



HAL
open science

Analytic normal forms for planar resonant saddle vector fields

Loïc Teyssier

► **To cite this version:**

| Loïc Teyssier. Analytic normal forms for planar resonant saddle vector fields. 2022. hal-03887570

HAL Id: hal-03887570

<https://hal-cnrs.archives-ouvertes.fr/hal-03887570>

Preprint submitted on 7 Dec 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial | 4.0 International License

ANALYTIC NORMAL FORMS FOR PLANAR RESONANT SADDLE VECTOR FIELDS

LOÏC TEYSSIER

ABSTRACT. We give essentially unique “normal forms” for germs of a holomorphic vector field of the complex plane in the neighborhood of an isolated singularity which is a $p : q$ resonant-saddle. Hence each vector field of that type is conjugate, by a germ of a biholomorphic map at the singularity, to a preferred element of an explicit family of vector fields. These model vector fields are polynomial in the resonant monomial.

This work is a followup of a similar result obtained for parabolic diffeomorphisms which are tangent to the identity, and solves the long standing problem of finding explicit local analytic models for resonant saddle vector fields.

Acknowledgement. The author is partially supported by the bilateral Hubert-Curien Cogito grant 2021-22.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The general question of finding a simpler form, or ultimately “the” simplest form, of a dynamical system through changes preserving its qualitative properties is central in the theory. A simpler form often means a better understanding of the behavior of the system, or of its analytic properties. This article is concerned with finding simple models for holomorphic dynamical systems given by the flow of a $p : q$ resonant-saddle vector field (eigenratio $-p/q$) near the origin of \mathbb{C}^2 . (Precise definitions are given later in the introduction.) We use intensively the appellation *normal form* for vector fields brought into these forms. Although the latter do not satisfy algebraic properties usually required in normal form theory, its usage is nonetheless spreading to refer to preferred forms which are *essentially unique* (say, up to the action of a finite-dimensional space).

It is possible to attach to a vector field $Z := A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y}$ two dynamical systems: the one induced by the flow, and the underlying foliation. In the former setting the objects of study are the trajectories of Z and their natural parameterization by the time, *i.e.* maximally-continued multivalued solutions of the autonomous differential system with complex time

$$(1.1) \quad \begin{cases} \dot{x}(t) &= A(x(t), y(t)) \\ \dot{y}(t) &= B(x(t), y(t)) \end{cases},$$

while in the latter only their images are of interest: the leaves of the foliation \mathcal{F}_Z are the integral curves of Z regardless of how they are parameterized. Save for

Date: November 2022.

2000 Mathematics Subject Classification. 34M35,37F75,34M50,34M30.

vertical leaves, they correspond to the graphs of solutions of the nonautonomous ordinary differential equation (since $\frac{dy}{dx} = \frac{y}{x}$)

$$A(x, y(x))y'(x) = B(x, y(x)).$$

Therefore two vector fields induce the same foliation when they differ by the multiplication with a nonvanishing function (a holomorphic unit).

1.1. A brief survey of the normal form problem for planar resonant singularities. Being given a (germ of a) holomorphic vector field, we seek to simplify its components by use of local analytic changes of coordinates. At first one would try and simplify the vector field using formal power series, and for planar vector field this process leads to polynomial formal normal forms [Bru89, Dul09]. Yet this formal approach does not always preserve the dynamics, as is particularly the case in the presence of resonances where divergence of the formal normalization is the rule.

Analyzing the divergence of these formal transforms provides many an information about the dynamics or the integrability (in the sense of Liouville) of the system. The theory of summability was used successfully by J. MARTINET and J.-P. RAMIS [MR82, MR83] to perform this task for saddle-node and resonant vector fields, ultimately yielding a complete set of functional invariants that classifies the foliation (called here the *orbital modulus*). However, their construction did not yield normal forms except in very exceptional (integrable) cases. Some years later the complete modulus of resonant and saddle-node vector fields (eigenratio 0) was described in [VG96, VM03, Tey04a] by appending to the orbital modulus another functional invariant, called here the *temporal modulus* and accounting for the multiplicative units that give rise to different vector fields while keeping the same foliation. Still no normal form was proposed. Analytic normal forms were announced in [Bru83] but no proof was subsequently published.

Building on an earlier work of P. ELIZAROV [Eli93], a prenormal (nonunique) form is presented in [Tey04c] that allows to decide in some cases whether two vector fields (or foliations) are not conjugate. At about the same time F. LORAY [Lor04] performed a cleverly simple geometric construction that yielded normal forms for codimension-1 saddle-node foliations, but only in the nongeneric case where half the orbital modulus is nontrivial (*convergent saddle-nodes*, admitting two analytic separatrices through the singularity). Loray's normal forms generalized the ones stated by J. ÉCALLE in [Éca85] (see the paper by D. SAUZIN [Sau09] for precise statements and proofs regarding the resurgent approach to saddle-node classification). In Écalle's terminology, convergent saddle-nodes are called *unilateral*, and save for that case no general normal forms were given. Later, Écalle refined his theory and techniques to propose a way of building a preferred representative in the analytic class of a given resonant vector field [Eca05], the *canonical-spherical synthesis*. Although uniqueness is reached, this approach does not provide an explicit family of vector fields that can be written down.

In joint work with R. SCHÄFKE, an altogether different approach was used in [ST15] to recover Loray normal forms, while extending them from foliations to vector fields and generalizing them to higher-codimension saddle-nodes. Based on a holomorphic fixed-point method, it was later reused with C. ROUSSEAU to encompass the parametric case in [RT21], in order to provide normal forms for convergent saddle-node bifurcations, while at the same time Loray's construction was

ported to parametric families. Yet it was not possible to drop the nongeneric assumption regarding the “unilaterality” of the orbital modulus. The reason behind this difficulty is explained later, let us for now simply state that the remedy lies in introducing a parameter, playing the same role as Écalle’s *twist* in the twisted resurgent monomials that serve as building blocks for the canonical-spherical synthesis. The trick was already used in [Tey22] to provide normal forms for general germs of a parabolic line biholomorphism which is tangent to the identity. The latter paper was written with the clear aim of overcoming the problem and porting the technique to general planar resonant vector fields, and I encountered Écalle’s work on twisted monomials during the final stage of its redaction.

The present paper is a blend of holomorphic fixed-point and twist parameter, and it achieves the task of finding a general explicit normal form family for resonant planar saddle vector fields.

1.2. Statement of the main result. Consider a planar holomorphic vector field Z near some isolated stationary point (or **singularity**), which we conveniently locate at $(0, 0)$ so that $Z(0, 0) = 0$. Its Jacobian matrix at that point admits two eigenvalues, λ_1 and λ_2 , at least one of which we assume nonzero (the origin is a nondegenerate singular point of Z), say $\lambda_2 \neq 0$. The **eigenratio** $\lambda := \frac{\lambda_1}{\lambda_2}$ encodes an important part of the dynamics. It is well known for instance that if $\lambda \notin \mathbb{R}$, then there exists a local biholomorphic mapping $\Psi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$, a property that we write $\Psi \in \text{Diff}(\mathbb{C}^2, 0)$, such that the pullback vector field

$$\Psi^*Z := D\Psi^{-1}(Z \circ \Psi)$$

is linear: $\Psi^*Z(x, y) = \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y}$. We say that Z is **analytically conjugate** to its linear part (or analytically linearizable).

