{u}_h^i\|^2_{L^2(\Omega_1)} \leq \mathcal{H}_i. \tag{69}$$

*Proof.* The proof is sketched because it is standard. By testing (46) with $(\mathbf{v}, q) = (\mathbf{u}_h^{n+1}, p_{2h}^{n+1})$ and using (39), (40), (43), and (44), we derive

$$\begin{aligned}
&\frac{1}{2\Delta t}\Big(\|\mathbf{u}_h^{n+1}\|^2_{L^2(\Omega_1)} - \|\mathbf{u}_h^n\|^2_{L^2(\Omega_1)} + \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|^2_{L^2(\Omega_1)}\Big) + \mu\, C_3 \|\mathbf{u}_h^{n+1}\|^2_X + C_4 \|p_{2h}^{n+1}\|^2_{M_2} \\
&\quad + \sum_{j=1}^{d-1} \|\frac{1}{G_j^{1/2}} \mathbf{u}_h^{n+1} \cdot \boldsymbol{\tau}_{12}^j\|^2_{L^2(\Gamma_{12})} \\
&\quad \leq C_7 \|\mathbf{u}_h^n\|_X \|\mathbf{u}_h^{n+1}\|^2_X + \|\mathbf{f}_1^{n+1}\|_{L^2(\Omega_1)} \|\mathbf{u}_h^{n+1}\|_{L^2(\Omega_1)} + \|f_2^{n+1}\|_{L^2(\Omega_2)} \|p_{2h}^{n+1}\|_{L^2(\Omega_2)}.
\end{aligned}$$

Then by applying the inequalities (37) and (36) with $r = 2$, and suitable Young's inequalities to the last two terms of the right-hand side, and using (68), we deduce

$$\frac{1}{2\Delta t}\Big(\|\mathbf{u}_h^{n+1}\|^2_{L^2(\Omega_1)} - \|\mathbf{u}_h^n\|^2_{L^2(\Omega_1)} + \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|^2_{L^2(\Omega_1)}\Big) \leq \frac{C_2^2}{2\mu\, C_3} \|\mathbf{f}_1^{n+1}\|^2_{L^2(\Omega_1)} + \frac{C_1^2}{4\, C_4} \|f_2^{n+1}\|^2_{L^2(\Omega_2)}.$$

The result follows by summing this inequality and using the fact that $\mathbf{u}_h^0 = \mathbf{0}$. Note that we can also derive a bound for $\Delta t \sum_{i=1}^{n+1} \|\mathbf{u}_h^i\|^2_X$ and $\Delta t \sum_{i=1}^{n+1} \|p_{2h}^i\|^2_{M_2}$ by increasing $\mathcal{H}_n$, but it does not seem to be useful in the rest of the analysis. ☐

Experience shows that a bound for $\|\mathbf{u}_h^n\|_X$ stems from a suitable bound for the discrete time derivative, (see formula (3.112) Chapter 3 in [23] or [18]), as expressed in the next lemma. To simplify, for a given sequence $(g^i)_i$, we denote the difference and discrete time-derivative by

$$\delta_i g = g^{i+1} - g^i, \quad \delta_t g^i = \frac{\delta_i g}{\Delta t} = \frac{g^{i+1} - g^i}{\Delta t}. \tag{70}$$

**Lemma 5.7.** *Define*

$$\mathcal{G}_1 = \frac{C_2^2}{\mu\, C_3} \|\mathbf{f}_1\|^2_{L^\infty(0,T;L^2(\Omega_1)^3)}, \quad \mathcal{G}_2 = \frac{C_1^2}{C_4} \|f_2\|^2_{L^\infty(0,T;L^2(\Omega_2))}. \tag{71}$$

*Let* $\mathbf{u}_h^0 = \mathbf{0}$ *and* $0 \le n \le N_T - 1$. *If for all* $i$, $0 \le i \le n$, (46) *has a solution* $(\mathbf{u}_h^i, p_{2h}^i)$ *satisfying*

$$\|\mathbf{u}_h^i\|_X \le \frac{\mu\, C_3}{4\, C_7}, \tag{72}$$

*then*

$$\begin{aligned} \|\mathbf{u}_h^{n+1}\|_X^2 \le& \frac{2}{\mu\, C_3}\Big(\mathcal{H}_{n+1}^{1/2}\|\delta_t \mathbf{u}_h^n\|_{L^2(\Omega_1)} + \mathcal{G}_1 + \frac{\mathcal{G}_2}{4}\Big),\\ \|p_{2h}^{n+1}\|_{M_2}^2 \le& \frac{2}{C_4}\Big(\mathcal{H}_{n+1}^{1/2}\|\delta_t \mathbf{u}_h^n\|_{L^2(\Omega_1)} + \frac{\mathcal{G}_1}{3} + \frac{\mathcal{G}_2}{2}\Big). \end{aligned} \tag{73}$$

*Proof.* Again, by testing (46) with $(\mathbf{v}, q) = (\mathbf{u}_h^{n+1}, p_{2h}^{n+1})$ and using (39), (40), (43), and (44), we obtain

$$\begin{aligned} \mu\, C_3\|\mathbf{u}_h^{n+1}\|_X^2 + C_4\|p_{2h}^{n+1}\|_{M_2}^2 + \sum_{j=1}^{d-1}\|\frac{1}{G_j^{1/2}}\mathbf{u}_h^{n+1}\cdot\boldsymbol{\tau}_{12}^j\|_{L^2(\Gamma_{12})}^2&\\ \le \|\delta_t\mathbf{u}_h^n\|_{L^2(\Omega_1)}\|\mathbf{u}_h^{n+1}\|_{L^2(\Omega_1)} + C_7\|\mathbf{u}_h^n\|_X\|\mathbf{u}_h^{n+1}\|_X^2&\\ + \|\mathbf{f}_1^{n+1}\|_{L^2(\Omega_1)}\|\mathbf{u}_h^{n+1}\|_{L^2(\Omega_1)} + \|f_2^{n+1}\|_{L^2(\Omega_2)}\|p_{2h}^{n+1}\|_{L^2(\Omega_2)}.& \end{aligned}$$

Next, by applying (37) and (36), and suitable Young's inequalities to the last two terms of the right-hand side, and using (72), we deduce

$$\begin{aligned} \frac{1}{2}\mu\, C_3\|\mathbf{u}_h^{n+1}\|_X^2 \le& \|\delta_t\mathbf{u}_h^n\|_{L^2(\Omega_1)}\|\mathbf{u}_h^{n+1}\|_{L^2(\Omega_1)} + \mathcal{G}_1 + \frac{\mathcal{G}_2}{4},\\ \frac{1}{2}C_4\|p_{2h}^{n+1}\|_{M_2}^2 \le& \|\delta_t\mathbf{u}_h^n\|_{L^2(\Omega_1)}\|\mathbf{u}_h^{n+1}\|_{L^2(\Omega_1)} + \frac{\mathcal{G}_1}{3} + \frac{\mathcal{G}_2}{2}. \end{aligned}$$

Then (73) follows by observing that (72) implies (69) and by substituting (69) into this inequality. □

In order to derive a bound for $\delta_t\mathbf{u}_h^n$, let us start with $n = 0$. As $\mathbf{u}_h^0 = \mathbf{0}$, $\delta_t\mathbf{u}_h^0$ reduces to $\frac{1}{\Delta t}\mathbf{u}_h^1$.

**Proposition 5.8.** *Let* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *be defined by* (71). *We have,*

$$\begin{aligned} \|\delta_t\mathbf{u}_h^0\|_{L^2(\Omega_1)} \le& \|\mathbf{f}_1\|_{L^\infty(0,T;L^2(\Omega_1)^3)} + \frac{C_{11}}{C_4^{1/2}}(\frac{\mathcal{G}_1}{2} + \mathcal{G}_2)^{1/2},\\ \frac{1}{\Delta t}\|\mathbf{u}_h^1\|_X^2 \le& \frac{1}{2\mu\, C_3}\Big(\|\mathbf{f}_1\|_{L^\infty(0,T;L^2(\Omega_1)^3)}^2 + \frac{C_{11}^2}{C_4}(\frac{\mathcal{G}_1}{2} + \mathcal{G}_2)\Big). \end{aligned} \tag{74}$$

*Proof.* As $\mathbf{u}_h^0 = \mathbf{0}$, (46) reduces to

$$\begin{aligned} \frac{1}{\Delta t}(\mathbf{u}_h^1, \mathbf{v})_{\Omega_1} + a_S(\mathbf{u}_h^1, \mathbf{v}) + a_D(p_{2h}^1, q) + (p_{2h}^1, \mathbf{v}\cdot\mathbf{n}_{12})_{\Gamma_{12}} - (\mathbf{u}_h^1\cdot\mathbf{n}_{12}, q)_{\Gamma_{12}}&\\ + \sum_{j=1}^{d-1}(\frac{1}{G_j}\mathbf{u}_h^1\cdot\boldsymbol{\tau}_{12}^j, \mathbf{v}\cdot\boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} = (\mathbf{f}_1^1, \mathbf{v})_{\Omega_1} + (f_2^1, q)_{\Omega_2}.& \end{aligned}$$

The proof of (74) is not completely straightforward because the factor $(\Delta t)^{-1}$ can only be controlled by canceling $\|\mathbf{u}_h^1\|_{L^2(\Omega_1)}$ on both sides of the above equation, and this entails the elimination of the data in $\Omega_2$. Therefore, (46) must be tested with $(\mathbf{v}, q) = (\mathbf{u}_h^1, 0)$. Then (39) leads to

$$\begin{aligned}\frac{1}{\Delta t}\|\mathbf{u}_h^1\|^2_{L^2(\Omega_1)} + \mu\, C_3 \|\mathbf{u}_h^1\|^2_X + \sum_{j=1}^{d-1} \|\frac{1}{G_j^{1/2}}\mathbf{u}_h^1 \cdot \boldsymbol{\tau}_{12}^j\|^2_{\Gamma_{12}} \\ \le \|\mathbf{f}_1^1\|_{L^2(\Omega_1)}\|\mathbf{u}_h^1\|_{L^2(\Omega_1)} + |(p_{2h}^1, \mathbf{u}_h^1\cdot \mathbf{n}_{12})_{\Gamma_{12}}| \\ \le \|\mathbf{f}_1^1\|_{L^2(\Omega_1)}\|\mathbf{u}_h^1\|_{L^2(\Omega_1)} + C_{11}\|p_{2h}^1\|_{M_2}\|\mathbf{u}_h^1\|_{L^2(\Omega_1)},\end{aligned} \tag{75}$$

where (49) is used to bound the last term. This leads to the intermediate bound

$$\frac{1}{\Delta t}\|\mathbf{u}_h^1\|_{L^2(\Omega_1)} \le \|\mathbf{f}_1^1\|_{L^2(\Omega_1)} + C_{11}\|p_{2h}^1\|_{M_2}. \tag{76}$$

By applying Young's inequality to the right-hand side of (76) so that the term $\frac{1}{\Delta t}\|\mathbf{u}_h^1\|^2_{L^2(\Omega_1)}$ is eliminated, we also obtain

$$\frac{1}{\Delta t}\|\mathbf{u}_h^1\|^2_X \le \frac{1}{2\mu\, C_3}\big(\|\mathbf{f}_1^1\|^2_{L^2(\Omega_1)} + C_{11}^2\|p_{2h}^1\|^2_{M_2}\big). \tag{77}$$

Finally, an easy variant of one step of the proof of Lemma 5.7 gives a bound for $\|p_{2h}^1\|^2_{M_2}$,

$$\|p_{2h}^1\|^2_{M_2} \le \frac{1}{C_4}\Big(\frac{C_2^2}{2\mu\, C_3}\|\mathbf{f}_1^1\|^2_{L^2(\Omega_1)} + \frac{C_1^2}{C_4}\|f_2^1\|^2_{L^2(\Omega_2)}\Big). \tag{78}$$

Then (74) follows by substituting (78) into (76) and (77). □

The following lemma treats the general case. Note that its sufficient condition (80) implies the previous ones, (65), (68), and (72).

