Equivariant spaces of matrices of constant rank

J M Landsberg, L Manivel

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J.M. LANDSBERG, L. MANIVEL

Abstract. We use representation theory to construct spaces of matrices of constant rank . These spaces are parametrized by the natural representation of the general linear group or the symplectic group. We present variants of this idea, with more complicated representations, and others with the orthogonal group. Our spaces of matrices correspond to vector bundles which are homogeneous but sometimes admit deformations to non-homogeneous vector bundles, showing that these spaces of matrices sometimes admit large families of deformations.

1. Introduction

It is a classical problem in both algebraic geometry and linear algebra to construct linear spaces of matrices of constant rank (constant outside the origin). This problem is presented in the language of algebraic geometry in [5] and most work on the problem since then has used this language and the tools it brings with it, including this paper.

There is a strong relationship with the study of vector bundles on projective space, another classical topic that attracted considerable attention. Indeed, a vector space of dimension $n+1$ of matrices of size $a \times b$ can be seen as a matrix with linear entries, or equivalently as a morphism of sheaves $\psi : \mathcal{O}_{\mathbb{P}^n}^{\oplus a} \longrightarrow \mathcal{O}(1)_{\mathbb{P}^n}^{\oplus b}$. If the rank is constant, equal to r, the image of this morphism is a vector bundle $\mathcal{E}$ of rank r. Letting $\mathcal{K}$ and $\mathcal{C}$ denote the kernel and cokernel bundles, we get the diagram below, where diagonals are short exact sequences. The vector bundle $\mathcal{E}$ has very special properties, in particular:

- $\mathcal{E}$ and $\mathcal{E}^\vee(1)$ are generated by global sections;
- as a consequence, $\mathcal{E}$ is uniform, in the sense that its restriction to every line $L \subset \mathbb{P}^n$ splits in the same way:

$$\mathcal{E}_{|L} \simeq \mathcal{O}_L(1)^{\oplus c_1(E)} \oplus \mathcal{O}_L^{\oplus(r-c_1(E))}.$$

Uniform vector bundles have been classified up to rank $r \leq n+1$: in this range, for $n \geq 3$ they are sums of line bundles and the tautological quotient bundle $\mathcal{Q}$ or its dual (see [6] and references therein; in that paper the same result is conjectured to hold for $n \geq 5$ and $r < 2n$). The general philosophy is that there should exist very few uniform vector bundles on $\mathbb{P}^n$ of small

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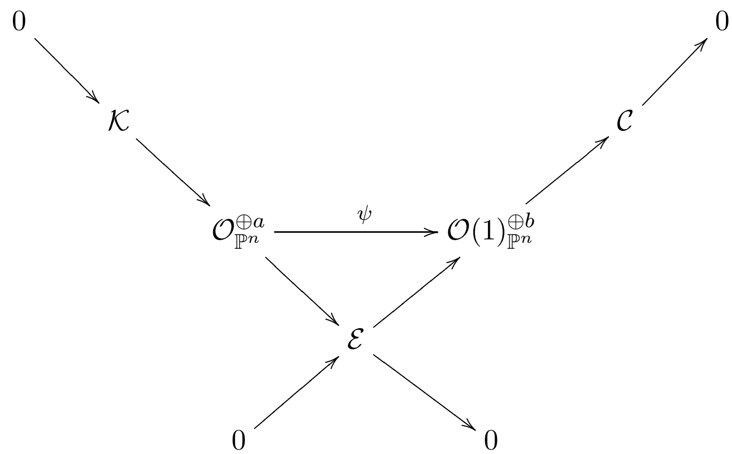
rank, but they are easier to construct when the rank is large. Conversely, if $\mathcal{E}$ is a rank r vector bundle on $\mathbb{P}^n$, such that $\mathcal{E}$ and $\mathcal{E}^\vee(1)$ are generated by global sections (so that in particular $\mathcal{E}$ is uniform), the natural morphism

$$\psi_{\mathcal{E}} : H^0(\mathbb{P}^n, \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow H^0(\mathbb{P}^n, \mathcal{E}^\vee(1))^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1)$$

has constant rank r.



Examples.

(1) If $\mathcal{E}$ is a sum of line bundles, one gets what is called in [1] a compression space.

(2) If $\mathcal{E} = \mathcal{Q}$, we have $H^0(\mathbb{P}^n, \mathcal{E}) = V$ and $H^0(\mathbb{P}^n, \mathcal{E}^\vee(1))^\vee = \wedge^2 V$. In this case the twist

$$\psi(-1) : V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \wedge^2 V \otimes \mathcal{O}_{\mathbb{P}^n}$$

is the obvious vector bundle map which at $[v] \in \mathbb{P}^n$ sends $w \otimes v$ to $w \wedge v$. More generally, let $\mathcal{E} = \wedge^p \mathcal{Q}$ for some $p > 0$, and we get the similar morphisms $\psi(-1) : \wedge^p V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \wedge^{p+1} V \otimes \mathcal{O}_{\mathbb{P}^n}$ from which the Koszul complex is constructed.

This last example is classical and dates at least back to Westwick [14]. The main goal of this paper is to present a wide generalization by considering maps $\psi : U \otimes \mathcal{O}_{\mathbb{P}V}(-1) \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}V}$, where U, W are G-modules (that we will suppose irreducible, for simplicity) of a classical group $G \subseteq GL(V)$, with a nonzero equivariant morphism $\Psi : U \otimes V \longrightarrow W$ and V denotes the natural representation.

Theorem. *If G is a general linear group, or a symplectic group, then ψ defines a linear space of matrices of constant rank.*

The associated bundle $\mathcal{E}$ is homogeneous and admits an explicit description via a module for the parabolic subgroup preserving a highest weight line in V.

The bundle $\mathcal{E}$ may arise from an arbitrarily long sequence of nontrivial extensions of explicit completely reducible homogeneous bundles, so its structure can be quite complicated. Being homogeneous, we expected it to be rigid, as is typically the case of $\mathcal{E} = \wedge^p \mathcal{Q}$ and of the first cases we checked. However, very quickly one obtains non-rigid bundles, whose deformation spaces can have large dimension. For the case we discuss in detail, we get vector bundles on $\mathbb{P}^n$ with $O(n^4)$ dimensions worth of moduli, see Proposition 14. As a consequence, ψ may be deformed into non-equivariant linear spaces of matrices of constant rank.

We make the following simple observation.

Proposition 1. *If $G = GL(V)$ or $G = Sp(V)$, and M, N are irreducible G-modules with $V \subset M^\vee \otimes N$, then the image of the morphism $\psi : V \longrightarrow Hom(M, N)$ is a linear space of constant rank.*

Proof. This immediately follows from the equivariance of the morphism, and that G acts transitively on $\mathbb{P}(V)$. □

Overview: After a section of preliminaries, in section 3 we revisit Westwick's examples and give a representation-theoretic discussion of the problem. In section 4, focusing on the general linear group we discuss the structure of the associated bundle $\mathcal{E}$; we explain why we get a rigid bundle in a simple case, and a bundle with a very large deformation space in a slightly more complicated situation. We also discuss how to extend Proposition 1 beyond the natural representation. In section 5 we observe that it can be extended to a simple statement that allows one to construct many spaces of matrices of constant rank starting from a given one. Then we consider the case of the symplectic group, and explicitly construct a six dimensional space of 14×14 matrices of rank nine. Finally we discuss what can be obtained for the orthogonal groups, and show that one can at least construct spaces of bounded rank; either from the natural representation in section 7, or the spin representations in section 8.

The reader may have observed that our constructions are closely connected to the general theory of Steiner vector bundles on projective spaces [2]. In a subsequent paper we present constructions of Steiner bundles giving rise to new classes of spaces of constant rank.

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2. Preliminaries

2.1. **Notation.** We work exclusively over the complex numbers.

V is a complex vector space of dimension $\mathbf{v} = n+1$. We give V basis $v_0, \dots, v_n$ and dual basis $\alpha^0, \dots, \alpha^n$.

For $\pi = (p_1, \dots, p_{n+1})$ a non-increasing sequence of integers, $S_\pi V$ denotes the corresponding irreducible $GL(V)$-module. If π contains repeated entries, we use exponents to denote them, e.g. for $(1,1,1,0,0,-1)$ we write $(1^3, 0^2, -1)$.

For a vector space W, we let $\underline{W} := W \otimes \mathcal{O}_{\mathbb{P}V}$ denote the corresponding trivial vector bundle on $\mathbb{P}V$ when it is clear from the context which projective space we are taking the bundle over.

If $\mathcal{E}, \mathcal{F} \subseteq \underline{W}$, then $\mathcal{E}\mathcal{F} \subset \underline{S^2 W}$ denotes the image of the multiplication map.

2.2. **Basic spaces.** There are several ways to avoid redundancies in the classification of spaces of bounded and constant rank.

A classical and essentially understood class of spaces of bounded rank are the *compression spaces*, those spaces of $\mathbf{b} \times \mathbf{c}$ matrices that in some choice of bases have a zero in the lower $\mathbf{b} - r_1 \times \mathbf{c} - r_2$ block. Such spaces are of bounded rank $r_1 + r_2$. Compression spaces can have constant rank, e.g.,

$$\begin{pmatrix} 0 & x_1 & \cdots & x_k \\ x_1 & & & \\ \vdots & & & \\ x_k & & & \end{pmatrix}$$

but note this is a subspace of the direct sum of two spaces of bounded rank one. To avoid this, call a space of the form

$$\begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix}$$

split and following [5], say a space is *strongly indecomposable* if it is not the projection $B' \otimes C' \to B \otimes C$ of a split space of the same rank. A space that is strongly indecomposable has associated $\mathcal{E}$ indecomposable [5].

2.3. **Rank criticality.** Maximal spaces of matrices of constant, or more generally bounded rank, are particularly interesting.