Of course when $\lambda \in \mathbb{R}$ it may happen that Z is not analytically linearizable but if $\lambda > 0$, then Z is analytically conjugate to a polynomial vector field [Dul09]. The difficult cases arise when $\lambda \leq 0$, and the really difficult cases (the ones that still seem out of reach) occur when λ is a negative irrational. In the sequel we suppose that $\lambda \in \mathbb{Q}_{<0}$.

Definition 1.1. We describe the class of **resonant** vector fields Z and their underlying resonant foliations \mathcal{F}_Z , assuming none of which can be put in a linear form by conjugacy. We say that Z admits a $p : q$ **saddle** singularity at that point (Z is a $p : q$ resonant vector field) if its eigenratio is $\lambda = -\frac{p}{q}$ for coprime positive integers p and q . If $\lambda = 0$ then Z is a **saddle-node** vector field.

Remark 1.2. There exists a deep link between resonant saddles and saddle-nodes, as we explain in Section 1.5.

The pioneering works of H. POINCARÉ and H. DULAC eventually yield the formal classification of all $p : q$ resonant vector fields: a codimension- k vector field Z , for $k \in \mathbb{Z}_{>0}$, is always formally conjugate to one of the **formal normal forms** $P(u)\widehat{X}_0$ where $u := x^q y^p$ is called the **resonant monomial**, P is a polynomial of degree at most k in the variable u with $P(0) \neq 0$, and

$$(1.2) \quad \widehat{X}_0(x, y) := u^k x \frac{\partial}{\partial x} + (1 + \mu u^k) \left(\lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), \quad \mu \in \mathbb{C}.$$

By this we mean that there exists an invertible formal power series $\Psi = (\Psi_1, \Psi_2)$, with $\Psi_j \in \mathbb{C}[[x, y]]$, such that $\Psi^*Z = P\widehat{X}_0$. This form is unique up to the action of linear changes of variables $(x, y) \mapsto (\alpha x, \beta y) \in \text{GL}_2(\mathbb{C})$ with $(\alpha^q \beta^p)^k = 1$. The couple (k, μ) is the **formal orbital modulus** coming from the Dulac-Poincaré normal form [Dul09], while P is the additional **formal temporal modulus** obtained by A. BRUNO [Bru89]. The formal modulus (k, μ, P) is left unchanged under formal changes of variables on Z : it is a (complete) formal invariant.

We introduce the functional space of germs of a holomorphic function in two complex variables u and y :

$$\mathbb{C}[u]_{\leq 2k}\{y\}_{>0} := \left\{ y \sum_{n=1}^{2k} u^n f_n(y) : f_n \in \mathbb{C}\{y\} \right\},$$

where $\mathbb{C}\{y\}$ is the algebra of germs of a holomorphic function at $0 \in \mathbb{C}$. Consider for every parameter $c > 0$ (Écalle's twist) the polynomial vector field

$$(1.3) \quad X_0(x, y) := u^k x \frac{\partial}{\partial x} + (c(1 - u^{2k}) + \mu u^k) Y(x, y)$$

where

$$Y(x, y) := -px \frac{\partial}{\partial x} + qy \frac{\partial}{\partial y}.$$

This is the main result of the article.

Normalization Theorem. *Let Z be a $p : q$ resonant vector field with formal modulus (k, μ, P) . There exist:*

- a bound $c(Z) > 1$ and, for each choice of the twist $c \geq c(Z)$,
- two germs $G, R \in \mathbb{C}[u]_{\leq 2k}\{y\}_{>0}$,
- a local holomorphic change of coordinates $\Psi \in \text{Diff}(\mathbb{C}^2, 0)$,

such that

$$(1.4) \quad \begin{aligned} \Psi^*Z &= Z_{G,R} := \frac{P}{1+PG} X_R \\ X_R &:= X_0 + RY. \end{aligned}$$

Moreover any two $Z_{G,R}$ and $Z_{\widetilde{G},\widetilde{R}}$ are analytically conjugate near $(0, 0)$ if and only if they are conjugate by some $(x, y) \mapsto (\alpha x, \beta y) \in \text{GL}_1(\mathbb{C})$ with $(\alpha^q \beta^p)^k = 1$.

Remark 1.3.

- (1) The reader should be aware at that point of a slight abuse of notation. When plugging the variables (x, y) in the expression above, the monomial u should be substituted with $u(x, y)$. For instance if $\mu = 0$, then

$$X_0(x, y) = y^{pk} x^{qk+1} \frac{\partial}{\partial x} + c(1 - x^{2qk} y^{2pk}) \left(-px \frac{\partial}{\partial x} + qy \frac{\partial}{\partial y} \right).$$

- (2) The proof actually asserts that the $2k$ functions $y \mapsto f_n(y)$ appearing in R and G are holomorphic and bounded on the disc $\{y : |y| < 2\}$.

Remark 1.4.

- (1) In [Lor04] it is proved that any germ of a saddle foliation at $(0,0)$ with eigenratio λ can be defined in a convenient local analytic chart by a vector field of the form

$$x \frac{\partial}{\partial x} + \lambda (f(y) + x)y \frac{\partial}{\partial y} \quad , \quad f \in 1 + y\mathbb{C}\{y\}.$$

Clearly this form is very simple, but it is not unique. The link between this form and the normal forms presented here is unclear.

- (2) In [Eca05] a methodological approach for “canonical spherical synthesis” of *e.g.* resonant foliations is proposed. Unfortunately it is not possible to directly extract from it an explicit form for the synthesized vector fields.
- (3) The form of R and G is satisfying and seems optimal in the sense that there is as many free $(2k)$ components in R (*resp.* G) than there is in the orbital modulus (*resp.* temporal modulus). Although the mapping

$$\text{modulus} \mapsto \text{normal form}$$

is certainly not as simple as in the case of unilateral moduli described in [ST15], where it is “triangular” and computable, the works of Écalle may provide a path to find an explicit expression for it.

Definition 1.5. We use the notations introduced in the Normalization Theorem.

- (1) The vector field $Z_{G,R}$ is called an **analytic normal form** of Z .
- (2) The vector field X_R is called an **analytic orbital normal form** of Z . The name is justified by the fact that \mathcal{F}_Z is analytically conjugate to \mathcal{F}_{X_R} .