**Lemma 5.9.** *Define the discrete time derivative of* $\mathcal{H}_n$,

$$\mathcal{F}_n = \frac{C_2^2}{\mu\, C_3}\Delta t \sum_{i=1}^{n}\|\delta_t \mathbf{f}_1^i\|^2_{L^2(\Omega_1)} + \frac{C_1^2}{2\, C_4}\Delta t\sum_{i=1}^{n}\|\delta_t f_2^i\|^2_{L^2(\Omega_2)}. \tag{79}$$

*Let* $\mathbf{u}_h^0 = \mathbf{0}$ *and let* $1 \le n \le N_T - 1$. *If for all* $i$, $0 \le i \le n$, (46) *has a solution* $(\mathbf{u}_h^i, p_{2h}^i)$ *satisfying*

$$\|\mathbf{u}_h^i\|_X \le \frac{\mu\, C_3}{4\big(C_7 + \frac{1}{2}C_{10}\big)}, \tag{80}$$

*where* $C_{10}$ *is the constant in* (48), *then*

$$\|\delta_t \mathbf{u}_h^n\|^2_{L^2(\Omega_1)} \le \|\delta_t \mathbf{u}_h^0\|^2_{L^2(\Omega_1)} + \frac{1}{2\Delta t}\mu\, C_3\|\delta_0 \mathbf{u}_h\|^2_X + \mathcal{F}_n. \tag{81}$$

*Proof.* Let $1 \le n \le N_T - 1$ and let us take the difference of equation (46) at steps $n+1$ and $n$ and test it with $\mathbf{v} = \delta_n \mathbf{u}_h$ and $q = \delta_n p_{2h}$. This leads to

$$\frac{1}{\Delta t}(\delta_n \mathbf{u}_h - \delta_{n-1}\mathbf{u}_h, \delta_n \mathbf{u}_h)_{\Omega_1} + \mu\, C_3 \|\delta_n \mathbf{u}_h\|_X^2 + C_4 \|\delta_n p_{2h}\|_{M_2}^2 + c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n; \mathbf{u}_h^{n+1}, \delta_n \mathbf{u}_h)$$
$$-c_{NS}(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \delta_n \mathbf{u}_h) + \sum_{j=1}^{d-1} \|\frac{1}{G_j^{1/2}} \delta_n \mathbf{u}_h \cdot \boldsymbol{\tau}_{12}^j\|_{\Gamma_{12}}^2 \le (\delta_n \mathbf{f}_1, \delta_n \mathbf{u}_h)_{\Omega_1} + (\delta_n f_2, \delta_n p_{2h})_{\Omega_2}.$$

To handle the nonlinear terms, we write

$$c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n; \mathbf{u}_h^{n+1}, \delta_n \mathbf{u}_h) - c_{NS}(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \delta_n \mathbf{u}_h)$$
$$= c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n; \delta_n \mathbf{u}_h, \delta_n \mathbf{u}_h) + c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n; \mathbf{u}_h^n, \delta_n \mathbf{u}_h) - c_{NS}(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \delta_n \mathbf{u}_h),$$

and use (43), (44), and (48). This gives

$$\begin{aligned} &\frac{1}{\Delta t}(\delta_n \mathbf{u}_h - \delta_{n-1}\mathbf{u}_h, \delta_n \mathbf{u}_h)_{\Omega_1} + \mu\, C_3 \|\delta_n \mathbf{u}_h\|_X^2 + C_4 \|\delta_n p_{2h}\|_{M_2}^2 \\ &\quad \le |(\delta_n \mathbf{f}_1, \delta_n \mathbf{u}_h)_{\Omega_1}| + |(\delta_n f_2, \delta_n p_{2h})_{\Omega_2}| + C_7 \|\mathbf{u}_h^n\|_X \|\delta_n \mathbf{u}_h\|_X^2 \\ &\qquad + C_{10} \|\mathbf{u}_h^n\|_X \|\delta_{n-1}\mathbf{u}_h\|_X \|\delta_n \mathbf{u}_h\|_X. \end{aligned} \tag{82}$$

Then, proceeding as in Lemma 5.6, and applying suitably Young's inequality, we infer

$$\begin{aligned} &\frac{1}{\Delta t}(\delta_n \mathbf{u}_h - \delta_{n-1}\mathbf{u}_h, \delta_n \mathbf{u}_h)_{\Omega_1} + \frac{1}{2}\mu\, C_3 \|\delta_n \mathbf{u}_h\|_X^2 - \big(\frac{1}{2}C_{10} + C_7\big) \|\mathbf{u}_h^n\|_X \|\delta_n \mathbf{u}_h\|_X^2 \\ &\quad \le \frac{1}{2}\Big(\frac{C_2^2}{\mu\, C_3} \|\delta_n \mathbf{f}_1\|_{L^2(\Omega_1)}^2 + \frac{C_1^2}{2C_4} \|\delta_n f_2\|_{L^2(\Omega_2)}^2 + C_{10} \|\mathbf{u}_h^n\|_X \|\delta_{n-1}\mathbf{u}_h\|_X^2\Big). \end{aligned}$$

As the assumption (80) permits to control both contributions of the nonlinear term, by summing this inequality over $n$, we derive

$$\begin{aligned} &\frac{1}{\Delta t} \|\delta_n \mathbf{u}_h\|_{L^2(\Omega_1)}^2 + \frac{1}{2}\mu\, C_3 \Big(\sum_{i=1}^n \|\delta_i \mathbf{u}_h\|_X^2 - \sum_{i=1}^n \|\delta_{i-1}\mathbf{u}_h\|_X^2\Big) \\ &\le \frac{1}{\Delta t} \|\delta_0 \mathbf{u}_h\|_{L^2(\Omega_1)}^2 + \frac{C_2^2}{\mu\, C_3} \sum_{i=1}^n \|\delta_i \mathbf{f}_1\|_{L^2(\Omega_1)}^2 + \frac{C_1^2}{2C_4} \sum_{i=1}^n \|\delta_i f_2\|_{L^2(\Omega_2)}^2, \end{aligned}$$

thus proving (81). $\square$

The results of this section are finalized in the following theorem.

**Theorem 5.10.** *Assume that for $1 \le n \le N_T - 1$, the data satisfy*

$$\begin{aligned} &\mathcal{H}_{n+1}^{\frac{1}{4}}\Big\{\frac{3}{2}\Big(\|\mathbf{f}_1\|_{L^\infty(0,T;L^2(\Omega_1)^3)} + \frac{C_{11}}{C_4^{1/2}}\big(\frac{\mathcal{G}_1}{2}+\mathcal{G}_2\big)^{1/2}\Big) + \mathcal{F}_n^{1/2}\Big\}^{1/2} + \big(\mathcal{G}_1 + \frac{\mathcal{G}_2}{4}\big)^{1/2} \\ &\le \frac{1}{\sqrt{2}}(\mu\, C_3)^{3/2}\frac{1}{4\big(C_7+\frac{1}{2}C_{10}\big)}. \end{aligned} \tag{83}$$

*Then, starting from $\mathbf{u}_h^0 = \mathbf{0}$, (46) defines a unique solution $(\mathbf{u}_h^n, p_{2h}^n)$ for all $n$, $1 \le n \le N_T$. This solution is bounded uniformly with respect to $n$ and $h$, the part $\mathbf{u}_h^n$ satisfies (80) for all $n$ and $\|p_{2h}^n\|_{M_2}$ satisfies a similar bound.*

*Proof.* By substituting (74) into (81), we find

$$\|\delta_t \mathbf{u}_h^n\|_{L^2(\Omega_1)} \le \frac{3}{2}\Big(\|\mathbf{f}_1\|_{L^\infty(0,T;L^2(\Omega_1)^3)} + \frac{C_{11}}{C_4^{1/2}}\big(\frac{\mathcal{G}_1}{2}+\mathcal{G}_2\big)^{1/2}\Big) + \mathcal{F}_n^{1/2}.$$

When substituted into (73), this bound gives, for $1 \le n \le N_T - 1$,

$$\begin{aligned} \|\mathbf{u}_h^{n+1}\|_X \le &\sqrt{\frac{2}{\mu\, C_3}}\Big\{\mathcal{H}_{n+1}^{1/4}\Big(\frac{3}{2}\big(\|\mathbf{f}_1\|_{L^\infty(0,T;L^2(\Omega_1)^3)} + \frac{C_{11}}{C_4^{1/2}}\big(\frac{\mathcal{G}_1}{2}+\mathcal{G}_2\big)^{1/2}\big) + \mathcal{F}_n^{1/2}\Big)^{1/2} \\ &+ \big(\mathcal{G}_1 + \frac{\mathcal{G}_2}{4}\big)^{1/2}\Big\}. \end{aligned} \tag{84}$$

When $n = 1$, we simply have

$$\|\mathbf{u}_h^1\|_X \le \sqrt{\frac{1}{\mu\, C_3}}\Big(\mathcal{G}_1 + \frac{\mathcal{G}_2}{2}\Big)^{1/2}, \tag{85}$$

which of course is smaller than the right-hand side of (84). Then the condition on the data (83) follows by prescribing that the right-hand side of (84) be smaller than the right-hand side of (80). Note that (83) also implies that the right-hand side of (85) is smaller than that of (80). This gives a uniform (in $n$ and $h$) upper bound for $\|\mathbf{u}_h^n\|_X$ for all $n \ge 1$. The same procedure leads to a uniform upper bound for $\|p_{2h}^n\|_{M_2}$; for the sake of brevity, we do not specify the factors of this last bound.