Definition 2. [3] *A space of bounded rank* $V \subset \mathrm{Hom}(B, C)$ *is rank-critical if any subspace of* $\mathrm{Hom}(B, C)$ *that strictly contains* V *contains morphisms of larger rank.*

Eisenbud and Harris [5] call rank critical spaces *unliftable*. Draisma gives an easily checked sufficient condition for rank criticality: for $L \subset Hom(U, W)$ a linear space of morphisms of generic rank r, define the space of *rank neutral directions*

$$\begin{aligned} RND(L) : &= \{B \in Hom(U, W),\ B(Ker(A)) \subset Im(A)\ \forall A \in L, rank(A) = r\} \\ &= \bigcap_{A \in L, rank(A) = r} \hat{T}_A \sigma_r(Seg(\mathbb{P}U \times \mathbb{P}W)). \end{aligned}$$

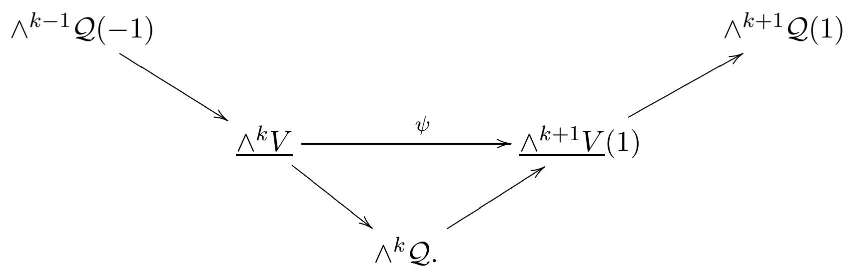
Applying [3, Prop. 8] to a slightly more general situation than [3, Prop. 3], $RND(L)$ always contains L and in case of equality, L is rank critical. If there is a group G acting on the set up and preserving L, then it must also preserve $RND(L)$.

3. Equivariant morphisms of constant or bounded rank

3.1. **A classical example revisited.** The prototypical equivariant morphisms of constant rank appear in the Koszul complex, as the morphisms

$$\psi : V \longrightarrow Hom(\wedge^k V, \wedge^{k+1} V).$$

For any nonzero vector v, the kernel (resp. image) of ψ_v is $\wedge^{k-1} V \wedge v$ (resp. $\wedge^k V \wedge v$). In other words, let $\mathcal{Q}$ be the tautological quotient bundle on $\mathbb{P}V$, we have exact sequences



So ψ has constant rank and the vector bundle $\mathcal{E} = \wedge^k \mathcal{Q}$ is uniform. Here we slightly abuse notation, using ψ to denote both the map between vector bundles and the inclusion of V into a space of homomorphisms.

All this is well-known, but we add the following observation:

Proposition 3. *$\psi(V)$ is rank-critical.*

Proof. We apply the results of [3] discussed in §2.2. Here $U = \wedge^k V$, $W = \wedge^{k+1} V$ and $L = \psi(V)$, and we assume $k \leq \frac{n}{2}$ to avoid redundancy. Then $Hom(U, W) = U^\vee \otimes W$ has the following $GL(V)$-module decomposition:

$$Hom(U, W) = \bigoplus_{k \geq a \geq 0} S_{1^{a+1}, 0^{n-2a}, -1^a} V.$$

Since there are no multiplicities, we are reduced to proving that $S_{1^a, 0^{n-2a}, -1^{a+1}} V$ cannot be contained in $RND(L)$ when $a > 0$. Equivalently, we need to check that a highest weight vector in $S_{1^{a+1} 0^{n-2a}, -1^a} V$ cannot be contained in $RND(L)$. A highest weight vector is given by

$$\sum_{|I| = k-a, a < i_1 < \cdots < i_{a-k} < n+1-a} v_0 \wedge \cdots \wedge v_a \wedge v_I \otimes \alpha_I \wedge \alpha_{n+1-a} \wedge \cdots \wedge \alpha_n$$

Then $X \in \wedge^k V$ maps to

$$\sum_{|I| = k-a, a \leq i_1 < \cdots < i_{a-k} < n+2-a} [\alpha_I \wedge \alpha_{n+1-a} \wedge \cdots \wedge \alpha_n(X)] v_0 \wedge \cdots \wedge v_a \wedge v_I$$

Take $v = v_n$ and $X = v_{n-k+1} \wedge \cdots \wedge v_n \in \ker \phi_v$, then

$$X \mapsto v_0 \wedge \cdots \wedge v_a \wedge v_{n-k+1} \wedge \cdots \wedge v_{n-a+1} \notin v_n \wedge \Lambda^k V = \mathrm{Im}(\phi_v).$$

This proves the claim. $\square$

3.2. The general equivariant case.

Question 4. *Given three G-modules U, V, W, and $T \in (U \otimes V \otimes W)^G$, when is one of the three associated spaces of constant, or simply, bounded rank?*

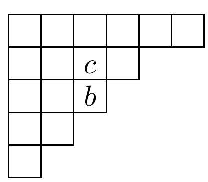
Another way two ask the same question is: Given G-modules U, W, for which submodules $V \subset U \otimes W$ is the corresponding space of bounded rank?

If one takes the Cartan component of $U \otimes W$, that is, if U has highest weight μ and W highest weight ν, the submodule of highest weight $\mu + \nu$, then the resulting space is not of bounded rank. Indeed, by the Borel-Weil theorem we can interpret our three representations as spaces of global sections of certain line bundles on the complete flag variety G/B, and our morphism is given by the pointwise product of such sections; in particular, it is always injective. Examples indicate that the submodules of lowest highest weight in $U \otimes W$ are good candidates.

4. General linear group

4.1. General remarks about embeddings $V \to \mathrm{Hom}(S_\mu V, S_\nu V)$. The irreducible representations of $GL(V)$ are the Schur modules $S_\mu V$, where μ can be supposed (after twisting by some character if necessary) to be a partition $\mu = (\mu_1, \ldots, \mu_{\mathbf{v}})$, with $\mu_1 \geq \cdots \geq \mu_{\mathbf{v}} \geq 0$ (in fact one may also assume $\mu_{\mathbf{v}} = 0$ but it will be convenient not to impose this).

By the Pieri rule, $S_\nu V$ is contained in $S_\mu V \otimes V$ if and only if $\nu = (\mu_1, \ldots, \mu_k + 1, \ldots, \mu_{\mathbf{v}})$ for some integer k such that $\mu_{k-1} > \mu_k$. The diagram of ν is then obtained by adding one box b to the diagram of μ at the extremity of the k-th row, as in the diagram below. For future use we denote the box immediately north of b by c, if there is one.



Again by the Pieri rule, there is a unique (up to scale) equivariant morphism

$$\phi : V \longrightarrow Hom(S_\mu V, S_\nu V).$$

Once we fix a nonzero vector $v \in V$, or the line $\ell = \mathbb{C}v \subset V$, the image and the kernel of ϕ_v are preserved by the action of the stabilizer of ℓ in $GL(V)$, which is a parabolic subgroup P. So we need to describe the P-module structure of $S_\mu V$ and $S_\nu V$. First, consider the action of the unipotent radical $P_u \subset P$, which is the subgroup acting trivially both on ℓ and on

V/ℓ; its (abelian) Lie algebra is $\mathfrak{p}_u = Hom(V/\ell, \ell) \subset End(V)$. The action of $\mathfrak{p}_u$ on $S_\mu V$ has a nontrivial kernel M_1, and by induction one obtains a canonical filtration

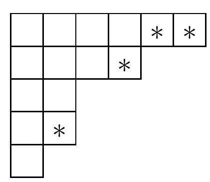
$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = S_\mu V,$$

such that $\mathfrak{p}_u(M_k) \subset M_{k-1}$. Consequently, the action of P_u on each quotient M_k/M_{k-1} is trivial, and the P-module structure of such a quotient is fully determined by its L-module structure, for L a Levi factor of P. Concretely, fixing L amounts to choosing a hyperplane H in V complementing ℓ, and then $L = GL(H) \times GL(\ell)$. As an L-module, $V = H \oplus \ell$ and the filtration of $S_\mu V$ that we have defined has associated grading determined by the ℓ degree in the summands of $S_\mu(H \oplus \ell)$.

The decomposition of the latter L-module may be described as follows:

$$S_\mu(H \oplus \ell) = \bigoplus_{k \geq 0} \bigoplus_{\mu \xrightarrow{k} \alpha} S_\alpha H \otimes \ell^k, \tag{1}$$

where the symbol $\mu \xrightarrow{k} \alpha$ means that the diagram of α can be obtained by deleting k boxes from the diagram of μ, deleting at most one box per column.



In particular, it is important to notice that this decomposition has no multiplicities bigger than one.

There is a similar decomposition for $S_\nu V$:

$$S_\nu(H \oplus \ell) = S_\nu H \oplus \bigoplus_{k \geq 0} \bigoplus_{\mu \xrightarrow{k+1} \beta} S_\beta H \otimes \ell^{k+1}.$$

The morphism ϕ_v is just multiplication by $v \in \ell$, so it sends a component $S_\alpha H \otimes \ell^k$ of $S_\mu V$ to $S_\alpha H \otimes \ell^{k+1}$ if this is a component of $S_\nu V$, and to zero otherwise. In particular the kernel of ϕ_v is the direct sum of the components $S_\alpha H \otimes \ell^k$ of $S_\mu V$ such that $S_\alpha H \otimes \ell^{k+1}$ does not appear in $S_\nu V$.

When does this happen? Recall that α is obtained by erasing k boxes from μ, at most one per column. Moreover, $\nu = \mu \cup \{b\}$ for one box b. So to obtain α from ν, one needs to erase two boxes from μ in the same column exactly when the box c immediately north to b does not belong to α; we need to erase both b and c.

Similarly, the cokernel of ϕ_v is, as an L-module, the sum of the factors $S_\beta H \otimes \ell^k$ such that the box b belongs to β. In summary:

Proposition 5. *As L-modules, the kernel, image and cokernel of ϕ_v are:*

$$\begin{aligned} Ker(\phi_v) &= \bigoplus_{k\geq 0} \bigoplus_{\mu \xrightarrow{k} \alpha, c\notin\alpha} S_\alpha H \otimes \ell^k, \\ Im(\phi_v) &= \bigoplus_{k\geq 0} \bigoplus_{\mu \xrightarrow{k} \alpha, c\in\alpha} S_\alpha H \otimes \ell^{k+1}, \\ Coker(\phi_v) &= \bigoplus_{k\geq 0} \bigoplus_{\nu \xrightarrow{k} \beta, b\in\beta} S_\beta H \otimes \ell^k. \end{aligned}$$

Corollary 6. *The morphism ϕ_v is injective (resp. surjective) exactly when the box b belongs to the first row (resp. the* **v***-th row).*

4.2. **Example: $\mu = (2)$ and $\nu = (2,1)$.** Given a decomposition $V = H \oplus \ell$ as above, we get

$$\begin{aligned} S_2 V &= S_2 H \quad \oplus H \otimes \ell \quad \oplus \ell^2, \\ S_{21} V &= S_{21} H \quad \oplus S^2 H \otimes \ell \quad \oplus \Lambda^2 H \otimes \ell \quad \oplus H \otimes \ell^2. \end{aligned}$$