1.3. Outline of the construction and structure of the article. The proof of the Normalization Theorem is done in three steps and relies on Martinet-Ramis orbital classification of resonant planar foliations [MR83], which is summarized in Section 3. That general scheme has already been used successfully in [ST15, RT21], although the technical intricacies differ from one case to the other and require specific arguments. In particular in the present situation we rely on results that have recently been obtained in [Tey22] for the realization of analytic class of parabolic germs. (Of course the link between the class of a saddle foliation and its holonomy is well known, but here we do not directly invoke such arguments.)

Orbital realization Being given the Martinet-Ramis modulus associated to a resonant vector field Z , we build a vector field X_R in normal form within the same formal class and with the same orbital modulus. This is done in Section 4 by a fixed-point method involving a Cauchy-Heine transform solving a nonlinear Cousin problem associated to a sectorial decomposition of the (u, y) -space (Section 2). The trickiest part in the construction is to find a model vector field X_0 whose orbit space in the intersection of consecutive sectors can be controlled. More precisely, by increasing the twist parameter c we are able to shrink the size of the orbit spaces so that they become adapted to the given orbital modulus. This is explained in Section 3. Without the introduction of the twist parameter it seems very dubious to provide normal forms: if one tries by another mean to reduce the size of the orbit space in one intersection to make it fit within the disc of convergence of a component of the modulus, then the size increases in the next intersection and may spill out of that of the corresponding component.

The notable exception comes from unilateral moduli, where only one out of two components are nontrivial and the same strategy as [ST15] would work for resonant saddle vector fields.

The process yields a vector field X_R where R is holomorphic on some «hollow» domain, which is not a neighborhood of the origin but contains the tube $\mathbb{C} \times \{1 < |y| < 2\}$. By studying the growth of $x \mapsto R(x, y)$ for fixed y we deduce that R has a polynomial form $\sum_{n=1}^{2k} u^n f_n(y)$, while the shape of the hollow domain forces each f_n to extend holomorphically to the whole disc $\{|y| < 2\}$.

Temporal realization So far we have found an analytic change of coordinates bringing Z to some UX_R with $U(0, 0) \neq 0$. Sending UX_R to $Z_{G,R} = \frac{P}{1+PG} X_R$ is done by solving the cohomological equation $X_R \cdot T = \frac{1}{U} - \frac{1}{P} - G$, where $X_R \cdot T$ is the Lie (directional) derivative of the function T along X_R . Being given U and P , there is a unique choice of G in normal form such that the equation has an analytic solution T . To understand this we need to exhibit an explicit cokernel for the derivation X_R , by providing a section of the period operator associated to X_R . Here again the main tool is the Cauchy-Heine transform (although it does not need to be iterated). This study is performed in Section 5 but we give more details in Section 1.4 to come, since knowing the cokernel is worthwhile and carries a lot of useful applications.

Uniqueness of the realization So far the family of vector fields $\{Z_{G,R}\}$ has been proved versal, thus it is natural to describe the automorphisms of the family to study its universality. Once the diagonal action $(x, y) \mapsto (\alpha, \beta y)$ with $(\alpha^q \beta^p)^k = 1$ has been factored out, it remains to prove that the only automorphism which is tangent to the identity is the identity itself. According to the discussion in the previous item, the function G is unique for a given orbital class X_R , therefore we must prove that if two foliations in normal form \mathcal{F}_{X_R} and $\mathcal{F}_{X_{\bar{R}}}$ are conjugate by some Ψ then $\Psi = \text{Id}$. First, we show that Ψ can be assumed to preserve the resonant monomial, *i.e.* $\Psi = (xe^{-pN}, ye^{qN})$ for some holomorphic germ N , then we relate the condition $\Psi^* \mathcal{F}_{X_R} = \mathcal{F}_{X_{\bar{R}}}$ to a cohomological equation involving N . The latter has only trivial solutions, thanks to the description of the cokernel that we have done. We give a full proof of the uniqueness statement in Section 6.

1.4. Cohomological equations, period operator and its natural section. Because $[X_0, Y] = 0$ we can follow the strategy laid out in [Tey04a].

- X_0 is (formally/analytically) conjugate to X_R if and only if there exists a (formal/analytic) solution of the cohomological equation

$$X_R \cdot N = -R.$$

In that case a conjugacy is obtained as $\Phi_Y^N : (x, y) \mapsto (x \exp(-pN(x, y)), y \exp(qN(x, y)))$, the flow at time N of the vector field Y .

- UX_R is (formally/analytically) conjugate to VX_R if and only if there exists a (formal/analytic) solution of the cohomological equation

$$X_R \cdot T = \frac{1}{U} - \frac{1}{V}.$$

In that case a conjugacy is obtained as $\Phi_{VX_R}^T$.

Such equations are called **cohomological equations**. The obstructions to solve formally or analytically these equations reasonably provides us with invariants of classification.

Cohomological Theorem. *Let X_R be an orbital normal form.*

- (1) Consider the cohomological equation $X_R \cdot F = G$ with $G \in \mathbb{C}\{x, y\}$.
 - (a) There exists a formal solution $F \in \mathbb{C}[[x, y]]$ if and only if the Taylor expansion of G at $(0, 0)$ does not contain terms u^n for $n \in \{0, \dots, k\}$ (in that case F is unique). We write $\mathbb{C}\{x, y\}_{>k}$ the space of all such germs, and we assume in the following that G belong to that space.
 - (b) There exists a neighborhood Ω of $(0, 0)$ on which G is holomorphic and bounded, a covering of $\Omega \setminus \{xy = 0\}$ into $2k$ "sectors" ${}^j\Omega^\sharp$, for $j \in \mathbb{Z}/k\mathbb{Z}$ and $\sharp \in \{+-, -+\}$, together with $2k$ bounded holomorphic functions ${}^jF^\sharp \in \text{Holo}_b({}^j\Omega^\sharp)$, such that $X_R \cdot {}^jF^\sharp = G$.
 - (c) The difference ${}^jF^{+-} - {}^jF^{-+}$ (resp. ${}^{j+1}F^{-+} - {}^jF^{+-}$) is constant on the leaves of \mathcal{F}_{X_R} and tends to 0 on $\{xy = 0\}$, therefore it defines a holomorphic function ${}^j f^- \in \mathbb{C}\left\{\frac{1}{h}\right\}_{>0}$ (resp. ${}^j f^+ \in \mathbb{C}\{h\}_{>0}$) of the leaf coordinate $h \in \mathbb{C}$.
- (2) We call **period operator** of X_R the linear mapping

$$\mathfrak{T}_R : G \in \mathbb{C}\{x, y\}_{>k} \mapsto ({}^j f^-, {}^j f^+)_{j \in \mathbb{Z}/k\mathbb{Z}} \in \left(\mathbb{C}\left\{\frac{1}{h}\right\}_{>0} \times \mathbb{C}\{h\}_{>0} \right)^{\times k}.$$

- (a) The formal solution F given in 1.(a) is a convergent power series if and only if $\mathfrak{T}_R(G) = 0$.
- (b) The period operator is surjective. More precisely, being given $f \in \left(\mathbb{C}\left\{\frac{1}{h}\right\}_{>0} \times \mathbb{C}\{h\}_{>0} \right)^{\times k}$ there exists a unique $G \in \mathbb{C}[u]_{\leq 2k}\{y\}_{>0}$ such that $\mathfrak{T}_R(G) = f$.