Finally, as (80) implies (65), Lemma 5.5 guarantees existence and uniqueness of the solution of (46) at each time step. □

From here on, we assume that the small data condition (83) holds, so that $\mathbf{u}_h^n$ satisfies (80) for all $n$.

## 6. Convergence estimate for the time–lagging scheme

Recall that the family of triangulations $\mathcal{E}_1^h$ and $\mathcal{E}_2^h$ are regular in the sense of (14).

6.1. **Interpolations and approximations.** As is usual in discretizing incompressible fluids, we need an approximation operator that preserves the discrete divergence.

**Lemma 6.1.** *There is an approximation operator* $\Pi_h \in \mathcal{L}(\mathbf{X}, \mathbf{X}^h)$ *satisfying*

$$b_S(\Pi_h(\mathbf{v}) - \mathbf{v}, q) = 0, \quad \forall \mathbf{v} \in \mathbf{X}, \quad \forall q \in M_1^h, \tag{86}$$

*and for all* $E$ *in* $\mathcal{E}_1^h$*, for all* $\mathbf{v}$ *in* $\mathbf{X} \cap W^{s,r}(E)^d$*,* $1 \le r \le \infty$*,* $1 \le s \le k_1 + 1$*,*

$$\|\Pi_h(\mathbf{v}) - \mathbf{v})\|_{L^r(E)} \le Ch^s |\mathbf{v}|_{W^{s,r}(\Delta_E)}, \quad \|\nabla(\Pi_h(\mathbf{v}) - \mathbf{v})\|_{L^r(E)} \le Ch^{s-1} |\mathbf{v}|_{W^{s,r}(\Delta_E)}, \tag{87}$$

*with a constant* $C$ *independent of* $h$ *and* $E$*, where* $\Delta_E \subset \Omega_1$ *is a macro-element used in the construction of* $\Pi_h(\mathbf{v})$ *in* $E$*. We also have for all* $s$*,* $1 \le s \le k_1 + 1$*,*

$$\forall \mathbf{v} \in \mathbf{X} \cap H^s(\Omega_1), \quad \|\Pi_h(\mathbf{v}) - \mathbf{v}\|_X \le Ch^{s-1} |\mathbf{v}|_{H^s(\Omega_1)}. \tag{88}$$

*Proof.* The proof proceeds by suitably correcting a standard approximation operator. Here the correction is done by the Raviart-Thomas operator [19] acting on the Scott-Zhang operator [22]. Let $\mathcal{S}_h \in \mathcal{L}(\mathbf{X}, \mathbf{X}^h \cap \mathbf{X})$ be the Scott-Zhang interpolation operator and set

$$\Pi_h(\mathbf{v}) = \mathcal{S}_h(\mathbf{v}) + \mathbf{c}_h(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X},$$

where $\mathbf{c}_h(\mathbf{v})$ belongs to the space $\mathrm{RT}_{k_1-1}(\Omega_1)$, which is the velocity space for the Raviart-Thomas element of order $k_1 - 1$. Let $\mathcal{RT}_h$ be the Raviart-Thomas operator that maps functions into the discrete space $\mathrm{RT}_{k_1-1}(\Omega_1)$. We choose

$$\mathbf{c}_h(\mathbf{v}) = \mathcal{RT}_h(\mathbf{v} - \mathcal{S}_h(\mathbf{v})).$$

By the properties of the Raviart-Thomas interpolant, we have

$$\big(q_h, \nabla \cdot \mathbf{c}_h(\mathbf{v})\big)_{\Omega_1} = \big(q_h, \nabla \cdot (\mathbf{v} - \mathcal{S}_h(\mathbf{v}))\big)_{\Omega_1}, \quad \forall q_h \in M_1^h,$$

$$(\{q_h\}, [\mathbf{c}_h(\mathbf{v})] \cdot \mathbf{n}_e)_e = 0, \quad \forall e \in \Gamma_1^h \cup \Gamma_1, \quad \forall q_h \in M_1^h.$$

This implies (86). For the error estimates, we write

$$\Pi_h(\mathbf{v}) - \mathbf{v} = \mathcal{S}_h(\mathbf{v}) - \mathbf{v} + \mathcal{RT}_h(\mathbf{v} - \mathcal{S}_h(\mathbf{v})),$$

use the local properties of the Raviart-Thomas interpolant, for any $r$, $1 \le r \le \infty$,

$$\forall E \in \mathcal{E}_1^h, \quad \|\mathcal{RT}_h(\mathbf{v} - \mathcal{S}_h(\mathbf{v}))\|_{L^r(E)} \le C h_E \|\nabla(\mathbf{v} - \mathcal{S}_h(\mathbf{v}))\|_{L^r(E)},$$
$$\forall E \in \mathcal{E}_1^h, \quad \|\nabla(\mathcal{RT}_h(\mathbf{v} - \mathcal{S}_h(\mathbf{v})))\|_{L^r(E)} \le C \|\nabla(\mathbf{v} - \mathcal{S}_h(\mathbf{v}))\|_{L^r(E)},$$

and the quasi-local properties of the Scott-Zhang operator. This yields (87). For the bound (88), we estimate the jumps of $\mathcal{RT}_h(\mathbf{v} - \mathcal{S}_h(\mathbf{v}))$ and use trace inequalities. □

To interpolate the pressures, we use for $p_1$ a local $L^2$ projection $\pi_h$ on $\mathbb{P}_{k_1-1}$ in each element $E$ of $\mathcal{E}_1^h$, and for $p_2$ the continuous Lagrange interpolant $L_h$ in $\mathbb{P}_{k_2}$. On one hand, the continuous Lagrange interpolant has zero jump and preserves the zero boundary value on $\Gamma_{2D}$. On the other hand, since $k_1 - 1 \ge 0$, the local projection preserves the mean value. Thus we indeed have $\pi_h(p_1)$ in $M_1^h$ and $L_h(p_2)$ in $M_2^h \cap M_2$. They satisfy the following approximation errors for all $r$, $1 \le r \le \infty$,

$$\forall E \in \mathcal{E}_1^h, \forall s \in [0, k_1], \forall p_1 \in W^{s,r}(E), \quad \|\pi_h(p_1) - p_1\|_{L^r(E)} \le C h^s |p_1|_{W^{s,r}(E)}, \tag{89}$$

$$\forall E \in \mathcal{E}_2^h, \forall s \in [2, k_2+1], \ell = 0, 1, \forall p_2 \in W^{s,r}(E), \quad |L_h(p_2) - p_2|_{W^{\ell,r}(E)} \le C h^{s-\ell} |p_2|_{W^{s,r}(E)}. \tag{90}$$

This last bound also holds when $s = 1$ and $r > d$, the restriction on $r$ arising from the continuity required to define pointwise values.

Before proceeding, let us recall two more useful properties of the nonlinear term. They are established by Proposition 4.1 in [16] in the case when $d = 2$, but the proof easily extends to $d = 3$.

**Proposition 6.2.** *There exists a constant $C$, independent of $h$, such that for all $\mathbf{u}$ in $\big(L^\infty(\Omega_1) \cap W^{1,3}(\Omega_1)\big)^d$, for all $\mathbf{v}_h \in \mathbf{V}^h$, and all $\mathbf{w}_h$ and $\mathbf{z}_h$ in $\mathbf{X}^h$,*

$$\big|c_{NS}(\mathbf{z}_h, \mathbf{v}_h; \mathbf{u}, \mathbf{w}_h)\big| \le C\big(\|\mathbf{u}\|_{L^\infty(\Omega_1)} + |\mathbf{u}|_{W^{1,3}(\Omega_1)}\big)\|\mathbf{v}_h\|_{L^2(\Omega_1)}\|\mathbf{w}_h\|_X. \tag{91}$$

*If in addition, $\mathbf{u}$ is in $H^{3/2}(\Omega_1)^d$, then*

$$\begin{aligned}\big|c_{NS}(\mathbf{z}_h, \mathbf{v}_h; \mathbf{u} - \Pi_h(\mathbf{u}), \mathbf{w}_h)\big| \le & C\big(\|\mathbf{u} - \Pi_h(\mathbf{u})\|_{L^\infty(\Omega_1)} + |\mathbf{u} - \Pi_h(\mathbf{u})|_{W^{1,3}(\Omega_1)} + |\mathbf{u}|_{H^{3/2}(\Omega_1)}\big) \\ & \times \|\mathbf{v}_h\|_{L^2(\Omega_1)}\|\mathbf{w}_h\|_X.\end{aligned} \tag{92}$$

When $\mathbf{u}$ is smoother, a positive power of $h$ multiplies the last term in parentheses above, but this is not required by the analysis below.

Let $\mathbf{u}^n, p_1^n, p_2^n$ denote the exact solutions evaluated at time $t^n$. We define the following errors:

$$\boldsymbol{\chi}^n = \mathbf{u}_h^n - \Pi_h(\mathbf{u}^n), \quad \xi_1^n = p_{1h}^n - \pi_h(p_1^n), \quad \xi_2^n = p_{2h}^n - L_h(p_2^n) \tag{93}$$

$$\boldsymbol{\zeta}^n = \mathbf{u}^n - \Pi_h(\mathbf{u}^n), \quad \eta_1^n = p_1^n - \pi_h(p_1^n), \quad \eta_2^n = p_2^n - L_h(p_2^n). \tag{94}$$

6.2. **Error estimates.** The following theorem proves an upper bound for the error on $\mathbf{u}_h$ and $p_{2h}$. The assumption on the triangulation is (14).

**Theorem 6.3.** *Under the assumption* (83), *there is a constant $C$ independent of $h$ and $\Delta t$ such that for all $1 \leq m \leq N_T$, we have,*

$$\|\mathbf{u}^m - \mathbf{u}_h^m\|_{L^2(\Omega_1)}^2 + \mu C_3 \Delta t \sum_{n=1}^{m} \|\mathbf{u}^n - \mathbf{u}_h^n\|_X^2 + C_4 \Delta t \sum_{n=1}^{m} \|p_2^n - p_{2h}^n\|_{M_2}^2$$
$$+ \Delta t \sum_{n=1}^{m} \sum_{j=1}^{d-1} \|\frac{1}{\sqrt{G^j}} (\mathbf{u}^n - \mathbf{u}_h^n) \cdot \boldsymbol{\tau}_{12}^j\|_{L^2(\Gamma_{12})}^2 \leq C(h^{2k_1} + h^{2k_2} + \Delta t^2).$$

*This bound is valid if the exact solution satisfies the following regularity assumptions:* $\mathbf{u} \in L^\infty(0,T; H^{k_1+1}(\Omega_1)^d)$, $\frac{\partial \mathbf{u}}{\partial t} \in L^2(0,T; L^\infty(\Omega_1)^d) \cap L^2(0,T; H^{k_1}(\Omega_1)^d)$, $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2((0,T) \times \Omega_1)^d$ *and* $p_2 \in L^\infty(0,T; H^{k_2+1}(\Omega_2))$.