Thus the kernel of ψ is isomorphic to ℓ^2, the image to $S_2 H \otimes \ell \oplus H \otimes \ell^2$ and the cokernel to $S_{21} H \oplus \wedge^2 H \otimes \ell$. This gives a matrix ψ of linear forms in $n+1$ variables, of size $a_n \times b_n$ and constant rank r_n, where

$$a_n = \frac{(n+2)(n+1)}{2}, \quad b_n = \frac{n(n+1)(n+2)}{3}, \quad r_n = \frac{n^2+3n}{2}.$$

Proposition 7. *$\psi(V)$ is not rank-critical.*

Proof. First observe that Draisma's criterion does not apply by computing $RND(L) = L$ for $L = \psi(V)$. Recall that $RND(L)$ must be a submodule of

$$Hom(S_2 V, S_{21} V) = S_{2,1,0^{n-2},-2} V \oplus S_{1,1,0^{n-2},-1} V \oplus S_{2,0^{n-1},-1} V \oplus V.$$

It suffices to consider their highest weight vectors, and the result of a straightforward computation is that $RND(L) = L \oplus S_{2,0^{n-1},-1} V$. This suggests considering $M = \langle L, \sigma \rangle$ for σ a highest weight vector in $S_{2,0^{n-1},-1} V$. This is a tensor of the form $\sigma = e^2 \otimes \alpha$ for $\alpha \in V^\vee$ a linear form vanishing on e. It sends v^2 to $\alpha(v)(e \wedge v) \otimes e = \alpha(v)\psi(v)\ (e^2)$ which belongs to $Im(\psi(v))$, so σ belongs to $RND(L)$. We claim that $\psi(v) + \sigma$ is never injective. Indeed, $\alpha(v)e^2 - v^2$ is contained in its kernel. □

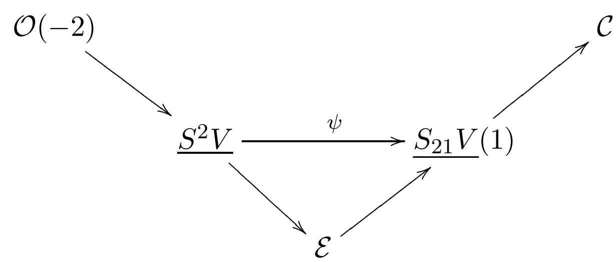
It is plausible that $\psi(V)$ is constant-rank-critical, in the sense that a bigger space of matrices will never have constant rank. Indeed having constant rank is not a closed condition, contrary to having bounded rank, so this does not follow from the previous discussion. For example, over $V \oplus \langle \sigma \rangle$, when $n = 2$ the map at σ has rank $2 < 5$.

Remark. When $n = 2$ one obtains a three dimensional space of 6×8 matrices of constant rank 5. Since $4 = 8 - 5 + 1$ does not divide $5 = 5!/4!$, the maximum possible dimension of such a space is four [14].

More intrinsically, the kernel bundle is the homogeneous line bundle $\mathcal{O}(-2)$, so in particular $c_1(\mathcal{E}) = 2$. The image bundle $\mathcal{E}$ and the cokernel bundle $\mathcal{C}$ fit into short exact sequences

$$0 \to \mathcal{Q}(-1) \to \mathcal{E} \to S^2 \mathcal{Q} \to 0, \qquad 0 \to \wedge^2 \mathcal{Q} \to \mathcal{C} \to S_{21} \mathcal{Q}(1) \to 0.$$

The situation can be summarized in the following diagram:



An easy application of Bott's theorem (see, e.g., [15]), recalling that $Ext^1(\mathcal{E},\mathcal{F}) = H^1(\mathcal{E}^*\otimes\mathcal{F})$, shows:

Lemma 8. *As a $\mathfrak{gl}_n$-module, $Ext^1(S^2\mathcal{Q},\mathcal{Q}(-1)) \simeq \mathfrak{gl}_n$.*

Lemma 8 suggests that $\mathcal{E}$ might be deformed by changing the extension class, but this is not the case:

Proposition 9. *$\mathcal{E}$ is rigid.*

Proof. Suppose $\mathcal{E}_t$ were a small deformation of $\mathcal{E}_0 = \mathcal{E}$. Since $h^q(\mathcal{E}) = 0$ for any $q > 0$, by semi-continuity this also holds for $\mathcal{E}_t$ for t close to zero, and therefore $h^0(\mathcal{E}_t) = h^0(\mathcal{E}) = 6$. Moreover the evaluation morphism $\underline{H^0(\mathcal{E}_t)} \to \mathcal{E}_t$ remains surjective since this is an open condition. The kernel is a line bundle and must be $\mathcal{O}(-2)$ since the first Chern class of $\mathcal{E}_t$ must be constant. The dual exact sequence

$$0 \to \mathcal{E}_t^\vee \to \underline{H^0(\mathcal{E}_t^\vee)} \to \mathcal{O}(2) \to 0$$

induces a morphism $H^0(\mathcal{E}_t) \to H^0(\mathcal{O}(2))$ which must be an isomorphism for t sufficiently small since it is for $t = 0$. Once we have identified these two spaces, we get the exact sequence whose kernel is $\mathcal{E}^\vee$, which is thus isomorphic with $\mathcal{E}_t^\vee$. □

Recall the *slope* of a coherent sheaf $\mathcal{F}$ is $\mu(\mathcal{F}) := c_1(\mathcal{F})/\mathrm{rank}(\mathcal{F})$, and that by definition a vector bundle $\mathcal{E}$ is *stable* if for all proper coherent subsheaves $\mathcal{F} \subset \mathcal{E}$ one has $\mu(\mathcal{F}) < \mu(\mathcal{E})$.

Proposition 10. *$\mathcal{E}$ is stable.*

Proof. The main result of [13] states that a homogeneous vector bundle $\mathcal{E}$ is stable if and only if for all homogeneous subbundles $\mathcal{F} \subset \mathcal{E}$ one has $\mu(\mathcal{F}) < \mu(\mathcal{E})$. In our case, such a subbundle yields a morphism $\mathcal{F} \to S^2\mathcal{Q}$ which by Schur's lemma must be zero or surjective. In the first case $\mathcal{F}$ is a subbundle of $\mathcal{Q}(-1)$, and again by Schur's lemma $\mathcal{F}$ being nonzero must coincide with $\mathcal{Q}(-1)$; it is then easy to check that $\mu(\mathcal{F}) < \mu(\mathcal{E})$. In the second case the kernel of the surjective map $\mathcal{F} \to S^2\mathcal{Q}$ must be a proper homogeneous subbundle of $\mathcal{Q}(-1)$, so for the same reasons it must be zero and $\mathcal{F} \simeq S^2\mathcal{Q}$ yields a splitting of $\mathcal{E}$, a contradiction. □

This confirms the expectation that this example yields an isolated point (up to change of basis) in the variety parametrizing matrices of constant rank. A straightforward computation gives the following 8×6-matrix where the rows are respectively labeled by $e_0^2, e_1^2, e_2^2, 2e_0e_1, 2e_0e_2, 2e_1e_3$ and the columns labeled by $e_0\wedge e_1\otimes e_0, e_0\wedge e_2\otimes e_0, e_0\wedge e_1\otimes e_1, e_0\wedge e_2\otimes e_2, e_1\wedge e_2\otimes e_1, e_1\wedge e_2\otimes e_2, e_0\wedge e_1\otimes e_2 + e_0\wedge e_2\otimes e_1, e_0\wedge e_1\otimes e_2 - e_1\wedge e_2\otimes e_0$:

$$\begin{pmatrix} -y & -z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & -z & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & y & 0 & 0 \\ x & 0 & -y & 0 & 0 & 0 & -z & z \\ 0 & x & 0 & -z & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 & y & -z & x & 0 \end{pmatrix}.$$

We summarize the information about the low rank example:

Proposition 11. *The inclusion $\mathbb{C}^3 \subset \mathrm{Hom}(S_2\mathbb{C}^3, S_{21}\mathbb{C}^3)$ is a space of constant rank five 6×8-matrices of dimension 3 that is rank critical and strongly indecomposable. The associated vector bundles have first Chern classes $c_1(\mathcal{E}) = 2$, $c_1(\mathcal{E}^\vee(1)) = 3$. The associated space has no nontrivial deformations. The associated tensor does not have minimal border rank.*

All the assertions have been proven except the last. To prove it, consider the Koszul flattening [10] $V^*\otimes S_2V \to \Lambda^2V^*\otimes S_{21}V$. This is a $GL(V)$-module map. It it must have rank at least 12 because the tensor is concise, but neither of the two irreducible modules in the source has dimension 12, so it must be full rank and thus the border rank of the tensor must be at least 9.

More generally, if $\mu = (a)$ and $\nu = (a,1)$, one obtains a space of constant corank one with $c_1(\mathcal{E}) = a$.

4.3. **Example:** $\mu = (2,2)$ **and** $\nu = (2,2,1)$**.** Decomposing $V = H\oplus \ell$ as above, we get

$$\begin{aligned} S_{22}V &= S_{22}H \quad \oplus S_{21}H\otimes \ell \quad \oplus S_2H\otimes \ell^2, \\ S_{221}V &= S_{221}H \quad \oplus (S_{22}H\oplus S_{211}H)\otimes \ell \quad \oplus S_{21}H\otimes \ell^2. \end{aligned}$$

Thus the kernel of ψ is isomorphic to $S_2H\otimes \ell^2$, and the image to

$$S_{22}H\otimes\ell \oplus S_{21}H\otimes \ell^2.$$

This gives a matrix ψ of linear forms in $n+1$ variables, of size $a_n\times b_n$ and constant rank r_n, where

$$a_n = \frac{n(n+1)^2(n+2)}{12}, \quad b_n = \frac{(n+2)(n+1)^2n(n-1)}{24}, \quad r_n = \frac{n(n^2-1)(n+4)}{12}.$$

The first nontrivial case is $n = 3$, where the image of the map $V\to Hom(S_{21}V, S_{221}V)$ is a four dimensional space of matrices of size 20×20, of constant rank 14.