Definition 1.6. The operator $\mathfrak{S}_R : f \mapsto G$ defined in 2. of the Cohomological Theorem is called the **natural section of the period operator** of X_R .

We may reformulate algebraically the previous theorem by saying that the following sequence of linear maps is exact:

$$\mathbb{C} \xrightarrow{\text{cst}} \mathbb{C}\{x, y\} \xrightarrow{X_R \cdot} \mathbb{C}\{x, y\}_{>k} \xrightarrow{\mathfrak{T}_R} \left(\mathbb{C}\left\{\frac{1}{h}\right\}_{>0} \times \mathbb{C}\{h\}_{>0} \right)^{\times k}$$

with a similar exact sequence at a formal level

$$\mathbb{C} \xrightarrow{\text{cst}} \mathbb{C}[[x, y]] \xrightarrow{X_R \cdot} \mathbb{C}[[x, y]] \xrightarrow{\Pi_k} \mathbb{C}[u]_{\leq k},$$

where $\Pi_k : \sum_{n, m \in \mathbb{Z}_{\geq 0}} g_{n, m} x^n y^m \mapsto \sum_{\ell \leq k} g_{q\ell, p\ell} u^\ell$ is the canonical projection coming from the power series expansion. We deduce from this theorem the following interpretation of the different moduli involved (see Corollary 5.6).

- The obstruction to solve formally $X_R \cdot \widehat{T} = \frac{1}{U}$ is located in $P := \Pi_k(U)$ and that gives the formal temporal modulus of Bruno.

- The obstruction to solve analytically $X_R \cdot T = \frac{1}{U} - \frac{1}{P}$ is embodied the period $\mathfrak{T}_R\left(\frac{1}{U} - \frac{1}{P}\right)$ and that gives the temporal modulus (t -shift) of Grintchy-Voronin.
- The obstruction to solve analytically $X_R \cdot N = -R$ is the period $\mathfrak{T}_R(-R)$ and that provides the logarithmic form of Martinet-Ramis orbital modulus.

Partial results pertaining to the Cohomological Theorem (namely, 2. (a)) were already obtained in [BL97].

Remark 1.7. Due to the particular structure of the leaf space of \mathcal{F}_{X_R} , the period operator is not onto the whole space of germs $\left(\mathbb{C}\left\{\frac{1}{h}\right\}_{>0} \times \mathbb{C}\{h\}_{>0}\right)^{\times k}$. Indeed, by taking a smaller neighborhood of $(0, 0)$ the size of the leaf space in the intersection of consecutive sectors does not shrink to a point. To realize a given element of that space as a period of X_R it is probably necessary to take a larger c (and thus another R while staying in the same orbital class). Precise statements are given in Theorem 5.7.

As an application of the Cohomological Theorem, the same reasoning as the one produced in [RT21] (that uses the natural section of the period operator) allows to generalize a result of M. BERTHIER and F. TOUZET [BT99, Proposition 5.5] to resonant saddle foliations: a resonant saddle foliation admits a Liouvillian first-integral if and only if its orbital normal form X_R is a Bernoulli vector field, that is $R(u, y) = y^d r(u)$ for some $d \in \mathbb{Z}_{\geq 0}$ and $r \in u\mathbb{C}[u]_{<2k}$. We leave details to the interested reader.

1.5. Summability and divergence. The Cohomological Theorem offers in 1.(b) “sectorial” sums to the (generally) divergent power series F . As will be made clear in Section 5, the divergence is concentrated in the resonant monomial. This property was already underlined in [MR83] for the orbital problem. In fact, when $G \in \mathbb{C}\{u, y\}_{>k}$ one can prove that the ${}^j F^\sharp$ come from holomorphic functions in the variable $(u, y) \in {}^j V^\sharp \times \{|y| < 2\}$ where u is replaced by $u(x, y)$ and ${}^j V^\sharp$ is a traditional sector in the u -variable. From Ramis-Sibuya’s theorem (see e.g. [LR16]), we can deduce that $F = \sum_{n \geq 0} f_n(y) u^n$ where the f_n are holomorphic and bounded on the disc $D := \{|y| < 2\}$ with $\|f_n\|_D = O(B^n (n!)^{1/k})$, i.e. that F is transversely k -summable in the variable u .

All these facts can also be deduced from corresponding properties already known for saddle-node vector fields. One may indeed observe that the resonant saddle vector fields we obtain as normal forms do come from saddle-node vector fields in the variables (u, y) . This is made apparent by considering the foliation \mathcal{F}_{X_R} as integrating the distribution of dual differential 1-forms:

$$u^k x dy + \left(c(1 - u^{2k}) + \mu u^k + y \sum_{n=1}^{2k} u^n f_n(y) \right) (p x dy + q y dx).$$

Recalling that $u = x^q y^p$, we deduce that \mathcal{F}_{X_R} also integrates the distribution given by

$$(1.5) \quad \omega_R(u, y) := u^{k+1} dy + y \left(c(1 - u^{2k}) + \mu u^k + y \sum_{n=1}^{2k} u^n f_n(y) \right) du.$$

This is exactly a saddle-node in the variables (u, y) with formal invariant (k, μ) . This correspondence is well known for formal normal forms ($R := 0$), and we just established it at an analytical level.

2. NOTATIONS AND BASIC TOOLS

Since a lot of objects of different natures mix up (germs at $(0, 0)$ of holomorphic objects, sectorial objects, Banach spaces of functions *etc.*), we must introduce numerous notations. This section provides the reader with a glossary of notations and conventions we stick to throughout the article.