*Proof.* Recall the discrete derivative $\delta_t$ defined in (70). The pointwise error equations of scheme (28)–(29) are,

$$\forall \mathbf{v} \in \mathbf{X}^h, \quad \Big(\delta_t \mathbf{u}_h^n - \delta_t \mathbf{u}^n, \mathbf{v}\Big)_{\Omega_1} + \Big(\delta_t \mathbf{u}^n - (\frac{\partial \mathbf{u}}{\partial t})^{n+1}, \mathbf{v}\Big)_{\Omega_1}$$
$$+ a_S(\mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}, \mathbf{v}) + a_D(p_{2h}^{n+1} - p_2^{n+1}, q_2) + b_S(\mathbf{v}, p_{1h}^{n+1} - p_1^{n+1})$$
$$+ c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n; \mathbf{u}_h^{n+1}, \mathbf{v}) - c_{NS}(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}; \mathbf{u}^{n+1}, \mathbf{v})$$
$$+ (p_{2h}^{n+1} - p_2^{n+1}, \mathbf{v} \cdot \mathbf{n}_{12})_{\Gamma_{12}} - ((\mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}) \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}}$$
$$+ \sum_{j=1}^{d-1} \Big(\frac{1}{G^j} (\mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}) \cdot \boldsymbol{\tau}_{12}^j, \mathbf{v} \cdot \boldsymbol{\tau}_{12}^j\Big)_{\Gamma_{12}} = 0,$$

$$\forall q_1 \in M_1^h, \quad b_S(\mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}, q_1) = 0.$$

We rewrite the error equations by inserting the interpolants and using the property (86)

$$\begin{aligned}
&\forall \mathbf{v} \in \mathbf{X}^h, \quad (\delta_t \boldsymbol{\chi}^n, \mathbf{v})_{\Omega_1} + a_S(\boldsymbol{\chi}^{n+1}, \mathbf{v}) + a_D(\xi_2^{n+1}, q_2) + b_S(\mathbf{v}, \xi_1^{n+1}) \\
&+ c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n; \mathbf{u}_h^{n+1}, \mathbf{v}) - c_{NS}(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}; \mathbf{u}^{n+1}, \mathbf{v}) \\
&+ (\xi_2^{n+1}, \mathbf{v}\cdot\mathbf{n}_{12})_{\Gamma_{12}} - (\boldsymbol{\chi}^{n+1}\cdot\mathbf{n}_{12}, q_2)_{\Gamma_{12}} + \sum_{j=1}^{d-1} \big(\frac{1}{G^j}\boldsymbol{\chi}^{n+1}\cdot\boldsymbol{\tau}_{12}^j, \mathbf{v}\cdot\boldsymbol{\tau}_{12}^j\big)_{\Gamma_{12}} \\
&= \Big( (\frac{\partial \mathbf{u}}{\partial t})^{n+1} - \delta_t \mathbf{u}^n, \mathbf{v}\Big)_{\Omega_1} + a_S(\boldsymbol{\zeta}^{n+1}, \mathbf{v}) + a_D(\eta_2^{n+1}, q_2) + b_S(\mathbf{v}, \eta_1^{n+1}) \\
&+ (\delta_t \boldsymbol{\zeta}^n, \mathbf{v})_{\Omega_1} + (\eta_2^{n+1}, \mathbf{v}\cdot\mathbf{n}_{12})_{\Gamma_{12}} - (\boldsymbol{\zeta}^{n+1}\cdot\mathbf{n}_{12}, q_2)_{\Gamma_{12}} + \sum_{j=1}^{d-1} \big(\frac{1}{G^j}\boldsymbol{\zeta}^{n+1}\cdot\boldsymbol{\tau}_{12}^j, \mathbf{v}\cdot\boldsymbol{\tau}_{12}^j\big)_{\Gamma_{12}},
\end{aligned} \tag{95}$$

$$\forall q_1 \in M_1^h, \quad b_S(\boldsymbol{\chi}^{n+1}, q_1) = 0. \tag{96}$$

Next we choose $\mathbf{v} = \boldsymbol{\chi}^{n+1}$, $q_1 = \xi_1^{n+1}$, and $q_2 = \xi_2^{n+1}$ in (95) and (96), apply coercivity of $a_S$ and $a_D$ to obtain.

$$\begin{aligned}
&\Big(\delta_t \boldsymbol{\chi}^n, \boldsymbol{\chi}^{n+1}\Big)_{\Omega_1} + \mu\, C_3 \|\boldsymbol{\chi}^{n+1}\|_X^2 + C_4 \|\xi_2^{n+1}\|_{M_2}^2 + \sum_{j=1}^{d-1} \|\frac{1}{\sqrt{G^j}}\boldsymbol{\chi}^{n+1}\cdot\boldsymbol{\tau}_{12}^j\|_{L^2(\Gamma_{12})}^2 \\
&+ c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n; \mathbf{u}_h^{n+1}, \boldsymbol{\chi}^{n+1}) - c_{NS}(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}; \mathbf{u}^{n+1}, \boldsymbol{\chi}^{n+1}) \\
&\le \Big( (\frac{\partial \mathbf{u}}{\partial t})^{n+1} - \delta_t \mathbf{u}^n, \boldsymbol{\chi}^{n+1}\Big)_{\Omega_1} + a_S(\boldsymbol{\zeta}^{n+1}, \boldsymbol{\chi}^{n+1}) + a_D(\eta_2^{n+1}, \xi_2^{n+1}) + b_S(\boldsymbol{\chi}^{n+1}, \eta_1^{n+1}) \\
&+ \big(\delta_t \boldsymbol{\zeta}^n, \boldsymbol{\chi}^{n+1}\big)_{\Omega_1} + (\eta_2^{n+1}, \boldsymbol{\chi}^{n+1}\cdot\mathbf{n}_{12})_{\Gamma_{12}} - (\boldsymbol{\zeta}^{n+1}\cdot\mathbf{n}_{12}, \xi_2^{n+1})_{\Gamma_{12}} \\
&+ \sum_{j=1}^{d-1} \big(\frac{1}{G^j}\boldsymbol{\zeta}^{n+1}\cdot\boldsymbol{\tau}_{12}^j, \boldsymbol{\chi}^{n+1}\cdot\boldsymbol{\tau}_{12}^j\big)_{\Gamma_{12}}.
\end{aligned}$$

We rewrite the term involving $c_{NS}$. Recall that $c_{NS}(\mathbf{z}, \mathbf{u}; \mathbf{v}, \mathbf{w})$ is linear in $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Further, when $\mathbf{v}$ has zero jump across elemental interfaces, the nonlinear part $n_h$ of $c_{NS}$ depending on $\mathbf{z}$ vanishes for any $\mathbf{z}$. For instance, because the exact solution has zero jump almost everywhere on faces, we have the following equality:

$$c_{NS}(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}; \mathbf{u}^{n+1}, \boldsymbol{\chi}^{n+1}) = c_{NS}(\mathbf{u}_h^n, \mathbf{u}^{n+1}; \mathbf{u}^{n+1}, \boldsymbol{\chi}^{n+1}). \tag{97}$$

The linearity of the last three arguments of $c_{NS}$ imply

$$
\begin{aligned}
&c_{NS}(\mathbf{u}_h^n,\mathbf{u}_h^n;\mathbf{u}_h^{n+1},\boldsymbol{\chi}^{n+1}) - c_{NS}(\mathbf{u}^{n+1},\mathbf{u}^{n+1};\mathbf{u}^{n+1},\boldsymbol{\chi}^{n+1})\\
&= c_{NS}(\mathbf{u}_h^n,\mathbf{u}_h^n;\boldsymbol{\chi}^{n+1},\boldsymbol{\chi}^{n+1}) + c_{NS}(\mathbf{u}_h^n,\boldsymbol{\chi}^n;\Pi_h(\mathbf{u}^{n+1}),\boldsymbol{\chi}^{n+1}) - c_{NS}(\mathbf{u}_h^n,\boldsymbol{\zeta}^n;\Pi_h(\mathbf{u}^{n+1}),\boldsymbol{\chi}^{n+1})\\
&\quad + c_{NS}(\mathbf{u}_h^n,\mathbf{u}^n-\mathbf{u}^{n+1};\Pi_h(\mathbf{u}^{n+1}),\boldsymbol{\chi}^{n+1}) - c_{NS}(\mathbf{u}_h^n,\mathbf{u}^{n+1};\boldsymbol{\zeta}^{n+1},\boldsymbol{\chi}^{n+1}).
\end{aligned}
$$

Using (43), (44), and (80), we then have

$$
\begin{aligned}
&\Big(\delta_t\boldsymbol{\chi}^n,\boldsymbol{\chi}^{n+1}\Big)_{\Omega_1} + \frac{3}{4}\mu\, C_3\|\boldsymbol{\chi}^{n+1}\|_X^2 + C_4\|\xi_2^{n+1}\|_{M_2}^2 + \sum_{j=1}^{d-1}\|\frac{1}{\sqrt{G^j}}\boldsymbol{\chi}^{n+1}\cdot\boldsymbol{\tau}_{12}^j\|_{L^2(\Gamma_{12})}^2\\
&\le |c_{NS}(\mathbf{u}_h^n,\boldsymbol{\chi}^n;\Pi_h(\mathbf{u}^{n+1}),\boldsymbol{\chi}^{n+1})| + |c_{NS}(\mathbf{u}_h^n,\boldsymbol{\zeta}^n;\Pi_h(\mathbf{u}^{n+1}),\boldsymbol{\chi}^{n+1})|\\
&+ |c_{NS}(\mathbf{u}_h^n,\mathbf{u}^n-\mathbf{u}^{n+1};\Pi_h(\mathbf{u}^{n+1}),\boldsymbol{\chi}^{n+1})| + |c_{NS}(\mathbf{u}_h^n,\mathbf{u}^{n+1};\boldsymbol{\zeta}^{n+1},\boldsymbol{\chi}^{n+1})|\\
&+ \Big((\frac{\partial\mathbf{u}}{\partial t})^{n+1} - \delta_t\mathbf{u}^n,\boldsymbol{\chi}^{n+1}\Big)_{\Omega_1} + a_S(\boldsymbol{\zeta}^{n+1},\boldsymbol{\chi}^{n+1}) + a_D(\eta_2^{n+1},\xi_2^{n+1}) + b_S(\boldsymbol{\chi}^{n+1},\eta_1^{n+1})\\
&+ \big(\delta_t\boldsymbol{\zeta}^n,\boldsymbol{\chi}^{n+1}\big)_{\Omega_1} + (\eta_2^{n+1},\boldsymbol{\chi}^{n+1}\cdot\mathbf{n}_{12})_{\Gamma_{12}} - (\boldsymbol{\zeta}^{n+1}\cdot\mathbf{n}_{12},\xi_2^{n+1})_{\Gamma_{12}}\\
&+ \sum_{j=1}^{d-1}\big(\frac{1}{G^j}\boldsymbol{\zeta}^{n+1}\cdot\boldsymbol{\tau}_{12}^j,\boldsymbol{\chi}^{n+1}\cdot\boldsymbol{\tau}_{12}^j\big)_{\Gamma_{12}} = T_1+\cdots+T_{12}.
\end{aligned}
\tag{98}
$$

The remainder of the proof is devoted to a brief derivation of bounds for each term $T_i$, $1\le i\le 12$. Many details are skipped because this derivation uses well-established techniques, see for instance the analysis in [16, 15]. To avoid particular cases, the estimates are derived when $d=3$. Since the left-hand side of (98) does not contain the energy norm of $\boldsymbol{\chi}^n$, this term has to appear in the upper-bound of the right-hand side of (98) in the $L^2$ norm, in order to be controlled by Gronwall's lemma.