The kernel bundle is the irreducible homogeneous bundle $\mathcal{K} = S^2\mathcal{Q}(-2)$, so $c_1(\mathcal{E}) = n-1$. The image bundle $\mathcal{E}$ fits into a short exact sequence

$$0\to S_{21}\mathcal{Q}(-1)\to \mathcal{E}\to S_{22}\mathcal{Q}\to 0.$$

In order to study the deformations of $\mathcal{E}$, we will need the following lemma:

Lemma 12. *The dual bundle $\mathcal{E}^\vee$ is acyclic.*

Proof. Start with the exact sequence

$$0 \to S^2\mathcal{Q}(-2) \to \underline{S_{22}V} \to \mathcal{E} \to 0. \tag{2}$$

Dualize (2) and apply Borel-Weil to see $H^0(S^2\mathcal{Q}^\vee(2)) = S_{22}V^\vee$. □

Proposition 13. *The vector bundle $\mathcal{E}$ is stable. It is not infinitesimally rigid, although it is rigid in the category of homogeneous bundles. Moreover $h^q(End(\mathcal{E})) = 0$ for $q > 1$, so its deformations are unobstructed.*

Proof. Twist (2) by $\mathcal{E}^\vee$ to get

$$0 \to S^2\mathcal{Q}(-2) \otimes \mathcal{E}^\vee \to S_{22}V \otimes \mathcal{E}^\vee \to End(\mathcal{E}) \to 0.$$

Since the central term is acyclic by Lemma 12, we deduce that

$$H^q(End(\mathcal{E})) \simeq H^{q+1}(S^2\mathcal{Q}(-2) \otimes \mathcal{E}^\vee).$$

In order to compute this, consider exact sequence

$$0 \to S^2\mathcal{Q}(-2) \otimes \mathcal{E}^\vee \to S^2\mathcal{Q}(-2) \otimes S_{22}V^\vee \to End(S^2\mathcal{Q}) \to 0.$$

By Bott's theorem $S^2\mathcal{Q}(-2)$ is acyclic, so

$$H^q(End(\mathcal{E})) \simeq H^q(End(S^2\mathcal{Q})).$$

Moreover, $End(S^2\mathcal{Q})$ has three irreducible components, namely $\mathcal{O}$, $S_{10\ldots0-1}\mathcal{Q}$ and $S_{20\ldots0-2}\mathcal{Q}$. By Bott's theorem again, the second component is acyclic. The last one has a nontrivial cohomology group in degree one, namely the module $S_{20\ldots0-1-1}V$, that is

$$H^1(End(\mathcal{E})) = Ker(S^2V \otimes \wedge^2V^\vee \to V \otimes V^\vee).$$

In particular $H^1(End(\mathcal{E}))^{SL(V)} = 0$, which means that $\mathcal{E}$ is infinitesimally rigid in the category of homogeneous bundles. This concludes the proof. □

Proposition 14. *The vector bundle $\mathcal{E}$ has an $\frac{(n+1)^2(n^2+2n-4)}{4} + 1$-dimensional space of deformations $\tilde{\mathcal{E}}$ which are not homogeneous bundles, but keep the property that $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}^\vee(1)$ are generated by global sections.*

Proof. Using the exact sequence

$$0 \to S_{21}\mathcal{Q}(-1) \to \mathcal{E} \to S_{22}\mathcal{Q} \to 0$$

and Bott's theorem, it is straightforward to check that $\mathcal{E}$ and $\mathcal{E}^\vee(1)$ have no higher cohomology. By semi-continuity this must remain true for a small deformation $\tilde{\mathcal{E}}$, and consequently $h^0(\tilde{\mathcal{E}})$ remains constant under small deformations. In this situation the condition to be generated by global sections is open, and the claim follows. □

The unexpected consequence, at least for us, is that in this case it is possible to deform ψ into a non-equivariant morphism of vector spaces, while keeping the property that the rank is constant.

4.4. **Example:** $\mu = (2^a, 1^b)$ **and** $\nu = (2^a, 1^{b+1})$**.** Here the kernel $\mathcal{K}$ of ϕ, its image $\mathcal{E}$ and its cokernel $\mathcal{C}$ fit into simple exact sequences, namely

$$0 \to S_{2^{a-1},1^b}Q(-2) \to \mathcal{K} \to S_{2^a,1^{b-1}}Q(-1) \to 0,$$
$$0 \to S_{2^{a-1},1^{b+1}}Q(-1) \to \mathcal{E} \to S_{2^a,1^b}Q \to 0,$$
$$0 \to S_{2^{a-1},1^{b+2}}Q \to \mathcal{C} \to S_{2^a,1^{b+1}}Q(1) \to 0.$$

We deduce

$$\begin{array}{ll} H^0(\mathcal{E}) = S_{2^a,1^b}V, & H^0(\mathcal{E}^\vee(1)) = S_{2^a,1^{b+1}}V^\vee, \\ H^0(\mathcal{K}(1)) = S_{[a,b-1]}V, & H^0(\mathcal{K}^\vee) = S_\mu V^\vee, \\ H^0(\mathcal{C}(-1)) = S_\nu V, & H^0(\mathcal{C}^\vee(2)) = S_{[a,b+2]}V^\vee. \end{array}$$

In particular all these homogeneous bundles must be indecomposable, and even stable by [13]. We expect that, as in the previous example, they can be deformed to non-homogeneous bundles.

Consider the special case $a = 1$, which is closest to the classical case $\Lambda^k V \to \Lambda^{k+1}V$. Here one has rank $r = \binom{n}{b+2} + \binom{n}{b}\frac{(n-b)(n-1)}{b+2}$. The first case is $n = 3$, $b = 1$ where one has a four dimensional subspace of $\mathbb{C}^{20}\otimes\mathbb{C}^{15}$ of rank 11. In this case $c_1(\mathcal{E}) = 7$, so $c_1(\mathcal{E}^\vee(1)) = 4$. (When $a = b = 1$, $c_1(\mathcal{E}) = (n^2 + 3n - 4)/2$.)

4.5. $GL(V)$**-equivariant spaces of bounded rank with base not** $\mathbb{P}V$**.** When the base space is not given by the natural representation of $GL(V)$, it is more difficult to construct equivariant spaces of matrices of constant, or even bounded rank. A well-known example is provided by the adjoint action of $\mathfrak{sl}(V)$ on itself; the general element being regular semisimple, its commutator is a Cartan subalgebra and the generic corank of the adjoint action is thus n; adjoint actions of simple Lie algebras are discussed in [3] and proved to be rank critical.

In this section we give additional examples of spaces of bounded rank, and ask if their main common feature may explain their existence.

Example 15. Let $V = \Lambda^2 A$, and let $\mathbf{a} = 2p$ be even. Then $S_{2^p}A$ is a submodule of $S^2(\Lambda^p A)$ while $S_{2^p 11}A$ is a submodule of $\Lambda^2(\Lambda^{p+1}A)$, both with multiplicity one. Consider

$$\phi : \Lambda^2 A \hookrightarrow \mathrm{Hom}(S_{2^p}A, S_{2^p 11}A)$$
$$v \wedge w \mapsto \Big(X^2 \mapsto (X \wedge v)\otimes(X \wedge w) - (X \wedge w)\otimes(X \wedge v)\Big),$$

and extending linearly.

If $z = v_1 \wedge w_1 + \cdots + v_p \wedge w_p \in \Lambda^2 A$ is a general element, then $X^2 \in \ker \phi_z$ when $X = v_1 \wedge v_2 \wedge \cdots \wedge v_p$. Such an X is a Plücker representative of a p-dimensional subspace V of A which is isotropic with respect to the skew-symmetric two-form ω that is dual to z. Conversely, such a V being given, one can choose another isotropic subspace W of A which is transverse to V, in which case ω restricts to a perfect duality between V and W. If $(v_1, \ldots, v_p)$ and $(w_1, \ldots, w_p)$ are dual basis of V and W, then we can write

$z = v_1 \wedge w_1 + \cdots + v_p \wedge w_p$, and $v_1 \wedge v_2 \wedge \cdots \wedge v_p$ is a Plücker representative for V.

We conclude that the linear span $K \cong V_{2\omega_p}^{Sp(2p)}$ of the second Veronese image of the Lagrangian Grassmannian $\langle v_2(LG_z(p, 2p))\rangle$ is contained in the kernel of ϕ_z. This is unexpected since

$$\dim S_{2^p} A = \frac{1}{p+1}\binom{\mathbf{a}}{p}\binom{\mathbf{a}+1}{p} < \dim S_{2^p 11} A = \frac{3}{\mathbf{a}-p+1}\binom{\mathbf{a}}{p}\binom{\mathbf{a}+1}{p+3}$$

for $p \geq 5$. The dimension of K can be computed from the Weyl dimension formula for the symplectic group $Sp(2p)$, which gives

$$\dim K = 24 \frac{(2p+1)!(2p+3)!}{p!(p+1)!(p+3)!(p+4)!}.$$

For $p = 5$ this gives matrix of size 19404×20790 of rank bounded by 14685.

Example 16. Let $V = \Lambda^2 A$, and let $\mathbf{a} = 2p + 1$ be odd. Consider an equivariant embedding $\Lambda^2 A \subset \mathrm{Hom}(S_\lambda A, S_\mu A)$. That such an embedding exists is equivalent to the condition that μ can be obtained by adding two boxes to λ, not on the same row. The embedding is unique up to scale.

A general element $z = v_1 \wedge w_1 + \cdots + v_p \wedge w_p \in \Lambda^2 A$ defines a unique hyperplane H_z of A such that z belongs to $\wedge^2 H_z$. The equivariance of ϕ implies that there is a commutative diagram

$$\begin{array}{ccc} S_\lambda A & \xrightarrow{\phi_z} & S_\mu A \\ \updownarrow & & \uparrow \\ S_\lambda H_z & \xrightarrow{\phi_z^{H_z}} & S_\mu H_z \end{array}$$

Set $s_\lambda(n) = \dim(S_\lambda \mathbb{C}^n)$. If

$$s_\lambda(2p) - s_\mu(2p) > s_\lambda(2p+1) - s_\mu(2p+1),$$

then ϕ has bounded rank because the factorization above shows that the kernel of ϕ_z contains the kernel of $\phi_z^{H_z}$.

Take for example $p = 2$, $\lambda = (32)$ and $\mu = (3211)$. Then $s_\lambda(5) = s_\mu(5) = 175$. Here $s_\lambda(4) = 60 > s_\mu(4) = 20$, so ϕ has at least a 40-dimensional kernel.

Example 17. Let A be seven dimensional and let $V = \wedge^3 A$. The action of $SL(A)$ on V yields a morphism $\mathfrak{sl}(A) \otimes V \longrightarrow V$ that we can see as a map $V \longrightarrow Hom(\mathfrak{sl}(A), V)$. Since $\dim \mathfrak{sl}(A) = 48 > \dim V = 35$, one could expect the generic point in the image of this morphism to be surjective but we claim this is not the case. Indeed, it is enough to check it at a general point $\omega \in V$. As already known to E. Cartan, the stabilizer of ω in $SL(A)$ is then a copy of G_2, and the associated morphism ϕ_ω in $Hom(\mathfrak{sl}(A), V)$ is $\mathfrak{g}_2$-equivariant. Compare the decompositions of $\mathfrak{sl}(A)$ and V into $\mathfrak{g}_2$-modules:

$$\mathfrak{sl}(A) = \mathfrak{g}_2 \oplus V_{2\omega_1} \oplus A, \qquad V = V_{2\omega_1} \oplus A \oplus \mathbb{C},$$

where $V_{2\omega_1}$ is a hyperplane in S^2A. The important point in these decomposition is that the trivial factor $\mathbb{C}$ appears on the right hand side but not on the left hand side. Therefore by Schur's Lemma it cannot be contained in the image of ϕ_ω. We thus get a 35-dimensional space $\phi(V^\vee)$ of 48×35-matrices of generic rank 34.