- It will be convenient to follow the convention

$$0^{+1} := 0 \quad \text{and} \quad 0^{-1} := \infty.$$

- We fix a pair (p, q) of coprime positive integers and we define the associated **resonant monomial**

$$u(x, y) := x^q y^p.$$

- Most constructions take place in the variables (u, y) and are pullbacked in the variables (x, y) through the **canonical embedding**

$$\iota : (x, y) \mapsto (u(x, y), y).$$

In order to keep notations as light as possible, we write u_* to stand for the value of $u(x, y)$ in expressions containing x and y , in order to distinguish it from the usage of u as a standalone symbol. We use a similar notation for functions. For instance starting from $f : (u, y) \mapsto f(u, y)$ we write f_* to stand for the function $\iota^* f = f \circ \iota$:

$$f_* : (x, y) \mapsto f(u(x, y), y).$$

2.1. Sector-related notations. We fix $k \in \mathbb{Z}_{>0}$. Undoubtedly the biggest source of notational heaviness comes from the decomposition of the (x, y) -space and u -space into $2k$ sectors. This decomposition is classical yet we wish to introduce notations that both contain all necessary contextual information and embodies the underlying dynamical structure. The vast majority of objects we introduce are collections O of $2k$ sectorial objects ${}^j O^\bullet$ indexed by $j \in \mathbb{Z}/k\mathbb{Z}$ and $\bullet \in \{+, -, +-, -+\}$. Here is the list of the conventions that are always used in the rest of the article:

- j is some (arbitrary) element of $\mathbb{Z}/k\mathbb{Z}$;
- the symbol \sharp is some (arbitrary) element of $\{+-, -+\}$;
- we simply write the collection $O = \left({}^j O^\sharp \right)_{j \in \mathbb{Z}/k\mathbb{Z}, \sharp \in \{+-, -+\}}$ as

$$O = \left({}^j O^\sharp \right),$$

and the collection $O = \left({}^j O^\star \right)_{j \in \mathbb{Z}/k\mathbb{Z}, \star \in \{-, +\}}$ as

$$O = \left({}^j O^\pm \right).$$

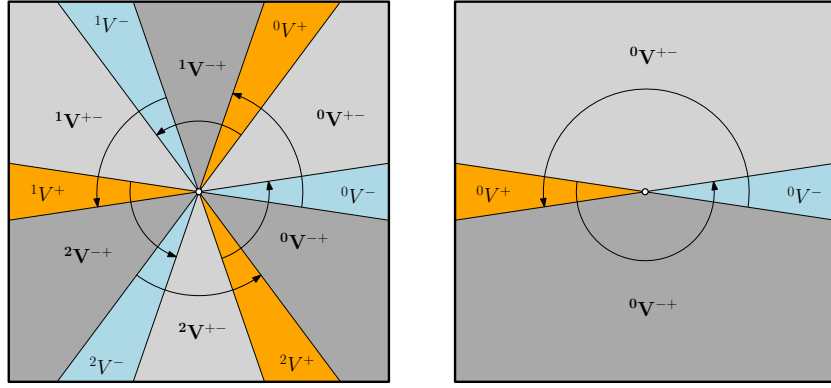


FIGURE 2.1. The sectorial decomposition near 0 in the case $k = 3$ (left) or $k = 1$ (right).

Definition 2.1. This should be read with the figure 2.1 in mind.

- (1) The **sectorial decomposition** of the u -space is the collection of $2k$ open, infinite sectors $(jV^\#)$ defined as

$$jV^{+-} := \left\{ u \neq 0 : \left| \arg u - \pi \frac{2j+1}{2k} \right| < \frac{5\pi}{8k} \right\},$$

$$jV^{-+} := \left\{ u \neq 0 : \left| \arg u + \pi \frac{2j+1}{2k} \right| < \frac{5\pi}{8k} \right\}.$$

- (2) The intersection of pairwise successive sectors is either one of the following sectorial domains (if $k > 1$)

$$jV^- := jV^{+-} \cap jV^{-+},$$

$$jV^+ := j^{+1}V^{-+} \cap jV^{+-}.$$

We extend this definition for $k = 1$ by considering the two connected components ${}^0V^-$ and ${}^0V^+$ of ${}^0V^{+-} \cap {}^0V^{-+}$ under the condition that ${}^0V^\pm$ contains $\pm\mathbb{R}_{<0}$.

- (3) Let us define:

$$j\mathcal{V}^{+-} := \{(u, y) \in \mathbb{C}^2 : |y| < 2, u \in jV^{+-}\}$$

$$j\mathcal{V}^{-+} := \{(u, y) \in \mathbb{C}^2 : 1 < |y| < 2, u \in jV^{-+}\},$$

as well as the corresponding intersections $j\mathcal{V}^\pm$ build in a similar fashion as in 2. The notation \mathcal{V} denotes the full collection $(j\mathcal{V}^\#)_{j,\#}$.

- (4) We pull-back these sectors by defining

$$j\mathcal{V}_*^\# := \iota^{-1}(j\mathcal{V}^\#) = \{(x, y) \in \mathbb{C}^2 : \iota(x, y) \in j\mathcal{V}^\#\},$$

and form the filled union

$$\mathcal{V}_* := \text{int} \left(\overline{\bigcup_{j,\#} j\mathcal{V}_*^\#} \right).$$

Remark 2.2. The domain \mathcal{V}_* is the disjoint union of $\bigcup_{j,\sharp} j\mathcal{V}_*^\sharp$ on the one hand and $\{(x, y) \in \mathcal{V}_* : xy = 0\}$ on the other hand. It is not a neighborhood of $(0, 0)$, and each $j\mathcal{V}_*^\sharp$ is not connected whenever $(p, q) \neq (1, 1)$.

2.2. Function spaces. We write $\mathbb{C}[z]_{d \geq \nu}$ the algebra of polynomials in z of degree at most d and valuation at least ν . We omit to write ν whenever it equals 0. The field \mathbb{C} may also be replaced with other commutative rings.

Definition 2.3.

- (1) Let $\mathcal{D} \subset \mathbb{C}^n$ be a domain. We denote by $\text{Holo}(\mathcal{D})$ the algebra of functions holomorphic on \mathcal{D} .
- (2) We define the Banach algebra $\text{Holo}_b(\mathcal{D})$ of all \mathbb{C} -valued bounded holomorphic functions on \mathcal{D} with continuous extension to the closure $\overline{\mathcal{D}}$, equipped with the norm:

$$\|f\|_{\mathcal{D}} := \sup_{z \in \mathcal{D}} |f(z)|.$$

- (3) Being given a finite collection $\mathcal{D} := (j\mathcal{D}^\pm)$ of $2k$ domains of \mathbb{C}^2 , we denote by $\text{Holo}_b(\mathcal{D})$ the product Banach space $\prod_{j,\pm} \text{Holo}_b(j\mathcal{D}^\pm)$ equipped with the product norm

$$f = (j f^\pm), \quad \|f\|_{\mathcal{D}} := \max_{j,\pm} \|j f^\pm\|_{j\mathcal{D}^\pm}.$$

- (4) Let $D^\pm \subset \mathbb{P}_1(\mathbb{C})$ be a domain containing $0^{\pm 1}$. We define the Banach algebra $\text{Holo}_b(D)'$ of all \mathbb{C} -valued bounded holomorphic functions on D , admitting a continuous extension to the closure \overline{D} and vanishing at $0^{\pm 1}$, equipped with the norm

$$\|f\|'_D := \sup_{z \in D} \frac{|f(z)|}{|z|^{\pm 1}}.$$

If D is a star-shaped domain centered at $0^{\pm 1}$, then $\|f\|'_D \leq \|f\|_D$.