1) The term $T_1$ is split as follows

$$
T_1 = c_{NS}(\mathbf{u}_h^n,\boldsymbol{\chi}^n;\Pi_h(\mathbf{u}^{n+1}),\boldsymbol{\chi}^{n+1}) = c_{NS}(\mathbf{u}_h^n,\boldsymbol{\chi}^n;\mathbf{u}^{n+1},\boldsymbol{\chi}^{n+1}) - c_{NS}(\mathbf{u}_h^n,\boldsymbol{\chi}^n;\boldsymbol{\zeta}^{n+1},\boldsymbol{\chi}^{n+1}).
$$

For the first term, we apply (91),

$$
\big|c_{NS}(\mathbf{u}_h^n,\boldsymbol{\chi}^n;\mathbf{u}^{n+1},\boldsymbol{\chi}^{n+1})\big| \le C\|\boldsymbol{\chi}^n\|_{L^2(\Omega_1)}\|\boldsymbol{\chi}^{n+1}\|_X\big(\|\mathbf{u}^{n+1}\|_{L^\infty(\Omega_1)} + |\mathbf{u}^{n+1}|_{W^{1,3}(\Omega_1)}\big),
$$

and for the second we apply (92),

$$\left|c_{NS}(\mathbf{u}_h^n, \boldsymbol{\chi}^n; \boldsymbol{\zeta}^{n+1}, \boldsymbol{\chi}^{n+1})\right| \leq C\|\boldsymbol{\chi}^n\|_{L^2(\Omega_1)}\|\boldsymbol{\chi}^{n+1}\|_X\Big(|\mathbf{u}^{n+1}|_{H^{3/2}(\Omega_1)} + \|\mathbf{u}^{n+1} - \Pi_h(\mathbf{u}^{n+1})\|_{L^\infty(\Omega_1)} + |\mathbf{u}^{n+1} - \Pi_h(\mathbf{u}^{n+1})|_{W^{1,3}(\Omega_1)}\Big).$$

Then the stability properties of $\Pi_h$ and a suitable application of Young's inequality yield the bound for $T_1$, for any positive $\delta_1$:

$$|T_1| \leq \delta_1\mu\, C_3\|\boldsymbol{\chi}^{n+1}\|_X^2 + \frac{C^2}{\delta_1\mu\, C_3}\|\boldsymbol{\chi}^n\|_{L^2(\Omega_1)}^2. \tag{99}$$

2) Now, we assume that $\mathbf{u}$ belongs to $L^\infty(0,T;H^{k_1+1}(\Omega_1)^d)$. A bound for the term $T_2$ is obtained by using the continuity (45) of the form $c_{NS}$,

$$|T_2| = |c_{NS}(\mathbf{u}_h^n, \boldsymbol{\zeta}^n; \Pi_h(\mathbf{u}^{n+1}), \boldsymbol{\chi}^{n+1})| \leq C_8\|\boldsymbol{\zeta}^n\|_X\, \|\Pi_h(\mathbf{u}^{n+1})\|_X\, \|\boldsymbol{\chi}^{n+1}\|_X.$$

Then (88) and Young's inequality imply, for any positive $\delta_2$:

$$|T_2| \leq \delta_2\mu\, C_3\|\boldsymbol{\chi}^{n+1}\|_X^2 + \frac{C^2}{\delta_2\mu\, C_3}h^{2k_1}|\mathbf{u}|_{L^\infty(0,T;H^{k_1+1}(\Omega_1)^d)}^2. \tag{100}$$

3) Here we assume in addition that the time derivative $\frac{\partial \mathbf{u}}{\partial t}$ belongs to $L^2(0,T;L^\infty(\Omega_1)^d)$. As $\mathbf{u}$ is divergence free and has no jumps, the term $T_3$ simplifies to

$$T_3 = c_{NS}(\mathbf{u}_h^n, \mathbf{u}^n - \mathbf{u}^{n+1}; \Pi_h(\mathbf{u}^{n+1}), \boldsymbol{\chi}^{n+1}) = \sum_{E\in\mathcal{E}_1^h}((\mathbf{u}^n - \mathbf{u}^{n+1})\cdot\nabla\,\Pi_h(\mathbf{u}^{n+1}), \boldsymbol{\chi}^{n+1})_E + \sum_{E\in\mathcal{E}_1^h}\big(|(\mathbf{u}^n - \mathbf{u}^{n+1})\cdot\mathbf{n}_E|(\Pi_h(\mathbf{u}^{n+1})^{\mathrm{int}} - \Pi_h(\mathbf{u}^{n+1})^{\mathrm{ext}}), \boldsymbol{\chi}^{n+1,\mathrm{int}}\big)_{\partial E_-(\mathbf{u}_h^n)\setminus\Gamma_{12}}.$$

But

$$\mathbf{u}^n - \mathbf{u}^{n+1} = -\int_{t^n}^{t^{n+1}}\frac{\partial\mathbf{u}}{\partial t}(s)ds.$$

Therefore

$$\left|\big((\mathbf{u}^n - \mathbf{u}^{n+1})\cdot\nabla\,\Pi_h(\mathbf{u}^{n+1}), \boldsymbol{\chi}^{n+1}\big)_E\right| \leq \int_{t^n}^{t^{n+1}}\|\frac{\partial\mathbf{u}}{\partial t}\|_{L^2(E)}\|\boldsymbol{\chi}^{n+1}\|_{L^6(E)}|\Pi_h(\mathbf{u}^{n+1})|_{W^{1,3}(E)},$$

and the stability of $\Pi_h$ implies that

$$\left|\big((\mathbf{u}^n - \mathbf{u}^{n+1})\cdot\nabla\,\Pi_h(\mathbf{u}^{n+1}), \boldsymbol{\chi}^{n+1}\big)_E\right| \leq C\sqrt{\Delta t}\|\frac{\partial\mathbf{u}}{\partial t}\|_{L^2(E\times]t^n,t^{n+1}[)}\|\boldsymbol{\chi}^{n+1}\|_{L^6(E)}|\mathbf{u}^{n+1}|_{W^{1,3}(\Delta_E)}.$$

By summing over $E$, this yields for the interior part of $T_3$,

$$\Big|\sum_{E\in\mathcal{E}_1^h}((\mathbf{u}^n-\mathbf{u}^{n+1})\cdot\nabla\,\Pi_h(\mathbf{u}^{n+1}),\boldsymbol{\chi}^{n+1})_E\Big|\le C\sqrt{\Delta t}\|\frac{\partial\mathbf{u}}{\partial t}\|_{L^2(\Omega_1\times]t^n,t^{n+1}[)}\|\boldsymbol{\chi}^{n+1}\|_X\|\mathbf{u}\|_{L^\infty(0,T;W^{1,3}(\Omega_1)^d)}. \tag{101}$$

For the part on faces, since $\mathbf{u}^{n+1}$ has zero jump, we write

$$|\Pi_h(\mathbf{u}^{n+1})^{\rm int}-\Pi_h(\mathbf{u}^{n+1})^{\rm ext}|=|[\Pi_h(\mathbf{u}^{n+1})]|=|[\Pi_h(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}]|.$$

Hence, for any element $E\in\mathcal{E}_1^h$

$$\begin{aligned}\big|\big(|(\mathbf{u}^n-\mathbf{u}^{n+1})\cdot\mathbf{n}_E|(\Pi_h(\mathbf{u}^{n+1})^{\rm int}-\Pi_h(\mathbf{u}^{n+1})^{\rm ext}),\boldsymbol{\chi}^{n+1,\rm int}\big)_e\big|&\le\sqrt{\Delta t}\|\frac{\partial\mathbf{u}}{\partial t}\|_{L^2(t^n,t^{n+1};L^\infty(e)^d)}\\ &\times\|[\Pi_h(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}]\|_{L^2(e)}\|\boldsymbol{\chi}^{n+1}|_E\|_{L^2(e)},\quad\forall e\subset\partial E.\end{aligned}$$

By applying a trace inequality to the jump term and an equivalence of norms to $\|\boldsymbol{\chi}^{n+1}\|_{L^2(e)}$, and by summing over all elements $E$, we obtain

$$\begin{aligned}&\Big|\sum_{E\in\mathcal{E}_1^h}\big(|(\mathbf{u}^n-\mathbf{u}^{n+1})\cdot\mathbf{n}_E|(\Pi_h(\mathbf{u}^{n+1})^{\rm int}-\Pi_h(\mathbf{u}^{n+1})^{\rm ext}),\boldsymbol{\chi}^{n+1,\rm int}\big)_{\partial E_-(\mathbf{u}_h^n)\setminus\Gamma_{12}}\Big|\\ &\quad\le C\sqrt{\Delta t}\|\frac{\partial\mathbf{u}}{\partial t}\|_{L^2(t^n,t^{n+1};L^\infty(\Omega_1)^d)}\|\boldsymbol{\chi}^{n+1}\|_X\|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega_1)^d)}.\end{aligned} \tag{102}$$

Then (101), (102), and Young's inequality imply a bound for $T_3$, for any positive $\delta_3$:

$$|c_{NS}(\mathbf{u}_h^n,\mathbf{u}^n-\mathbf{u}^{n+1};\Pi_h(\mathbf{u}^{n+1}),\boldsymbol{\chi}^{n+1})|\le\delta_3\mu\,C_3\|\boldsymbol{\chi}^{n+1}\|_X^2+\frac{C^2}{\delta_3\mu\,C_3}\Delta t\|\frac{\partial\mathbf{u}}{\partial t}\|^2_{L^2(t^n,t^{n+1};L^\infty(\Omega_1)^d)}. \tag{103}$$

4) The bound for $T_4$ follows closely the bound for $T_3$, but without the derivative in time. We skip the details and state the result

$$|T_4|=|c_{NS}(\mathbf{u}_h^n,\mathbf{u}^{n+1};\boldsymbol{\zeta}^{n+1},\boldsymbol{\chi}^{n+1})|\le\delta_4\mu\,C_3\|\boldsymbol{\chi}^{n+1}\|_X^2+h^{2k_1}\frac{C^2}{\delta_4\mu\,C_3}. \tag{104}$$