A similar argument applies when A is eight dimensional and $V = \wedge^3 A$. Since dim $\mathfrak{sl}(A) = 63 >$ dim $V = 56$, one could again expect the generic point in the image of the morphism $V \longrightarrow Hom(\mathfrak{sl}(A), V)$ to be surjective, and again this is not the case. To see this, it suffices to observe that the stabilizer in $SL(A)$ of a generic element ω in V is a copy of $SL(3)$. Then as before V contains a trivial $SL(3)$-module (generated by ω) while $\mathfrak{sl}(A)$ does not, and therefore ϕ_ω cannot be surjective.

Note finally that when $\dim A > 8$, the dimension of $\mathfrak{sl}(A)$ gets smaller than the dimension of $V = \wedge^3 A$, so the argument no longer applies.

These are all the examples we are aware of. Hence the following:

Question. *Suppose that $V = S_\pi A$ admits a $GL(A)$-embedding as a space of bounded rank. Is the stabilizer in $GL(A)$ of a general element of V positive dimensional?*

5. Big matrices from small ones

5.1. **A motivating example.** Consider a matrix M of size $a \times (a+b)$ of linear forms, with constant rank $a < a+b$. This will be the case of a general matrix whose entries are general combinations of $c+1$ linear forms, with $c \leq b$. In this case, there is an Eagon-Northcott complex [4] of sheaves on $\mathbb{P}^c$ which is everywhere exact, and can be written as

$$0 \to S^b A \otimes \mathcal{O}(-b) \to S^{b-1} A \otimes B \otimes \mathcal{O}(-b+1) \to \cdots$$
$$\cdots \to A \otimes \wedge^{b-1} B \otimes \mathcal{O}(-1) \to \wedge^b B \otimes \mathcal{O} \to 0.$$

Here A has dimension a, B has dimension $a+b$ and M is interpreted as a element in $Hom(A, B)$ whose entries are linear forms. The morphisms in the complex, up to twist, are

$$e_k(M) : S^{k+1} A \otimes \wedge^{b-k-1} B \otimes \mathcal{O}(-1) \to S^k A \otimes \wedge^{b-k} B \otimes \mathcal{O}$$

and can easily be defined by contraction with M. Since the Eagon-Norton complex is in this case an exact complex of vector bundles, the rank of $e_k(M)$ must be constant. In other words we obtain for each M and each k, a matrix $e_k(M)$ which is of constant rank.

This observation can be widely generalized.

5.2. **How to build big matrices.** Consider partitions λ, λ' and μ, μ' such that λ' is obtained by suppressing one box of λ, and μ' is obtained by adding one box to μ. There is then a unique up to scale equivariant morphism

$$\Theta : Hom(A, B) \otimes S_\lambda A \otimes S_\mu B \longrightarrow S_{\lambda'} A \otimes S_{\mu'} B.$$

So any morphism $X \in Hom(A,B)$ induces a morphism $\Theta_X : S_\lambda A \otimes S_\mu B \longrightarrow S_{\lambda'} A \otimes S_{\mu'} B$.

Proposition 18. *The rank of Θ_X only depends on the rank of X. As a consequence, if $M \subset Hom(A,B)$ is a space of morphisms of constant rank, then Θ_M also is.*

Proof. Since the construction is equivariant, the rank of Θ_X is constant on the $GL(A) \times GL(B)$-orbits, which are indexed by the matrix rank. □

The same argument yields a more general version with $GL(A) \times GL(B)$-modules that are not necessarily irreducible. This is just an expansion of Proposition 1, but the greater generality allows one to construct infinitely many spaces of matrices of constant rank just from one.

5.3. **A simple example of Eagon-Northcott type.** Consider one of the simplest morphisms appearing in an Eagon-Northcott complex, namely:

$$\Theta_X : S^2 A \otimes B \longrightarrow A \otimes \wedge^2 B$$

defined by $X \in Hom(A,B)$.

Proposition 19. *Suppose that X has rank r, then*

$$\operatorname{rank} \Theta_X = abr - a\binom{r+1}{2} - b\binom{r}{2} + 2\binom{r+1}{3}.$$

Proof. Choose splittings $A = \ker(X) \oplus A'$ and $B = B' \oplus B''$, where $B' = \operatorname{Im}(X) \cong A'$. Then Θ_X decomposes as a sum of the following maps, whose various images we examine separately:

$$\begin{aligned}
S^2 \ker X \otimes B &\to 0 \\
(\ker X) \cdot A' \otimes B' &\to \ker X \otimes \Lambda^2 B' \\
(\ker X) \cdot A' \otimes B'' &\to \ker X \otimes B' \wedge B'' \\
S^2 A' \otimes B'' &\to A' \otimes B' \wedge B'' \\
S^2 A' \otimes B' &\to A' \otimes \Lambda^2 B'
\end{aligned}$$

This yields a block decomposition of Θ_X, whose rank is therefore the sum of the ranks of the maps above. Those ranks are easy to compute: the first map is zero, the second one is surjective, the third one is an isomorphism, the fourth one is injective; the last one can be identified to the morphism $S^2 A' \otimes A' \to A' \otimes \Lambda^2 A'$, whose image is $S_{21} A'$. Thus the rank of Θ_X is

$$(a-r)\binom{r}{2} + (a-r)r(b-r) + \binom{r+1}{2}(b-r) + \frac{r^3 - r}{3},$$

which after a slight rewriting yields our claim. □

6. Symplectic group

6.1. **General set-up.** Branching rules for restrictions of irreducible representations from $Sp(2n)$ to $Sp(2p) \times Sp(2q)$, where $n = p + q$, are given in [8]. Irreducible representations of $Sp(2n)$ are indexed by partitions λ of length at most n; we denote them by $S_{\langle\lambda\rangle}W$ where W is the natural representation of dimension $2n$. The representation $S_{\langle\lambda\rangle}W$ is the submodule of $S_\lambda W$ consisting of the common kernel of all the possible contractions by the symplectic form. Specializing formula [8, (4.15)] to $p = n - 1$ and $q = 1$ we obtain:

$$S_{\langle\lambda\rangle}(\mathbb{C}^{2p} \oplus \mathbb{C}^2) = \bigoplus_{\ell(\zeta)\leq 2} S_{\langle\lambda/\zeta\rangle}\mathbb{C}^{2p} \otimes S_\zeta \mathbb{C}^2, \tag{3}$$

where $S_{\langle\lambda/\zeta\rangle}U = \bigoplus_\eta c^\lambda_{\zeta\eta} S_{\langle\eta\rangle}U$ and the $c^\lambda_{\zeta\eta}$ are the Littlewood-Richardson coefficients.

The stabilizer P in $Sp(W)$ of a line $\ell \in \mathbb{P}(W)$ is a parabolic subgroup that also preserves the hyperplane $\ell^\perp$. Its unipotent radical P_u is the subgroup that acts trivially on the three factors $\ell, \ell^\perp/\ell, W/\ell^\perp$. A Levi factor L is obtained by choosing a decomposition $W = \ell \oplus \ell' \oplus H$, where H is the orthogonal complement (with respect to the symplectic form) to $\ell \oplus \ell'$, so that $\ell^\perp = \ell \oplus H$. In particular (3) yields the decomposition of $S_\lambda W$ as an L-module.

It is then straightforward to find a criterion ensuring that the natural morphism $\phi : W \longrightarrow Hom(S_{\langle\mu\rangle}W, S_{\langle\nu\rangle}W)$ has bounded rank, for two partitions μ, ν such that ν/μ is one box b.

x	x	x	x	x	x	x
x	x	x	x	x	x	
x	x	c				
x	x	b				
x	x					
x						

Proposition 20. *The morphism ϕ_v is never surjective. It is not injective as long as the box b does not belong to the first row.*

Proof. Let $\ell = \mathbb{C}v$, and fix an orthogonal decomposition $W = H \oplus (\ell \oplus \ell')$ defining a Levi factor L of the parabolic subgroup $P \subset Sp(W)$ that fixes ℓ. Let $K = \ell \oplus \ell'$. As L-modules,

$$S_{\langle\mu\rangle}(H \oplus K) = \bigoplus_{\ell(\zeta)\leq 2} S_{\langle\mu/\zeta\rangle}H \otimes S_\zeta K,$$

$$S_{\langle\nu\rangle}(H \oplus K) = \bigoplus_{\ell(\delta)\leq 2} S_{\langle\nu/\delta\rangle}H \otimes S_\delta K.$$

The multiplication by v acts on the K-factors and leaves the H-factors untouched. All the partitions appearing in μ/ζ are contained in μ, in particular

ν is not one of them, while it appears in ν/δ for the empty partition δ. This proves that ϕ_v cannot be surjective.

Conversely, to prove that ϕ_v is not injective, we make the following observation. Let $m > 0$ be the number of boxes of μ to the east of c, including it. Let θ be the partition obtained by suppressing these boxes from the diagram of μ. Since these boxes all belong to the same row, θ appears in some μ/ζ if and only if $\zeta = (m)$. Similarly, the Littlewood-Richardson rule implies that θ appears in some ν/δ if and only if $\delta = (m, 1)$. So ϕ_v has to send $S_{\langle\theta\rangle}H \otimes S^m K$ to $S_{\langle\theta\rangle}H \otimes S_{m,1}K$. But $S_{\langle\theta\rangle}H \otimes v^m$ is in the kernel of this map. $\square$

6.2. **Example:** $\mathbb{C}^6 \subset Hom(\wedge^{\langle 2\rangle}\mathbb{C}^6, \wedge^{\langle 3\rangle}\mathbb{C}^6)$**.** Consider $G = Sp_6$ and let W denote the natural six-dimensional representation. Then the image of the map $W \to Hom(\wedge^{\langle 2\rangle}W, \wedge^{\langle 3\rangle}W)$ is a six dimensional space of matrices of size 14×14, of constant rank 9.