2.3. Vector fields.

- Z is a resonant-saddle or saddle-node vector field near $(0, 0)$. The notation X is in general reserved for vector fields with a trivial temporal component.
- $Z \cdot F$ stands for the **Lie derivative** along Z , acting on $F \in \mathbb{C}[[x, y]]$ or on $F \in \mathbb{C}\{x, y\}$.
- We let $\Phi_Z^t(x, y)$ be the **flow** at time t of Z , i.e. the local holomorphic solution of 1.1 with initial condition (x, y) . It is locally holomorphic in the variables (x, y, t) taken sufficiently close to $(0, 0, 0)$.
- A **first-integral** H of Z is a holomorphic function such that $Z \cdot H = 0$.
- $(k, \mu) \in \mathbb{N}_{>0} \times \mathbb{C}$ is the **formal orbital modulus** of Z while $P \in \mathbb{C}[u]_{\leq k}$ with $P(0) \neq 0$ is its **formal temporal modulus**. The complete formal modulus is (k, μ, P) .

- The **formal orbital normal form** associated to the formal modulus (k, μ, P) is the polynomial vector field depending on the twist parameter $c > 1$:

$$(2.1) \quad \begin{aligned} X_0(x, y) &:= u_*^k x \frac{\partial}{\partial x} + \left(c(1 - u_*^{2k}) + \mu u_*^k \right) Y(x, y) \\ Y(x, y) &:= -px \frac{\partial}{\partial x} + qy \frac{\partial}{\partial y}. \end{aligned}$$

3. MARTINET-RAMIS ORBITAL MODULUS OF A RESONANT SADDLE

Finding formal models for the dynamics of 2-dimensional vector fields is easy enough. When these formal normalizations fail to be analytic, one must perform a finer study to obtain the analytical classification. In the orbital case, *i.e.* that of foliations, this amounts to endowing the leaf space with a holomorphic structure and describing the analytic diffeomorphisms between these manifolds. The now-classical strategy for resonant foliations is to build adapted sector-like areas whose closure is a neighborhood of the singularity, and to find normalizing sectorial maps conjugating the dynamics with that of the formal model \mathcal{F}_{X_0} .

For expository reasons, in this section we briefly explain how both tasks are achieved for 1 : 1 foliations, following the ideas of J. MARTINET and J.-P. RAMIS [MR82, MR83] and introducing some material needed later on. In Section 4.4 we stress the slight modifications that are needed to make the general theory for $p : q$ resonant foliations work.

3.1. Study of the formal model and making of the sectors. Here we investigate the global dynamical properties, for a fixed value of $c > 0$, of the vector field X_0 given by (2.1). We are particularly interested in describing its orbit space, which can be achieved through the study of the Liouvillian first-integral

$$(3.1) \quad \begin{aligned} H(u, y) &:= y \widehat{H}(u), \\ \widehat{H}(u) &:= u^{-\mu} \exp \frac{c(u^{-k} + u^k)}{k}. \end{aligned}$$

By letting $X_0 \cdot$ stand for the Lie directional derivative along X_0 , an elementary computation yields

$$X_0 \cdot H_* = 0.$$

(In fact X_0 is built as the dual vector field of the rational 1-form $\frac{dH}{H}$.) This identity tells us that level sets of H_* coincide with trajectories of X_0 . That is, an equation of a leaf of \mathcal{F}_{X_0} is given by

$$H_*(x, y) = \text{cst.}$$

Yet, because \widehat{H} is multivalued when $\mu \notin \mathbb{Z}$, some care needs to be taken; in the sequel we use the determination of the argument of u on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ to compute the actual value of $u^{-\mu}$: when crossing the boundary from ${}^0V^{-+}$ to ${}^0V^{+-}$ the value of $u^{-\mu}$ is multiplied with $e^{-2i\pi\mu}$. We wish to distribute evenly this change over all sectors ${}^jV^\#$.

Definition 3.1. Set

$$\sigma := \exp \frac{i\pi\mu}{k}.$$

We define the **model** (sectorial) **first-integrals** as ${}^j_0\mathcal{H}^\sharp$ where:

$${}^j_0\mathcal{H}^\sharp(u, y) := \sigma^n y \times \widehat{H}(u) \text{ with } n := \begin{cases} 2j & \text{if } \sharp = +- \\ 2j-1 & \text{if } \sharp = -+ \text{ and } j > 0. \\ 2k-1 & \text{if } \sharp = -+ \text{ and } j = 0 \end{cases}$$

The third item of the next lemma is the key property that allows the rest of the construction to be worked out. The vector field X_0 has been designed so that it holds.

Lemma 3.2.

- (1) Let \mathcal{V} be a small domain containing $(0, 0)$ and take $M, \widetilde{M} \in \mathcal{V} \cap {}^j\mathcal{V}^\sharp$. The identity ${}^j_0\mathcal{H}^\sharp(M) = {}^j_0\mathcal{H}^\sharp(\widetilde{M})$ holds if and only if there exists a leaf \mathcal{L} of the restricted foliation $\mathcal{F}_{X_0}|_{\mathcal{V} \cap {}^j\mathcal{V}^\sharp}$ such that $M \in \mathcal{L}$ and $\widetilde{M} \in \mathcal{L}$.
- (2) ${}^j_0\mathcal{H}^\sharp({}^j\mathcal{V}^\sharp) = \mathbb{C}^\times$.
- (3) ${}^j_0\mathcal{H}^\sharp({}^j\mathcal{V}^\pm)$ is a punctured neighborhood of $0^{\pm 1}$. More precisely, there exists $A = A(k, \mu) > 0$ and $c(k, \mu) > 0$ such that if $c > c(k, \mu)$, then:

$$\left\| \left({}^j_0\mathcal{H}^\sharp \right)^{\pm 1} \right\|_{j\mathcal{V}^\pm} \leq Ae^{-c/k}.$$

Proof. Most of the assertions can be found in [MR83]. Especially 1. is proved in [MR83, p593, p598] for the usual first-integral $\widehat{H}_0(x, y) := y u_*^{-\mu} \exp u_*^{-k}/k$ of the standard model \widehat{X}_0 (see (1.2)). Using the fact that the critical points of $u^k \mapsto u^k + u^{-k}$ lie on the unit circle in the u -variable, we deduce our claim by requiring that the size of \mathcal{V} be so small as to ensure $\|u\|_{\mathcal{V}} < 1$. Item 2. comes from the fact that $u \in {}^j\mathcal{V}^\sharp \mapsto \widehat{H}(u)$ admits an essential singularity at 0 while the values reached by $u^k + u^{-k}$ cover a punctured neighborhood of 0.