5) Regarding $T_5$, Taylor's expansion yields for any function $v$ of one variable $t$,

$$\delta_t v^n=v'(t^{n+1})-\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}(s-t^n)v''(s)ds.$$

Therefore, assuming that $\frac{\partial^2\mathbf{u}}{\partial t^2}$ belongs to $L^2(\Omega_1\times(0,T))^d$, $T_5$ has the bound, for any positive $\delta_5$:

$$|T_5|\le C\sqrt{\Delta t}\|\boldsymbol{\chi}^{n+1}\|_X\|\frac{\partial^2\mathbf{u}}{\partial t^2}\|_{L^2(\Omega_1\times(t^n,t^{n+1}))}\le\delta_5\mu\,C_3\|\boldsymbol{\chi}^{n+1}\|_X^2+\frac{C^2}{\delta_5\mu\,C_3}\Delta t\|\frac{\partial^2\mathbf{u}}{\partial t^2}\|^2_{L^2(\Omega_1\times(t^n,t^{n+1}))}. \tag{105}$$

6) For the interior linear terms $T_6$, $T_7$ and $T_8$, we easily derive for any positive $\delta_6, \delta_7, \delta_8$ (see for instance [20]):

$$\begin{aligned} |T_6| \le& \delta_6 \mu C_3 \|\boldsymbol{\chi}^{n+1}\|_X^2 + \frac{(\mu C)^2}{\delta_6 C_3} h^{2k_1}, \\ |T_7| \le& \delta_7 C_4 \|\xi_2^{n+1}\|_{M_2}^2 + \frac{C^2}{\delta_7 C_4} h^{2k_2}, \\ |T_8| \le& \delta_8 \mu C_3 \|\boldsymbol{\chi}^{n+1}\|_X^2 + \frac{C^2}{\delta_8 \mu C_3} h^{2k_1}, \end{aligned} \tag{106}$$

assuming in addition that $p_2$ belongs to $L^\infty(0, T; H^{k_2+1}(\Omega_2))$.

7) For the interior term $T_9$ that involves a discrete time derivative, we write, as for $T_3$,

$$\delta_t \boldsymbol{\zeta}^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial t}(\Pi_h(\mathbf{u}(s)) - \mathbf{u}(s)) ds.$$

Hence for any positive $\delta_9$:

$$|T_9| \le \delta_9 \mu C_3 \|\boldsymbol{\chi}^{n+1}\|_X^2 + \frac{C^2}{\delta_9 \mu C_3} \frac{h^{2k_1}}{\Delta t} \|\frac{\partial \mathbf{u}}{\partial t}\|_{L^2(t^n, t^{n+1}; H^{k_1}(\Omega_1)^d)}^2, \tag{107}$$

provided $\frac{\partial \mathbf{u}}{\partial t}$ belongs to $L^2(0, T; H^{k_1}(\Omega_1)^d)$.

8) There remains to consider the face terms $T_{10}$, $T_{11}$ and $T_{12}$. As in the case of the face terms in $c_{NS}$ above, they are treated by trace inequalities and equivalence of norms. Thus, for any positive $\delta_{10}, \delta_{11}, \delta_{12}$:

$$\begin{aligned} |T_{10}| \le& \delta_{10} \mu C_3 \|\boldsymbol{\chi}^{n+1}\|_X^2 + \frac{C^2}{\delta_{10} \mu C_3} h^{2k_2}, \\ |T_{11}| \le& \delta_{11} C_4 \|\xi_2^{n+1}\|_{M_2}^2 + \frac{C^2}{\delta_{11} C_4} h^{2k_1}, \\ |T_{12}| \le& \delta_{12} \mu C_3 \|\boldsymbol{\chi}^{n+1}\|_X^2 + \frac{C^2}{\delta_{12} \mu C_3} h^{2k_1}. \end{aligned} \tag{108}$$

The rest of the argument is standard. We combine all the bounds above and choose suitably the parameters $\delta_i$ for all $1 \le i \le 12$, so that the contributions of $\boldsymbol{\chi}^{n+1}\|_X^2$ and $\|\xi_2^{n+1}\|_{M_2}^2$ are balanced by the corresponding terms in the left-hand side of (98). We sum the equations from $n = 0$ to $n = m - 1$, multiply by $2\Delta t$, use the fact that $\boldsymbol{\chi}^0 = \mathbf{0}$ and conclude with Gronwall's lemma. □

We end this section with a brief discussion on the Navier-Stokes pressure error. We only consider the case when $a_S$ is symmetric, as the non-symmetric formulation yields suboptimal convergence rates. As usual, deriving an error estimate on the pressure $p_{1h}^n$ is more delicate because it cannot be dissociated from the error on $\delta_t \mathbf{u}_h^n$; this can be seen by inspecting the error equation (95). An estimate for $\delta_t \boldsymbol{\chi}^n$, more precisely

$$\Delta t \sum_{n=0}^{m} \|\delta_t \boldsymbol{\chi}^n\|^2_{L^2(\Omega_1)},$$

can be obtained by testing (95) with $\delta_t \boldsymbol{\chi}^n$, multiplying by $\Delta t$ and summing over $n$ from 0 to $m-1$, for $1 \le m \le N_T$. Owing to the symmetry of $a_S$, its contribution to the sum is

$$\Delta t \sum_{n=0}^{m-1} a_S(\boldsymbol{\chi}^{n+1}, \delta_t \boldsymbol{\chi}^n) = \frac{1}{2}\Big(a_S(\boldsymbol{\chi}^m, \boldsymbol{\chi}^m) + \sum_{n=0}^{m-1} a_S(\boldsymbol{\chi}^{n+1} - \boldsymbol{\chi}^n, \boldsymbol{\chi}^{n+1} - \boldsymbol{\chi}^n)\Big).$$

Regarding the right-hand side, the factor $\delta_t \boldsymbol{\chi}^n$ must be bounded in the $L^2$ norm because this is the only norm available in the left-hand side. When this is not possible, for instance in the case of $\Delta t\, a_S(\boldsymbol{\zeta}^{n+1}, \delta_t \boldsymbol{\chi}^n)$, we apply a summation by parts that transfers the discrete time derivative on $\boldsymbol{\zeta}^n$, which can then be handled by supposing that the exact solution is sufficiently smooth in time. For brevity, we skip the details. The only difficulty concerns the nonlinear term. Its contribution can be split as

$$\begin{aligned} c_{NS}(\mathbf{u}_h^n, &\mathbf{u}^n; \mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}, \delta_t \boldsymbol{\chi}^n) + c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n - \mathbf{u}^n; \mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}, \delta_t \boldsymbol{\chi}^n) \\ &+ c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n - \mathbf{u}^n; \mathbf{u}^{n+1}, \delta_t \boldsymbol{\chi}^n) - c_{NS}(\mathbf{u}_h^n, \delta \mathbf{u}^n; \mathbf{u}^{n+1}, \delta_t \boldsymbol{\chi}^n). \end{aligned}$$

Assuming that the exact solution is sufficiently smooth, the above terms are easily bounded except the second term,

$$c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n - \mathbf{u}^n; \mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}, \delta_t \boldsymbol{\chi}^n).$$

To see this, it suffices to consider one interior term; the terms on faces require a slightly more technical treatment but lead to similar results. Thus, we consider

$$\big|\big((\mathbf{u}_h^n - \mathbf{u}^n).\nabla(\mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}), \delta_t \boldsymbol{\chi}^n\big)_E\big| \le \|\delta_t \boldsymbol{\chi}^n\|_{L^2(E)} |\mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}|_{W^{1,3}(E)} \|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^6(E)}.$$

After summation over $E$ and over $n$, we have

$$\begin{aligned} \big|\Delta t &\sum_{n=0}^{m-1} c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n - \mathbf{u}^n; \mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}, \delta_t \boldsymbol{\chi}^n)\big| \\ &\le \Delta t \sum_{n=0}^{m-1} \|\delta_t \boldsymbol{\chi}^n\|_{L^2(\Omega_1)} \|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^6(\Omega_1)} \big(\sum_{E \in \mathcal{E}_1^h} |\mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}|^3_{W^{1,3}(E)}\big)^{1/3} \\ &\le C \Delta t \sum_{n=0}^{m-1} \|\delta_t \boldsymbol{\chi}^n\|_{L^2(\Omega_1)} \|\mathbf{u}_h^n - \mathbf{u}^n\|_X \big(\sum_{E \in \mathcal{E}_1^h} |\mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}|^3_{W^{1,3}(E)}\big)^{1/3}, \end{aligned}$$

after using (37) with $r = 6$ in the second factor. Since we have no bound for the last factor, we break it into two terms by inserting the interpolant $\Pi_h(\mathbf{u}^{n+1})$, applying an inverse inequality to

$$|\mathbf{u}_h^{n+1} - \Pi_h(\mathbf{u}^{n+1})|_{W^{1,3}(E)} = |\boldsymbol{\chi}^{n+1}|_{W^{1,3}(E)},$$

and using the approximation properties of $\Pi_h$, see (87), to bound the other part, $|\boldsymbol{\zeta}^{n+1}|_{W^{1,3}(E)}$, assuming that the exact solution is smooth enough. Now, since the inverse inequality brings the factor $|E|^{-1/6} \sim h_E^{-1/2}$, after applying Jensen's inequality, this factor is bounded by,

$$\frac{1}{\min_{E\in\mathcal{E}_1^h} \sqrt{h_E}} \Big( \sum_{E\in\mathcal{E}_h} |\boldsymbol{\chi}^{n+1}|^3_{H^1(E)} \Big)^{1/3} \le \frac{C}{\min_{E\in\mathcal{E}_1^h} \sqrt{h_E}} \|\boldsymbol{\chi}^{n+1}\|_X.$$

Summarizing, we have to deal with

$$C \sum_{n=0}^{m-1} \frac{\Delta t}{\min_E \sqrt{h_E}} \|\delta_t \boldsymbol{\chi}^n\|_{L^2(\Omega_1)} \|\boldsymbol{\chi}^n\|_X \|\boldsymbol{\chi}^{n+1}\|_X$$
$$\le \frac{\delta}{2} \Delta t \sum_{n=0}^{m-1} \|\delta_t \boldsymbol{\chi}^n\|^2_{L^2(\Omega_1)} + \frac{C^2}{2\delta} \frac{1}{\min_{E\in\mathcal{E}_1^h} h_E} \max_{1\le n\le m-1} (\|\boldsymbol{\chi}^n\|_X^2) \sum_{n=1}^{m} \Delta t \|\boldsymbol{\chi}^n\|^2_X,$$

for some parameter $\delta > 0$ to be chosen further on. The factor $\frac{1}{\min_E h_E}$ must be compensated by the error bound on the velocity derived in Theorem 6.3. In the worst case, $k_1 = k_2 = 1$ (which is the smallest degree), Theorem 6.3 states that

$$\Delta t \sum_{n=1}^{m} \|\mathbf{u}_h^n - \mathbf{u}^n\|_X^2 \le C(h^2 + (\Delta t)^2).$$