To be more explicit, a line ℓ in W determines a hyperplane $H = \ell^\perp$, and in $\wedge^2 W$ one obtains two flags as follows

$$\begin{array}{ccccccc} & & & \wedge^2 H & & & \\ & & \nearrow & & \searrow & & \\ 0 \longrightarrow & \ell\wedge H & & & & \ell\wedge W + \wedge^2 H & \longrightarrow \wedge^2 W \\ & & \searrow & & \nearrow & & \\ & & & \ell\wedge W & & & \end{array}$$

There are two different ways to get a subspace in $\wedge^{\langle 2\rangle}W$ from a subspace U of $\wedge^2 W$, either by taking the intersection with $\wedge^{\langle 2\rangle}W$, or by considering the projection $\overline{U}$ according to the decomposition $\wedge^2 W = \wedge^{\langle 2\rangle}W \oplus \mathbb{C}\omega$, where ω denotes the (dual) symplectic form. We obtain a diagram of the same shape:

$$\begin{array}{ccccccc} & & & \wedge^{\langle 2\rangle} H & & & \\ & & \nearrow & & \searrow & & \\ 0 \longrightarrow & \overline{\ell\wedge H} & & & & \overline{\wedge^2 H} & \longrightarrow \wedge^{\langle 2\rangle} W \\ & & \searrow & & \nearrow & & \\ & & & \overline{\ell\wedge W} & & & \end{array}$$

It is straightforward to compute the kernel and image of ϕ in this case, which are respectively the five-dimensional space $\overline{\ell \wedge W}$ and the nine-dimensional space $\ell \wedge \wedge^{\langle 2\rangle}W$.

In terms of vector bundles, we deduce the following result. The quotient $\mathcal{N} = \mathcal{L}^\perp/\mathcal{L} \simeq \mathcal{N}^\vee$ is the *null-correlation bundle* (see, e.g., [12, I.4.2]). This is a rank four bundle with an invariant symplectic form, induced by the one on W. It satisfies $c(\mathcal{N}) = 1 + h^2 + h^4 + \cdots + h^{n-1}$.

Lemma 21. *The image $\mathcal{E}$ of ϕ is a homogeneous bundle of rank nine, fitting into an extension*

$$0 \to \wedge^{\langle 2\rangle}\mathcal{N} \to \mathcal{E} \to \mathcal{N}(1) \to 0.$$

By the computations in the proof that follows, $Ext^1(\mathcal{N}(1), \wedge^{\langle 2\rangle}\mathcal{N}) = \mathbb{C}$, so this extension is unique. In fact there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O} & = & \mathcal{O} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \wedge^2\mathcal{N} & \longrightarrow & \wedge^2\mathcal{Q} & \longrightarrow & \mathcal{N}(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \wedge^{\langle 2\rangle}\mathcal{N} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{N}(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and the middle vertical exact sequence defines $\mathcal{E}$ as the quotient of $\wedge^2\mathcal{Q}$ by its global section defined by the (dual) symplectic form $\omega \in \wedge^2 W$.

In contrast to the previous examples, we have:

Proposition 22. *$\mathcal{E}$ is stable and not rigid.*

Proof. Tensoring the middle vertical sequence with $\mathcal{E}^\vee$ gives

$$0 \to \mathcal{E}^\vee \to \mathcal{E}^\vee \otimes \wedge^2\mathcal{Q} \to End(\mathcal{E}) \to 0.$$

By Bott's theorem $\wedge^2\mathcal{Q}^\vee$ is acyclic, the only nonzero cohomology group of $\mathcal{E}^\vee$ is $H^1(\mathcal{E}^\vee) = \mathbb{C}$. To compute the cohomology of the second term we use the dual of the middle vertical sequence tensored with $\wedge^2 Q$ to get

$$0 \to \mathcal{E}^\vee \otimes \wedge^2\mathcal{Q} \to \wedge^2\mathcal{Q}^\vee \otimes \wedge^2\mathcal{Q} \to \wedge^2\mathcal{Q} \to 0.$$

The tensor product $\wedge^2\mathcal{Q} \otimes \wedge^2 \mathcal{Q}^\vee$ has three components, one of which is trivial and the other two are acyclic. We conclude $\mathcal{E}^\vee \otimes \wedge^2\mathcal{Q}$ has only one nonzero cohomology group, namely $H^1(\mathcal{E}^\vee \otimes \wedge^2\mathcal{Q}) = \wedge^{\langle 2\rangle} W$. We deduce

$$H^0(End(\mathcal{E})) = \mathbb{C}, \quad H^1(End(\mathcal{E})) = \wedge^{\langle 2\rangle} W, \quad H^q(End(\mathcal{E})) = 0 \text{ for } q > 1.$$

So $\mathcal{E}$ is simple but not infinitesimally rigid. The stability follows from [13]. □

In this case the non-rigidity is explained by the action of $SL(W)$, since our bundle $\mathcal{E}$ is only $Sp(W)$-homogeneous, but not $SL(W)$-homogeneous. By varying the symplectic form one obtains a family of bundles parametrized by $SL(W)/Sp(W)$, whose tangent space at the identity is precisely $\wedge^{\langle 2\rangle} W$.

We exhibit our 14×14 matrix by chosing an adapted basis $e_1, \dots, e_6$ of W, in which the (dual) symplectic form is $\omega = e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6$. We get the following constant rank matrix ψ_v, depending on $v = (x_1, \dots, x_6) \in W$:

$$\begin{pmatrix}
x_3 & 0 & -x_2 & 0 & 0 & x_1 & 0 & 0 & 0 & -x_6 & x_5 & 0 & 0 & 0 \\
x_4 & 0 & 0 & -x_2 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 & -x_6 & -x_5 \\
x_5 & -x_5 & 0 & 0 & -x_2 & 0 & 0 & 0 & x_1 & 0 & x_4 & 0 & -x_3 & 0 \\
x_6 & -x_6 & 0 & 0 & 0 & -x_2 & 0 & 0 & 0 & x_1 & 0 & x_4 & 0 & -x_3 \\
0 & x_1 & x_4 & -x_3 & -x_6 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_2 & 0 & 0 & 0 & 0 & x_4 & -x_3 & -x_6 & x_5 & 0 & 0 & 0 & 0 \\
0 & 0 & x_5 & 0 & -x_3 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 \\
0 & 0 & x_6 & 0 & 0 & -x_3 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 \\
0 & 0 & 0 & x_5 & -x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\
0 & 0 & 0 & x_6 & 0 & -x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \\
0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & -x_3 & 0 & x_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_6 & 0 & 0 & -x_3 & 0 & x_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & -x_4 & 0 & 0 & 0 & x_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_6 & 0 & -x_4 & 0 & 0 & 0 & x_2
\end{pmatrix}$$

This has a curious consequence for the Koszul map

$$\bar{\psi} : W \longrightarrow Hom(\wedge^2 W, \wedge^3 W),$$

which is of constant rank 10. Once we have chosen a symplectic form on W, we get direct sum decompositions $\wedge^2 W = \wedge^{\langle 2 \rangle} W \oplus \mathbb{C}$, $\wedge^3 W = \wedge^{\langle 3 \rangle} W \oplus W$ which induces a decomposition of $\bar{\psi}$ into blocks as follows:

$$\bar{\psi}_v = \left(\begin{array}{c|c}
 & 0 \\
 & 0 \\
 & 0 \\
\psi_v & 0 \\
 & 0 \\
 & 0 \\
 & 0 \\
 & 0 \\
\hline
 & x_1 \\
 & x_2 \\
\theta_v & x_3 \\
 & x_4 \\
 & x_5 \\
 & x_6
\end{array} \right)$$

where θ^t is the matrix of the morphism $W \longrightarrow Hom(W, \wedge^{\langle 2 \rangle} W)$. The string of zeroes is explained by the fact that there is no nonzero equivariant map $W \longrightarrow Hom(\mathbb{C}, \wedge^{\langle 3 \rangle} W)$. The rank of $\bar{\psi}_v$ must be at least equal to the rank of ψ_v plus one, and in fact there is always equality. In particular the space

$$\begin{pmatrix} \psi \\ \theta \end{pmatrix} : W \to \mathrm{Hom}(\wedge^{\langle 2 \rangle} W, \wedge^3 W)$$

is of constant rank 9 so ψ is expandable.

Question 23. *Are all $SP(W)$ spaces with base W expandable to $SL(W)$-spaces? If not, how to distinguish which are?*

7. Orthogonal groups: tensorial representations

In the case where $G = SO(W)$ is a special orthogonal group Proposition 1 will in general fail to hold, as one expects the morphism ϕ to degenerate along the invariant quadric $Q \subset \mathbb{P}(W)$. This is not always the case, obvious counter-examples arise from $SO(W)$-modules that are restrictions of $SL(W)$-modules. We discuss two non-obvious examples where the base space is $\mathbb{P}(W)$.

Irreducible representations of $SO(m)$ with support on the first $\lfloor m/2 \rfloor - 2$ fundamental weights are indexed by partitions λ of length at most $m/2 - 2$ when m is even and $(m-1)/2 - 1$ when m is odd; we denote them by $S_{[\lambda]}W$ where W is the natural representation of dimension m. As in the symplectic case, the representation $S_{[\lambda]}W$ can be defined in $S_\lambda W$ as the common kernel of all the possible contractions by the invariant quadratic form. Similarly, if the partition ν is obtained by adding a box b to a partition μ, there is a unique (up to scale) equivariant morphism $\phi : W \longrightarrow Hom(S_{[\mu]}W, S_{[\nu]}W)$.