The estimates appearing in 3. follow from elementary calculus. In [Tey22, Corollary 4.5] a value for $c(k, \mu)$ is determined and a bound $\|\widehat{H}\|_{j\mathcal{V}^\pm}^{\pm 1} < m e^{-c/2k}$ is proved for some explicit $m = m(k, \mu)$. The cited paper deals with the case $k = 1$ and the factor $u^{-\mu}$ is slightly different, but the general case follows from what has been carried out there; details are left to the reader. If $|y| \geq 1$ and $u \in \mathbb{C}$, then

$$\left| \frac{1}{{}^j_0\mathcal{H}^\sharp(u, y)} \right| = \frac{1}{|y|} |\sigma^n \widehat{H}(u)|^{-1} \leq |\sigma|^{-n} |\widehat{H}(u)|^{-1}, \quad n \in \{0, \dots, 2k-1\}.$$

A similar bound can be established when $|y| \leq 2$ and $u \in \mathbb{C}$ since $\left| {}^j_0\mathcal{H}^\sharp(u, y) \right| = |y| |\sigma^n \widehat{H}(u)| \leq 2 |\sigma|^n |\widehat{H}(u)|$ for some $n \in \{0, \dots, 2k-1\}$. The proof is complete. \square

We can interpret 1. of the Lemma by saying that the values h of ${}^j_0\mathcal{H}^\sharp$ provide a natural coordinate on the sectorial orbit space of X_0 near $(0, 0)$, and that 2. makes the sectorial orbit space a punctured sphere $\overline{\mathbb{C}} \setminus \{0, \infty\}$. The discarded values 0 and ∞ correspond to the two separatrices $\{xy = 0\} \setminus \{(0, 0)\}$. Taking 3. into account, we deduce that the orbit space of X_0 outside $\{xy = 0\}$ is obtained by identifying the $2k$ successive spheres about their poles by linear maps, since the choices of

determination of H we made over the intersections ${}^jV^\pm$ imply:

$$\begin{aligned} (\forall (x, y) \in {}^j\mathcal{V}_*^-) \quad & {}^j_0\mathcal{H}_*^{+-}(x, y) = \sigma \times {}^j_0\mathcal{H}_*^{-+}(x, y), \\ (\forall (x, y) \in {}^j\mathcal{V}_*^+) \quad & {}^{j+1}_0\mathcal{H}_*^{-+}(x, y) = \sigma \times {}^j_0\mathcal{H}_*^{+-}(x, y). \end{aligned}$$

J. MARTINET and J.-P. RAMIS called this configuration the *chapelet de sphères* (rosary of spheres), which we prefer to call the *orbital necklace* of \mathcal{F}_{X_0} as in [RT21]. The orbital modulus of Martinet-Ramis is obtained in the case of a general resonant foliation by replacing the linear polar identifications with nonlinear perturbations.

3.2. Sectorial normalization and sectorial first-integral. Start with a 1 : 1 resonant vector field $X_R = X_0 + RY$ with R holomorphic and $R(0,0) = 0$ (following [Dul09, Dul04] any 1 : 1 resonant saddle foliation can be brought into that form by choosing suitable local analytic coordinates). According to [MR83, Theorem 6.2.1], there exists a neighborhood \mathcal{V} of $(0,0)$ and a collection of functions $\mathcal{N} = ({}^j\mathcal{N}^\sharp)$ with ${}^j\mathcal{N}^\sharp \in \text{Hol}_b(\{(x, y) \in \mathcal{V} : u_* \in {}^jV^\sharp\})$ such that, if one defines ${}^j\Psi^\sharp := \Phi_Y^{j\mathcal{N}^\sharp}$, then

$$({}^j\Psi^\sharp)^* X_0 = X_R.$$

Remark 3.3. This result is a byproduct of Martinet-Ramis synthesis theorem. We do not need it in our present study, we simply invoke it for the purpose of our exposition of their classification. We revisit this assertion in Section 5 by providing it with a more geometric flavor.

Because $X_0 \cdot {}^j_0\mathcal{H}_*^\sharp = 0$ we have $X_R \cdot ({}^j_0\mathcal{H}_*^\sharp \circ {}^j\Psi^\sharp) = 0$, where ${}^j_0\mathcal{H}_*^\sharp$ is the first-integral of the formal model X_0 defined in Section 3.1. Let us describe in more details these sectorial first-integrals of X_R , since they provide a natural coordinate on the sectorial leaf space of \mathcal{F}_{X_R} .

Lemma 3.4. *The functions ${}^j\mathcal{N}^\sharp$ satisfy the following properties.*

- (1) ${}^j_0\mathcal{H}_*^\sharp \circ {}^j\Psi^\sharp = {}^j_0\mathcal{H}_*^\sharp \exp j\mathcal{N}^\sharp$.
- (2) $X_R \cdot j\mathcal{N}^\sharp = -R$.
- (3) *If X_R is in normal form then ${}^j\mathcal{N}^\sharp = {}^j\mathcal{N}_*^\sharp$ for some sectorial function in (u, y) -space.*

Proof.

- (1) Because ${}^j\Psi^\sharp(x, y)$ is given by the flow of Y starting from (x, y) and with time ${}^j\mathcal{N}^\sharp(x, y)$, we have:

$${}^j\Psi^\sharp(x, y) = (x \exp(-{}^j\mathcal{N}^\sharp(x, y)), y \exp({}^j\mathcal{N}^\sharp(x, y))).$$

In particular $u \circ {}^j\Psi^\sharp = u$, which gives the conclusion.

- (2) From the identities

$$\begin{aligned} Y \cdot u &= 0 \\ Y \cdot y &= y, \end{aligned}$$

we derive the fact that $Y \cdot {}^j_0\mathcal{H}_*^\sharp = {}^j_0\mathcal{H}_*^\sharp$. Since $X_0 \cdot {}^j_0\mathcal{H}_*^\sharp = 0$ we conclude, by taking logarithmic derivatives:

$$\begin{aligned} \frac{{}^jX^\sharp \cdot {}^j_N\mathcal{H}^\sharp}{{}^j_N\mathcal{H}^\sharp} &= {}^jX^\sharp \cdot \left(\log {}^j_0\mathcal{H}_*^\sharp + {}^jN^\sharp \right) \\ &= R + {}^jX^\sharp \cdot {}^jN^\sharp = 0 \end{aligned}$$

as required.

(3) See Corollary 4.7 1. □

Hence, we are led to give the following definition.

Definition 3.5.

(1) Define the Banach space of functions in the variables (u, y) :

$${}^j\mathcal{A}^\sharp := \text{Holo}_b({}^j\mathcal{V}^\sharp),$$

as well as the product algebra $\mathcal{A} := \prod_{j,\sharp} {}^j\mathcal{A}^\sharp$.

(2) Let $N \in \mathcal{A}$ be a collection of $2k$ sectorial, bounded and holomorphic functions. We define

$${}^j_N\mathcal{H}^\sharp := {}^j_0\mathcal{H}^\sharp \times \exp {}^jN^\sharp.$$

The collection $({}^j_N\mathcal{H}_*^\sharp)$ is called the **canonical sectorial first-integral** associated to N .