In this case, we require that the mesh satisfies for some constant $D > 0$, independent of $h$ and $\Delta t$, to be chosen later,

$$\frac{h^2 + (\Delta t)^2}{\min_{E\in\mathcal{E}_1^h} h_E} \le D, \tag{109}$$

which is more restrictive than (14), but milder than quasi uniformity in space. With this assumption, after suitable applications of Young's inequality and choices of the parameters, and supposing that the solution is sufficiently smooth, the error equation yields the following inequality:

$$\begin{aligned} &\alpha_1 \Delta t \sum_{n=0}^{m-1} \|\delta_t \boldsymbol{\chi}^n\|^2_{L^2(\Omega_1)} + \alpha_2 \mu C_3 \|\boldsymbol{\chi}^m\|^2_X + \alpha_3 C_4 \Delta t \sum_{n=1}^{m} \|\xi_2^{n+1}\|^2_{M_2} \\ &\quad \le K_1\big(h^{2k_1} + h^{2k_2} + (\Delta t)^2\big) + K_2 D \max_{1\le n\le N_T-1} (\|\boldsymbol{\chi}^n\|^2_X), \end{aligned} \tag{110}$$

where $\alpha_1$, $\alpha_2$, $\alpha_3$, $K_1$ and $K_2$ are constants that depend on the choice of the parameters, but are independent of $h$ and $\Delta t$. Let $n_0$ be an index where the maximum of $\|\boldsymbol{\chi}^n\|_X^2$ is attained; (110) with $n_0$ instead of $m$ implies in particular that

$$\alpha_2 \mu C_3 \|\boldsymbol{\chi}^{n_0}\|_X^2 \le K_1\big(h^{2k_1} + h^{2k_2} + (\Delta t)^2\big) + K_2 D \|\boldsymbol{\chi}^{n_0}\|_X^2.$$

Therefore, it suffices to choose for instance

$$D \le \frac{\alpha_2}{2K_2} \mu C_3,$$

to derive that

$$\max_{1 \le n \le N_T - 1} (\|\boldsymbol{\chi}^n\|_X^2) \le \frac{2K_1}{\alpha_2 \mu C_3}\big(h^{2k_1} + h^{2k_2} + (\Delta t)^2\big).$$

Thus we have the following theorem; to simplify we do not specify the precise regularity of the solution.

**Theorem 6.4.** *Assume that in addition to* (14)*, the mesh and time step satisfy* (109)*. Then, if* $\epsilon_1 = -1$*, there is a constant* $C$ *independent of* $h$ *and* $\Delta t$ *such that for all* $1 \le m \le N_T$*, we have*

$$\Delta t \sum_{n=0}^{m-1} \|\delta_t(\mathbf{u}^n - \mathbf{u}_h^n)\|_{L^2(\Omega_1)}^2 \le C(h^{2k_1} + h^{2k_2} + \Delta t^2). \tag{111}$$

Finally, the following error estimate on the pressure is easily derived from this result and the inf-sup condition in Lemma 6.1: with the assumption (109), there is a constant $C$ independent of $h$ and $\Delta t$ such that for all $1 \le m \le N_T$,

$$\Delta t \sum_{n=1}^{m} \|p_1^n - p_{1h}^n\|_{M_1}^2 \le C(h^{2k_1} + h^{2k_2} + \Delta t^2). \tag{112}$$

## 7. Numerical results

In this section, we verify the convergence results numerically. The computational domain $\Omega \in \mathbb{R}^2$ is subdivided into a Navier-Stokes region $\Omega_1 = (0,1)\times(0,1)$ and a Darcy region $\Omega_2 = (0,1)\times(-1,0)$. The solution $(\mathbf{u}, p_1, p_2)$ is chosen to satisfy the model (1)–(6) and the interface jump conditions (7)–(9). The initial velocity does not vanish, and our analysis above can be extended to this case. The exact solutions are given as follows:

$$\mathbf{u}(t,x,y) = ((y^2 - 2y + 2x - 4y^3 x - 3)\cos(\pi t),\ (x^2 - x - 2y + y^4)\cos(\pi t)),$$

$$p_1(t,x,y) = (-x^2 y + xy + y^2 - 4 + 8y^3)\cos(\pi t), \quad p_2(x,y) = (-x^2 y + xy + y^2)\cos(\pi t).$$

First, we derive spatial rates of convergence by computing the solution on a sequence of uniformly refined meshes. The scheme (28)–(29) is used with parameters $\epsilon_1 = \epsilon_2 = -1$, which corresponds to the symmetric formulations for $a_S$ and $a_D$. The penalty parameter is chosen constant, $\sigma_e = 40$. The time step is $\Delta t = 10^{-4}$ and $N_T = 100$. We vary the polynomial approximations. In the first scenario, we use discontinuous linear polynomials for the Navier-Stokes velocity and Darcy pressure and we use piecewise constants for the Navier-Stokes pressure. In other words, $k_1 = k_2 = 1$. Table 1 shows the numerical errors in the Navier-Stokes subdomain and Table 2 shows the errors in the Darcy subdomain. The notation $\nabla_h$ is used for the elementwise gradient operator, also called broken gradient. We also report the convergence rates. We obtain optimal rates as predicted by our theory. In addition, the results indicate that the errors for the Navier-Stokes velocity and Darcy pressure converge optimally in the $L^2$ norm.

| $h$ | $\|\|\mathbf{u} - \mathbf{u}_h\|\|_{L^2(\Omega_1)}$ | Conv. | $\|\|\nabla_h(\mathbf{u} - \mathbf{u}_h)\|\|_{L^2(\Omega_1)}$ | Conv. | $\|\|p_1 - p_{1h}\|\|_{L^2(\Omega_1)}$ | Conv. |
|---|---|---|---|---|---|---|
| 1/2 | 2.0216e-01 | | 1.5412e+00 | | 2.6151e+00 | |
| 1/4 | 5.4695e-02 | 1.89 | 8.6260e-01 | 0.84 | 2.4187e+00 | 0.11 |
| 1/8 | 1.6506e-02 | 1.73 | 4.5047e-01 | 0.94 | 1.9402e+00 | 0.32 |
| 1/16 | 4.8106e-03 | 1.78 | 2.2324e-01 | 1.01 | 1.2947e+00 | 0.58 |
| 1/32 | 1.3381e-03 | 1.85 | 1.0930e-01 | 1.03 | 7.5122e-01 | 0.79 |

TABLE 1. Spatial convergence rates for the Navier-Stokes velocity and pressure with the choice $k_1 = k_2 = 1$ and $\epsilon_1 = \epsilon_2 = -1$.

| $h$ | $\|\|p_2 - p_{2h}\|\|_{L^2(\Omega_2)}$ | Conv. | $\|\|\nabla_h(p_2 - p_{2h})\|\|_{L^2(\Omega_2)}$ | Conv. |
|---|---|---|---|---|
| 1/2 | 6.0193e-02 | | 4.5512e-01 | |
| 1/4 | 1.5648e-02 | 1.94 | 2.4628e-01 | 0.89 |
| 1/8 | 4.0403e-03 | 1.95 | 1.2674e-01 | 0.96 |
| 1/16 | 1.0346e-03 | 1.97 | 6.4049e-02 | 0.98 |
| 1/32 | 2.6144e-04 | 1.98 | 3.2157e-02 | 0.99 |

TABLE 2. Spatial convergence rates for the Darcy pressure with the choice $k_1 = k_2 = 1$ and $\epsilon_1 = \epsilon_2 = -1$.

Next, we increase the polynomial degree and choose $k_1 = k_2 = 2$. We repeat the experiments and show the errors and rates in Table 3 and Table 4. They are optimal..

| $h$ | $\|\|\mathbf{u} - \mathbf{u}_h\|\|_{L^2(\Omega_1)}$ | Conv. | $\|\|\nabla_h(\mathbf{u} - \mathbf{u}_h)\|\|_{L^2(\Omega_1)}$ | Conv. | $\|\|p_1 - p_{1h}\|\|_{L^2(\Omega_1)}$ | Conv. |
|---|---|---|---|---|---|---|
| 1/2 | 2.6944e-02 | | 4.3835e-01 | | 4.9743e-01 | |
| 1/4 | 4.3001e-03 | 2.65 | 1.1511e-01 | 1.93 | 1.6088e-01 | 1.63 |
| 1/8 | 5.8130e-04 | 2.89 | 2.8713e-02 | 2.00 | 4.6854e-02 | 1.78 |
| 1/16 | 7.3852e-05 | 2.98 | 7.0518e-03 | 2.03 | 1.2636e-02 | 1.89 |
| 1/32 | 1.0249e-05 | 2.85 | 1.7388e-03 | 2.02 | 3.2828e-03 | 1.94 |

TABLE 3. Spatial convergence rates for the Navier-Stokes velocity and pressure for the choice $k_1 = k_2 = 2$ and $\epsilon_1 = \epsilon_2 = -1$,

| $h$ | $\|\|p_2 - p_{2h}\|\|_{L^2(\Omega_2)}$ | Conv. | $\|\|\nabla_h(p_2 - p_{2h})\|\|_{L^2(\Omega_2)}$ | Conv. |
|---|---|---|---|---|
| 1/2 | 3.1803e-03 | | 5.4874e-02 | |
| 1/4 | 4.9972e-04 | 2.67 | 1.4217e-02 | 1.95 |
| 1/8 | 6.8328e-05 | 2.87 | 3.6107e-03 | 1.98 |
| 1/16 | 8.8834e-06 | 2.94 | 9.0948e-04 | 1.99 |
| 1/32 | 1.1711e-06 | 2.92 | 2.2822e-04 | 1.99 |

TABLE 4. Spatial convergence rates for the Darcy pressure for the choice $k_1 = k_2 = 2$ and $\epsilon_1 = \epsilon_2 = -1$.