Once we fix a non-isotropic vector $v \in W$, we get an orthogonal decomposition $W = \ell \oplus \ell^\perp$, where $\ell = \mathbb{C}v$. Moreover $\ell^\perp$ inherits an invariant quadratic form giving rise to a copy of $SO(m-1)$ inside $SO(m)$, that acts trivially on ℓ. In particular the morphism ϕ_v is $SO(m-1)$-invariant. Formula [8, (4.12)] gives the following decomposition:

$$S_{[\mu]}(\ell \oplus \ell^\perp) = \bigoplus_{k \geq 0} S_{[\mu/k]}(\ell^\perp) \otimes \ell^k. \tag{4}$$

Formally this is the same decomposition as (1) that we used for $SL(W)$, and we can just mimic the proof of Proposition 1 to obtain:

Proposition 24. *The morphism ϕ_v is never surjective. It is not injective as long as the box b does not belong to the first row.*

7.1. **Case $\mu = (2)$ and $\nu = (2,1)$.** Here $S_{[2]}W$ is the hyperplane of S_2W generated by the squares of the isotropic vectors; its invariant complement is generated by the dual $\hat{q}$ of the quadratic form. The natural composition $W \otimes \hat{q} \to W \otimes S_2W \to S_{2,1}W$ is an embedding, and the cokernel is a copy of $S_{[2,1]}W$. We may therefore describe the map $\phi : W \to Hom(S_{[2]}W, S_{[2,1]}W)$ in terms of $\phi : W \to Hom(S_2W, S_{2,1}W)$ by sending $v \in W$ to the composition

$$\phi_v : S_{[2]}W \hookrightarrow S_2W \xrightarrow{\psi_v} S_{2,1}W \longrightarrow S_{[2,1]}W.$$

In order to compute the kernel of ϕ_v, we note that any $\kappa \in S_2W$ can be written as $\kappa = \sum_i \kappa_i e_i^2$ for some q-orthonormal basis $e_1, \dots, e_m$ of W; it belongs to $S_{[2]}W$ when $\sum_i \kappa_i = 0$. Since $\hat{q} = \sum_i e_i^2$, the kernel of the projection $S_{2,1}W \longrightarrow S_{[2,1]}W$ is the space of tensors of the form $\sum_i w \wedge e_i \otimes e_i$, for $w \in W$. So $\phi_v(\kappa) = 0$ if and only if there exists $w \in W$ such that

$$\sum_i \kappa_i v \wedge e_i \otimes e_i = \sum_i w \wedge e_i \otimes e_i,$$

which means that for each i, we have $w = \kappa_i v + \mu_i e_i$ for some scalar μ_i. If there exists two indices $i \neq j$ such that $\kappa_i \neq \kappa_j$, we deduce that v and w belong to $\langle e_i, e_j \rangle$. Then for $k \neq i, j$ we must have $\mu_k = 0$ and $w = \kappa_k v$, hence $(\kappa_k - \kappa_i)v = \mu_i e_i$ and $(\kappa_k - \kappa_j)v = \mu_j e_j$. So necessarily, up to changing i and j, $\mu_j = 0$. Then we conclude that v and e_i must be colinear and that κ_k is independent of $k \neq i$, which implies that κ must be a linear combination of $\hat{q}$ and v^2. We conclude:

Proposition 25. *The kernel of ϕ_v is the line generated by $v^2 - q(v)\hat{q}$, which is nonzero for all $v \in W$. In particular $\phi : W \to Hom(S_{[2]}W, S_{[2,1]}W)$ yields a $\frac{m^2+m-2}{2} \times \frac{m^3-4m}{3}$ matrix of linear forms of constant rank $(m^2+m-4)/2$.*

When $m = 3$ we get a 5×5 space of constant rank 4. Since $\mathfrak{so}_3 = \mathfrak{sl}_2$ we may see the space in terms of SL_2, as $S^2\mathbb{C}^2 \subset \mathrm{Hom}(S^4\mathbb{C}^2, S^4\mathbb{C}^2)$. The inclusion on decomposable elements is $\ell^2 \mapsto (m^4 \mapsto (\ell \wedge m)\ell m^3)$. On a rank one element ℓ^2 the kernel is spanned by ℓ^4. Let x, y be a unimodular basis of $\mathbb{C}^2$, then at $x^2 + y^2$, the kernel is $x^4 + y^4 - 2x^2y^2 = (x^2 - y^2)^2$. I.e., over all points $v \in S^2\mathbb{C}^2$, the kernel is $(v^\perp)^2$. The associated kernel bundle is thus $\mathcal{O}_{\mathbb{P}^2}(-2)$, hence $c_1(\mathcal{E}) = 2$.

This space is a specialization of $\Lambda^2\mathbb{C}^5 \subset \mathbb{C}^5 \otimes \mathbb{C}^5$ because it corresponds to the representation $\rho : \mathfrak{so}_3 \to \mathrm{End}(\mathbb{C}^5)$ with image in $\mathfrak{so}_5 \cong \wedge^2\mathbb{C}^5$. However, unlike $\Lambda^2\mathbb{C}^5$, it is of constant rank. In fact the $m = 3$ case generalizes to all odd dimensional representations of $\mathfrak{so}_3$, they all map to spaces of corank one, and are just specializations of the skew-symmetric matrices in odd dimensions. The interesting point here is that one obtains constant rank matrices.

When $m = 4$ we get a 9×16 space of constant rank 8. Since $\mathfrak{so}_4 = \mathfrak{sl}_2 \times_2$ we may see the space in terms of two spaces A, B of dimension two, with $W = A \otimes B$. Then

$$S_{[2]}W = S_2A \otimes S_2B, \quad S_{[31]}W = S_3A \otimes S_{21}B \oplus S_{21}A \otimes S_3B,$$

and the resulting morphisms are of the type discussed in Proposition 18.

Remark 26. A related example was studied in [7], where it was observed that the unique (up to scale) equivariant morphism

$$\psi : S^3\mathbb{C}^2 \longrightarrow Hom(S^{3d}\mathbb{C}^2, S^{3d+1}\mathbb{C}^2),$$

which on powers of linear forms is $m^3 \mapsto (\ell^{3d} \mapsto (m \wedge \ell)m^2\ell^{3d-1})$, has constant corank one (and this is no longer true if one replaces $3d$ by some integer not divisible by three). This leads to an interesting SL_2-equivariant instanton on $\mathbb{P}^3$.

7.2. **Case $\mu = (3,1,1)$ and $\nu = (3,2,1)$.** For v non-isotropic, (5) gives

$$Ker(\phi_v) = \wedge^3\ell^\perp \otimes \ell^2 \oplus \wedge^2\ell^\perp \otimes \ell^3.$$

On the other hand, when v is isotropic, a calculation similar to the previous case gives

$$Ker(\phi_v) \simeq \wedge^3 H \oplus (\wedge^2 H)^{\oplus 2} \oplus H \simeq \wedge^3\ell^\perp \oplus \wedge^2\ell^\perp.$$

Thus the two kernels have the same dimension, and we conclude:

Proposition 27. *The map $\phi : W \to Hom(S_{[3,1,1]}W, S_{[3,2,1]}W)$ yields a matrix of linear forms of constant corank $\binom{m-1}{3} + \binom{m-1}{2}$.*

7.3. **Problem: determine which $SO(W)$-inclusions of the standard representation have constant rank.** To solve this problem, it is enough to compare the two possible values of the rank of ϕ_v, obtained for v isotropic, or non-isotropic. In the latter case, the analysis above allows one to extend Proposition 5 to the orthogonal case, and yields the analogue formula for the kernel of ϕ_v:

$$(5) \qquad Ker(\phi_v) = \bigoplus_{k\geq 0} \bigoplus_{\mu \xrightarrow{k} \alpha, c\notin\alpha} S_{[\alpha]}\ell^\perp \otimes \ell^k.$$

Now consider the case where v is isotropic. Then its stabilizer is a parabolic subgroup P of $SO(W)$ and as in the symplectic case, choosing a Levi subgroup L of P is equivalent to fixing an orthogonal decomposition $W = H \oplus (\ell \oplus \ell')$, so that $\ell^\perp = H \oplus \ell$. A natural approach would be to try to use a branching formula from $SO(W)$ to $SO(H) \times SO(\ell \oplus \ell')$ to describe the kernel of ϕ_v, and to compare the result with (5).

8. Spin representations

Let Δ_+ and Δ_- denote the two half-spin representations of $Spin(2n)$. As before, W denotes the natural representation, of dimension $2n$. There is a natural map

$$\phi : W \longrightarrow Hom(\Delta_+, \Delta_-)$$

and it is well-known that ϕ_v is an isomorphism when v is not isotropic, while the rank ϕ_v is half the dimension of $\Delta_\pm$ when v is isotropic. In fact the kernel and cokernel of ϕ_v give rise to the spinor bundles on the invariant quadric.

Instead of ϕ, consider the following equivariant morphism arising from the same tensor

$$\psi : \Delta_+ \longrightarrow Hom(W, \Delta_-).$$

In order to understand this morphism more concretely, recall that the spin representations can be constructed by choosing a decomposition $W = E \oplus F$ into a direct sum of maximal isotropic spaces. In particular F is naturally identified with the dual of E. Then we can respectively define Δ_+ and Δ_- as the even and odd degree parts in the exterior algebra of E. The map ϕ, and equivalently ψ, is then obtained by letting E act by wedge product, and F by contraction.

Proposition 28. *Suppose $n = 5$. Then for general $\delta \in \Delta_+$, the kernel of ψ_δ is one dimensional. More precisely, if we decompose δ as $(\delta_0, \delta_2, \delta_4)$, where $\delta_k \in \wedge^k E$, then*

$$Ker(\psi_\delta) = \mathbb{C}(\delta_2 \lrcorner\, \delta_4^* \oplus (\delta_0\delta_4 - \frac{1}{2}\delta_2 \wedge \delta_2)^{\#}).$$

Here we identified $\wedge^4 E$ with $E^\vee \otimes \det E$, and δ^* is the image of δ under this identification. Similarly, using the quadratic form, $\delta_0\delta_4 - \frac{1}{2}\delta_2 \wedge \delta_2$ can be considered as an element of $E^\vee \otimes \det E \simeq F \otimes \det E$ and we let $\delta^{\#}$ denote the image of δ under this identification. We thus get a line in $(E \oplus F) \otimes \det E$, which is the same as a line in W.

Proof. A vector $v = e + f$ is in the kernel of ψ_δ when the following equations are satisfied:

$$\delta_0 e + f \lrcorner\, \delta_2 = 0, \quad e \wedge \delta_2 + f \lrcorner\, \delta_4 = 0, \quad e \wedge \delta_4 = 0.$$

These equations respectively take values in E, $\wedge^3 E$ and $\wedge^5 E \simeq \mathbb{C}$. For $\delta_0 \neq 0$ the first equation determines e as a function of f. Plugging this relation into the second equation, and using the identity $f.(\delta_2 \wedge \delta_2) = 2(f.\delta_2) \wedge \delta_2$, we get the relation

$$f \lrcorner\, (\delta_0\delta_4 - \frac{1}{2}\delta_2 \wedge \delta_2) = 0.$$

When $\delta_0\delta_4 - \frac{1}{2}\delta_2 \wedge \delta_2$ is nonzero, such an equation determines f up to a unique scalar. Indeed

$$\delta_0\delta_4 - \frac{1}{2}\delta_2 \wedge \delta_2 \in \wedge^4 E \cong E^\vee \simeq F,$$

so it can be considered as an element δ_F of F, and f must be a multiple of δ_F. We claim that the first two equations imply the last equation $(f \lrcorner \delta_2) \wedge \delta_4 = 0$. Indeed, since $\delta_2 \wedge \delta_4 = 0$ for degree reasons, it is equivalent to $(f \lrcorner \delta_4) \wedge \delta_2 = 0$. But $f \lrcorner \delta_4$ is a multiple of $(f \lrcorner \delta_2) \wedge \delta_2$ by the first two equations, so our equation reduces to $(f \lrcorner\, \delta_2) \wedge \delta_2^{\wedge 2} = 0$, or equivalently $f \lrcorner\, \delta_2^{\wedge 3} = 0$, which is trivially verified since $\delta_2^{\wedge 3}$ belongs to $\wedge^6 E = 0$. □

Proposition 29. *$\psi(\Delta_+)$ is rank-critical.*

Proof. We proceed as for the proof of Proposition 3, applying the results of [3] and showing that for $L = \psi(\Delta_+)$, the space of rank neutral directions $RND(L)$ coincides with L. This is particularly easy in this case because of the decomposition

$$Hom(W, \Delta_-) = \Delta_+ \oplus W_{\omega_1+\omega_4},$$

where $W_{\omega_1+\omega_4}$ is the irreducible $SO(W)$-module of highest weight $\omega_1 + \omega_4$ using fundamental weight notation. Were $RND(L)$ strictly bigger than L, being a G-module it would have to be the whole $Hom(W, \Delta_-)$, which is absurd. □

To obtain an explicit matrix, choose a basis $e_1, \ldots, e_5$ of E and decompose $\delta = \sum_{|I|\ even} \delta_I e_I$. Then the matrix of ψ_δ has entry $\pm\delta_I$ on the row indexed i and column indexed $I \cup \{1\}$ when $1 \notin I$, on the row indexed i^* and column indexed $I - \{1\}$ when $1 \in I$, and zeroes everywhere else. We let $\theta_m = \pm\delta_{ijk\ell}$, with the $\pm$ the sign of the permutation $mijkl$ of 12345. This yields the following matrix

$$M_\delta = \begin{pmatrix}
\delta_\emptyset & 0 & 0 & 0 & 0 & 0 & -\delta_{12} & -\delta_{13} & -\delta_{14} & -\delta_{15} \\
0 & \delta_\emptyset & 0 & 0 & 0 & \delta_{12} & 0 & -\delta_{23} & -\delta_{24} & -\delta_{25} \\
0 & 0 & \delta_\emptyset & 0 & 0 & \delta_{13} & \delta_{23} & 0 & -\delta_{34} & -\delta_{35} \\
0 & 0 & 0 & \delta_\emptyset & 0 & \delta_{14} & \delta_{24} & \delta_{34} & 0 & -\delta_{45} \\
0 & 0 & 0 & 0 & \delta_\emptyset & \delta_{15} & \delta_{25} & \delta_{35} & \delta_{45} & 0 \\
\delta_{23} & -\delta_{13} & \delta_{12} & 0 & 0 & 0 & 0 & 0 & -\theta_5 & \theta_4 \\
\delta_{24} & -\delta_{14} & 0 & \delta_{12} & 0 & 0 & 0 & \theta_5 & 0 & -\theta_3 \\
\delta_{25} & -\delta_{15} & 0 & 0 & \delta_{12} & 0 & 0 & -\theta_4 & \theta_3 & 0 \\
\delta_{34} & 0 & -\delta_{14} & \delta_{13} & 0 & 0 & -\theta_5 & 0 & 0 & \theta_2 \\
\delta_{35} & 0 & -\delta_{15} & 0 & \delta_{13} & 0 & \theta_4 & 0 & -\theta_2 & 0 \\
\delta_{45} & 0 & 0 & -\delta_{15} & \delta_{14} & 0 & -\theta_3 & \theta_2 & 0 & 0 \\
0 & \delta_{34} & -\delta_{24} & \delta_{23} & 0 & \theta_5 & 0 & 0 & 0 & -\theta_1 \\
0 & \delta_{35} & -\delta_{25} & 0 & \delta_{23} & -\theta_4 & 0 & 0 & \theta_1 & 0 \\
0 & \delta_{45} & 0 & -\delta_{25} & \delta_{24} & \theta_3 & 0 & -\theta_1 & 0 & 0 \\
0 & 0 & \delta_{45} & -\delta_{35} & \delta_{34} & -\theta_2 & \theta_1 & 0 & 0 & 0 \\
\theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

The blocking is $(E, F) \times (\Lambda^1 E, \Lambda^3 E, \Lambda^5 E)$.

The image of this matrix in $\mathbb{C}^{10}$ is the hyperplane orthogonal to the vector $h = \sum(h_i e_i + h_i^\vee e_i^\vee)$ with

$$h_i = \sum_{j>i} \delta_{ij}\theta_j - \sum_{j<i} \delta_{ij}\theta_j, \qquad h_i^\vee = \delta_\emptyset \theta_i + \delta_{jk}\delta_{\ell m} - \delta_{j\ell}\delta_{km} + \delta_{jm}\delta_{k\ell}.$$

Remark. For $n = 5$ there exists a unique equivariant morphism

$$a : Sym^2(\Delta_+) \to W,$$

and the kernel of ψ_δ is generated by $a(\delta)$ when the latter is nonzero. The condition $a(\delta) = 0$ is a collection of ten quadratic equations, which are the generators of the ideal of the spinor variety $\mathbb{S}_{10} \subset \mathbb{P}^{15}$.

From this perspective, Proposition 29 is no surprise if one observes that it is related with the minimal resolution of this spinor variety. This minimal resolution was computed in [9] and it has the following form:

$$\begin{aligned}0 \to \mathcal{O}_{\mathbb{P}^{15}}(-8) \to \mathcal{O}_{\mathbb{P}^{15}}(-6)^{\oplus 10} \to \mathcal{O}_{\mathbb{P}^{15}}(-5)^{\oplus 16} \\ \to \mathcal{O}_{\mathbb{P}^{15}}(-3)^{\oplus 16} \to \mathcal{O}_{\mathbb{P}^{15}}(-2)^{\oplus 10} \to \mathcal{O}_{\mathbb{P}^{15}} \to \mathcal{O}_{\mathbb{S}_{10}} \to 0.\end{aligned}$$

This shows that the 10×16 matrix ψ_v of linear forms can be interpreted as the matrix of linear syzygies between the ten quadrics. (The middle matrix of quadrics is also interesting.)

Question 30. *Do the the larger spinor varieties have property N_2 (meaning that the syzygies between their quadratic equations are only linear)?*

What is known is that these varieties, like all homogeneous varieties, have ideal generated in degree two and, as in the case for Grassmannians, the space of quadratic equations is an irreducible module only in small dimensions.

For $\mathbb{S}_{12} \subset \mathbb{P}^{31}$, the space of quadratic equations is isomorphic with $\mathfrak{so}_{12} \simeq \wedge^2 W$, where $W = W_{\omega_1}$ is the natural representation. The space of linear syzygies between these quadrics is the irreducible module $W_{\omega_1+\omega_5}$. In particular the natural equivariant map $\Delta_+ \longrightarrow Hom(W_{\omega_1+\omega_5}, \wedge^2 W)$ yields a 32-dimensional space of 352×66-matrices of bounded rank.

For $\mathbb{S}_{14} \subset \mathbb{P}^{63}$, the space of quadratic equations is still irreducible, being isomorphic with $W_{\omega_3} = \wedge^3 W$. The space of linear syzygies between these quadrics is reducible, being isomorphic with $U = \Delta_- \oplus W_{\omega_2+\omega_7}$. In particular the natural equivariant map $\Delta_+ \longrightarrow Hom(U, \wedge^3 W)$ yields a 4992×364-matrix of bounded rank of linear forms in 64-variables.

Question 31. *For $n > 5$, is the morphism $\psi : \Delta_+ \longrightarrow Hom(W, \Delta_-)$ of bounded rank?*

In general, the spinor variety $\mathbb{S}_{2n} \subset \mathbb{P}(\Delta_+)$ is cut-out by a space of quadratic equations that contains $\wedge^{n-4} W$, with multiplicity one [11]. Hence there is a unique (up to scale) equivariant map

$$a : Sym^2(\Delta_+) \longrightarrow \wedge^{n-4} W.$$

Proposition 32. *There exists a unique (up to scale) nonzero equivariant morphism*

$$\psi : \Delta_+ \to Hom(W_{\omega_{n-4}}, W_{\omega_{n-5}+\omega_{n-1}}),$$

and this morphism yields a matrix of linear forms of bounded rank.

Indeed, for any $\delta \in \Delta_+$ we have

$$\psi_\delta(a(\delta)) = 0.$$

Proof. Recall that $W_{\omega_i} = \wedge^i W$ for $i \leq n-2$, and the remaining two fundamental representations are $\Delta_+ = W_{\omega_n}$ and $\Delta_- = W_{\omega_{n-1}}$. We have also seen that there exist equivariant morphisms $W \otimes \Delta_\pm \longrightarrow \Delta_\mp$. By duality, we get morphisms $\Delta_\mp \longrightarrow \Delta_\mp \otimes W$ (the half-spin representations are either self-dual, or dual one of the other according to the parity of n, but this does not affect our conclusion). By [11, Proposition 3] there exists a unique component of $\Delta_+ \otimes W_{\omega_{n-4}}$ isomorphic to $W_{\omega_{n-5}+\omega_{n-1}}$, and this yields the morphism ψ. Combining it with a we get an equivariant morphism

$$Sym^3 \Delta_+ \longrightarrow W_{\omega_{n-5}+\omega_{n-1}}.$$

But according to [11, Theorem 2] there is no such morphism! Hence the formula $\psi_\delta(a(\delta)) = 0$ for any $\delta \in \Delta_+$, and consequently ψ_δ has a nontrivial kernel. □

The matrices we obtain depend on 2^{n-1} parameters, and their size $a_n \times b_n$ is also huge. The Weyl dimension formula may be used to show:

$$a_n = \binom{2n}{n-4} \simeq \alpha \frac{2^{2n}}{n^{3/2}}, \qquad b_n \simeq \beta \frac{2^{6n}}{n^{31/2}}$$

for some positive constants α, β.

Remark 33. It would be interesting to decide whether the unexpected kernel of this example is one-dimensional, or bigger.

Question 34. *The method of using the fact that an equivariant morphism with certain constraints must be zero to force another morphism to have non-trivial kernel when it is not expected to seems rather robust, it only relies on the vanishing of certain multiplicities in tensor products or plethysms. Can it be used to exhibit other matrices of bounded rank?*

The connection with syzygies is not surprising, since syzygies were already identified in [5] as a wide source of examples of spaces of matrices of bounded, or even constant rank. We plan to explore this topic further.

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Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA
Email address: jml@math.tamu.edu

Institut de Mathématiques de Toulouse ; UMR 5219, Université de Toulouse & CNRS, F-31062 Toulouse Cedex 9, France
Email address: manivel@math.cnrs.fr