In the following experiments, we choose the parameters $\epsilon_1 = \epsilon_2 = 1$, which corresponds to the non-symmetric formulations of $a_S$ and $a_D$. We repeat the tests above by first considering $k_1 = k_2 = 1$ and then $k_1 = k_2 = 2$. Tables 5, 6, 7, 8 show the numerical errors and their rates. The rates in the broken gradient norm for the Navier-Stokes velocity and the Darcy pressure are optimal, as predicted by our theory. The rates for the $L^2$ norm for the Navier-Stokes pressure are also optimal. The proof remains an open question since the bound (112) is only valid for the case $\epsilon_1 = -1$. We also note that rates in $L^2$ for the Darcy pressure are suboptimal, which is consistent with the classical theory for the non-symmetric interior penalty Galerkin method [20].

| $h$ | $\|\|\mathbf{u}-\mathbf{u}_h\|\|_{L^2(\Omega_1)}$ | Conv. | $\|\|\nabla_h(\mathbf{u}-\mathbf{u}_h)\|\|_{L^2(\Omega_1)}$ | Conv. | $\|\|p_1-p_{1h}\|\|_{L^2(\Omega_1)}$ | Conv. |
|---|---|---|---|---|---|---|
| 1/2 | 2.0771e-01 | | 1.4522e+00 | | 3.3226e+00 | |
| 1/4 | 9.5577e-02 | 1.12 | 8.2012e-01 | 0.82 | 1.4329e+00 | 1.21 |
| 1/8 | 3.8831e-02 | 1.30 | 4.4140e-01 | 0.89 | 5.2375e-01 | 1.45 |
| 1/16 | 1.1797e-02 | 1.72 | 2.1486e-01 | 1.04 | 2.5820e-01 | 1.02 |
| 1/32 | 3.1715e-03 | 1.90 | 1.0454e-01 | 1.04 | 1.2895e-01 | 1.00 |

Table 5. Spatial convergence rates for the Navier-Stokes velocity and pressure for the choice $k_1 = k_2 = 1$ and $\epsilon_1 = \epsilon_2 = 1$.

| $h$ | $\|\|p_2-p_{2h}\|\|_{L^2(\Omega_2)}$ | Conv. | $\|\|\nabla_h(p_2-p_{2h})\|\|_{L^2(\Omega_2)}$ | Conv. |
|---|---|---|---|---|
| 1/2 | 8.1224e-02 | | 4.8229e-01 | |
| 1/4 | 1.6212e-02 | 2.32 | 2.2337e-01 | 1.11 |
| 1/8 | 3.6515e-03 | 2.15 | 1.0779e-01 | 1.05 |
| 1/16 | 8.7554e-04 | 2.06 | 5.3084e-02 | 1.02 |
| 1/32 | 2.1485e-04 | 2.03 | 2.6360e-02 | 1.01 |

Table 6. Spatial convergence rates for the Darcy pressure for the choice $k_1 = k_2 = 1$ and $\epsilon_1 = \epsilon_2 = 1$

| $h$ | $\|\|\mathbf{u}-\mathbf{u}_h\|\|_{L^2(\Omega_1)}$ | Conv. | $\|\|\nabla_h(\mathbf{u}-\mathbf{u}_h)\|\|_{L^2(\Omega_1)}$ | Conv. | $\|\|p_1-p_{1h}\|\|_{L^2(\Omega_1)}$ | Conv. |
|---|---|---|---|---|---|---|
| 1/2 | 5.4021e-02 | | 4.6690e-01 | | 7.9517e-01 | |
| 1/4 | 1.2614e-02 | 2.10 | 1.5174e-01 | 1.62 | 1.9891e-01 | 2.00 |
| 1/8 | 2.1786e-03 | 2.53 | 4.2195e-02 | 1.85 | 5.1365e-02 | 1.95 |
| 1/16 | 3.2926e-04 | 2.73 | 1.1113e-02 | 1.92 | 1.2526e-02 | 2.04 |
| 1/32 | 4.8185e-05 | 2.77 | 2.8689e-03 | 1.95 | 3.2818e-03 | 1.93 |

Table 7. Spatial convergence rates for the Navier-Stokes velocity and pressure for the choice $k_1 = k_2 = 2$ and $\epsilon_1 = \epsilon_2 = 1$.

To establish the temporal rates of convergence, we fix a fine mesh and we vary the time step size. The errors are computed at the final time $T = 1$. The errors and rates are reported in Tables 9-10. We recover optimal first order convergence rates.

| $h$ | $\|\|p_2 - p_{2h}\|\|_{L^2(\Omega_2)}$ | Conv. | $\|\|\nabla_h(p_2 - p_{2h})\|\|_{L^2(\Omega_2)}$ | Conv. |
|---|---|---|---|---|
| 1/2 | 1.1081e-02 | | 7.5663e-02 | |
| 1/4 | 1.6310e-03 | 2.76 | 1.7015e-02 | 2.15 |
| 1/8 | 2.8895e-04 | 2.50 | 4.0381e-03 | 2.08 |
| 1/16 | 6.0262e-05 | 2.26 | 9.8285e-04 | 2.04 |
| 1/32 | 1.3846e-05 | 2.12 | 2.4252e-04 | 2.02 |

TABLE 8. Spatial convergence rates for the Darcy pressure for the choice $k_1 = k_2 = 2$ and $\epsilon_1 = \epsilon_2 = 1$

| $\Delta t$ | $\|\|\mathbf{u} - \mathbf{u}_h\|\|_{L^2(\Omega_1)}$ | Conv. | $\|\|\nabla_h(\mathbf{u} - \mathbf{u}_h)\|\|_{L^2(\Omega_1)}$ | Conv. | $\|\|p_1 - p_{1h}\|\|_{L^2(\Omega_1)}$ | Conv. |
|---|---|---|---|---|---|---|
| 1/2 | 8.0870e-02 | | 1.1961e+00 | | 3.7816e+00 | |
| 1/4 | 3.4933e-02 | 1.21 | 3.5960e-01 | 1.73 | 1.4666e+00 | 1.37 |
| 1/8 | 1.6292e-02 | 1.10 | 1.1048e-01 | 1.70 | 6.4018e-01 | 1.20 |
| 1/16 | 8.0506e-03 | 1.02 | 4.3830e-02 | 1.33 | 3.1530e-01 | 1.02 |
| 1/32 | 4.0441e-03 | 0.99 | 2.2349e-02 | 0.97 | 1.5985e-01 | 0.98 |

TABLE 9. Temporal convergence rates for the Navier-Stokes velocity and pressure.

| $\Delta t$ | $\|\|p_2 - p_{2h}\|\|_{L^2(\Omega_2)}$ | Conv. | $\|\|\nabla_h(p_2 - p_{2h})\|\|_{L^2(\Omega_2)}$ | Conv. |
|---|---|---|---|---|
| 1/2 | 4.1680e-03 | | 3.7806e-02 | |
| 1/4 | 2.2823e-03 | 0.87 | 2.0532e-02 | 0.88 |
| 1/8 | 1.1844e-03 | 0.95 | 1.0637e-02 | 0.95 |
| 1/16 | 5.9923e-04 | 0.98 | 5.4063e-03 | 0.98 |
| 1/32 | 3.0627e-04 | 0.97 | 2.8348e-03 | 0.93 |

TABLE 10. Temporal convergence rates for the Darcy pressure.

## 8. Conclusions

We have obtained the numerical analysis of a discontinuous Galerkin method in space combined with backward Euler in time for solving the time-dependent Navier-Stokes and Darcy equations. The analysis presented here can be easily adapted for various finite-element discretizations.

## References


[1] D.N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19:742–760, 1982.

[2] L. Badea, M. Discacciati, and A. Quarteroni. Numerical analysis of the Navier–Stokes/Darcy coupling. *Numerische Mathematik*, 115(2):195–227, 2010.

[3] G.S. Beavers and D.D. Joseph. Boundary conditions at a naturally impermeable wall. *J. Fluid. Mech*, 30:197–207, 1967.

[4] S. C. Brenner. Poincaré-Friedrichs inequalities for piecewise $H^1$ functions. *SIAM J. Numer. Anal.*, 41(1):306–324 (electronic), 2003.

[5] S. C. Brenner. Korn's inequalities for piecewise $H^1$ vector fields. *Math. Comp.*, 73(247):1067–1087 (electronic), 2004.

[6] A. Cesmelioglu, V. Girault, and B. Riviere. Time-dependent coupling of Navier–Stokes and Darcy flows. *ESAIM: Mathematical Modelling and Numerical Analysis*, 47(2):539–554, 2013.

[7] A. Cesmelioglu and B. Riviere. Analysis of time-dependent Navier-Stokes flow coupled with Darcy flow. *Journal of Numerical Mathematics*, 16:249–280, 2008.

[8] A. Cesmelioglu and B. Riviere. Primal discontinuous Galerkin methods for time-dependent coupled surface and subsurface flow. *Journal of Scientific Computing*, 40:115–140, 2009.

[9] P. Chidyagwai and B. Riviere. On the solution of the coupled Navier-Stokes and Darcy equations. *Computer Methods in Applied Mechanics and Engineering*, 198:3806–3820, 2009.

[10] P. Chidyagwai and B. Riviere. Numerical modelling of coupled surface and subsurface flow systems. *Advances in Water Resources*, 33:92–105, 2010.

[11] P. G. Ciarlet. *The finite element method for elliptic problems*. North-Holland, Amsterdam, 1978.

[12] M. Discacciati, E. Miglio, and A. Quarteroni. Mathematical and numerical models for coupling surface and groundwater flows. *Appl. Numer. Math.*, 43:57–74, 2001.

[13] M. Discacciati and A. Quarteroni. Analysis of a domain decomposition method for the coupling of Stokes and Darcy equations. In Brezzi et al, editor, *Numerical Analysis and Advanced Applications - ENUMATH 2001*, pages 3–20. Springer, Milan, 2003.

[14] M. Discacciati and A. Quarteroni. Navier-Stokes/Darcy coupling: modeling, analysis, and numerical approximation. *Revista Matemática Complutense*, 22(2):315–426, 2009.

[15] V. Girault and B. Riviere. DG approximation of coupled Navier-Stokes and Darcy equations by Beaver-Joseph-Saffman interface condition. *SIAM Journal on Numerical Analysis*, 47(3):2052–2089, 2009.

[16] V. Girault, B. Riviere, and M.F. Wheeler. A splitting method using discontinuous Galerkin for the transient incompressible Navier-Stokes equations. *ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique*, 39(6):1115–1147, 2005.

[17] Vivette Girault, Béatrice Rivière, and Mary Wheeler. A discontinuous Galerkin method with nonoverlapping domain decomposition for the Stokes and Navier-Stokes problems. *Mathematics of Computation*, 74(249):53–84, 2005.

[18] J. L. Lions. *Equations différentielles opérationelles et problèmes aux limites*. Springer-Verlag, 1961.

[19] P.A. Raviart and J.M. Thomas. A mixed finite element method for 2nd order elliptic problems. In *Mathematical Aspects of the Finite Element Method, Lecture Notes in Mathematics*, volume 606, pages 292–315. Springer-Verlag, New York, 1977.

[20] B. Riviere. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*. SIAM, 2008.

[21] P. Saffman. On the boundary condition at the surface of a porous media. *Stud. Appl. Math.*, 50:292–315, 1971.

[22] L.R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Mathematics of Computation*, 54(190):483–493, 1990.

[23] R. Temam. *Navier-Stokes equations: theory and numerical analysis*, volume 343. American Mathematical Soc., 2001.