Because of the choices made for ${}^j_0\mathcal{H}^\sharp$, we have

$${}^j_N\mathcal{H}^\sharp \in \text{Holo}({}^j\mathcal{V}^\sharp).$$

It is straightforward to show that the conclusions of Lemma 3.2 hold for these first-integrals but for the presence of the perturbations ${}^jN^\sharp$. For the sake of brevity, the next proposition is only written down for foliations in normal form (1.4), but it can be adapted in a straightforward manner to encompass the case of a general resonant vector field $X_0 + RY$. We leave the details to the reader.

Proposition 3.6. *Let $N \in \mathcal{A}$.*

(1) *Let \mathcal{V} be a small domain containing $(0,0)$ and take $M, \tilde{M} \in \mathcal{V} \cap {}^j\mathcal{V}^\sharp$. Assume that N comes from a resonant foliation \mathcal{F}_{X_R} as in Lemma 3.4. The relation*

${}^j_N\mathcal{H}_^\sharp(M) = {}^j_N\mathcal{H}_*^\sharp(\tilde{M})$ holds if and only if there exists a leaf \mathcal{L} of the restricted foliation $\mathcal{F}_{X_R}|_{{}^j\mathcal{V}^\sharp}$ such that $M \in \mathcal{L}$ and $\tilde{M} \in \mathcal{L}$.*

(2) ${}^j_N\mathcal{H}^\sharp({}^j\mathcal{V}^\sharp) = \mathbb{C}^\times$.

(3) ${}^j_0\mathcal{H}^\sharp({}^j\mathcal{V}^\pm)$ is a punctured neighborhood of $0^{\pm 1}$. More precisely, there exists $A = A(k, \mu) > 0$ and $c(k, \mu) > 0$ such that if $c > c(k, \mu)$, then:

$$\left\| \left({}^j_N\mathcal{H}^\sharp \right)^{\pm 1} \right\|_{{}^j\mathcal{V}^\pm} \leq A e^{-c/k + \|N\|_{{}^j\mathcal{V}^\sharp}}.$$

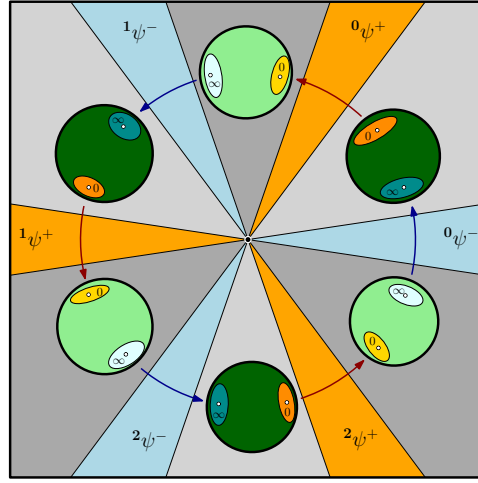


FIGURE 3.1. An orbital necklace.

3.3. Orbital necklaces and Martinet-Ramis orbital modulus.

Definition 3.7.

- (1) An **orbital necklace** of order k is the manifold (in general, non-Hausdorff) obtained by gluing $2k$ Riemann spheres $\overline{\mathbb{C}}$ near their poles 0 and ∞ by members of a collection of transition maps $(^j\psi^\pm)$, consisting in $2k$ germs of a diffeomorphism near $0^{\pm 1}$, one after the other in the circular order on the indices given by Definition 2.1.
- (2) We can choose a linear coordinate on each sphere in such a way that each $^j\psi^\pm$ is tangent to the identity but the last one $^{k-1}\psi^+$, which in turn is tangent to a linear map $h \mapsto e^{2i\pi\alpha}h$ for some $\alpha \in \mathbb{C}/\mathbb{Z}$. Let us call α the **residue** of the necklace.
- (3) The **Martinet-Ramis orbital modulus** of (a germ of) a $1 : 1$ resonant foliation \mathcal{F} is the orbital necklace defined by $(^j\psi^\pm)$ obtained in the following way. Choose a neighborhood \mathcal{V} of $(0,0)$ on which \mathcal{F} admits a holomorphic representative. Pick a point $M \in ^j\mathcal{V}^\pm$; depending on the considered intersection $^j\mathcal{V}^+$ or $^j\mathcal{V}^-$, and according to Proposition 3.6, there exist unique corresponding values $h^{+-}, h^{-+} \in \mathbb{C}$ of the respective sectorial first-integrals. Set

$$(3.2) \quad ^j\psi^-(h^{-+}) := h^{+-} \text{ or } ^j\psi^+(h^{+-}) := h^{-+}.$$

Remark 3.8.

- (1) By making another choice of the spherical coordinates we may assume that this residue is distributed evenly between all transition mappings $^j\psi^\pm$, as explained below for Martinet-Ramis modulus.
- (2) Following Proposition 3.6 1., the mapping $h \mapsto ^j\psi^\pm(h)$ coming from a resonant foliation is injective on the domain $^j_N\mathcal{H}^\#(^j\mathcal{V}^\pm)$. Moreover, the choice

of the leaf coordinate ${}^j_0\mathcal{H}^\sharp$ (and of the nonzero number σ) made in Definition 3.1 implies

$$(3.3) \quad \begin{aligned} {}^j\psi^+(h) &= h(\sigma + \mathcal{O}(h)) \\ {}^j\psi^-(h) &= h(\sigma + \mathcal{O}(1/h)), \end{aligned}$$

so that μ is the residue of the corresponding orbital necklace and it is distributed evenly.

By construction if \mathcal{O} is an orbital necklace, then $\text{Diff}(\mathcal{O}) \simeq \text{GL}_1(\mathbb{C}) \times \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Indeed, once the obvious action of $\mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by translation of the indices $j \mapsto j + \theta$ and exchange of polarity $+ \leftrightarrow -$ is factored out, an element $\psi \in \text{Diff}(\mathcal{O})$ must define an automorphism of each sphere fixing both poles. Each such mapping takes the form $h \mapsto ah$, and the nonzero constant a must be the same on each sphere for the global conformal structure to be preserved. We are now ready to state the following fundamental classification theorem upon which the present work is based.

Martinet-Ramis Theorem [MR83]. *The space of all equivalence classes (up to local analytic changes of coordinates) of germs of a 1 : 1 resonant saddle foliation belonging to a given formal class (k, μ) , is isomorphic to the space of orbital necklaces of order k and residue μ up to the action of $\text{GL}_1(\mathbb{C}) \times \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

4. BUILDING A RESONANT SADDLE WITH PRESCRIBED ORBITAL NECKLACE

With hindsight, the heuristic for building the orbital modulus as an orbital necklace seems rather natural: the analytic class of a foliation is determined by the conformal structure of its leaf space. What is less intuitive to grasp is how to solve the inverse problem (or synthesis problem, or realization problem) for foliations.

Problem. Being given an (abstract) orbital necklace, to prove that it comes from a $p : q$ resonant foliation.

What J. MARTINET and J.-P. RAMIS did to solve the orbital inverse problem was to use a powerful geometric black-box, Newlander-Niremberg theorem, at the expense of loosing their grip on an explicit realization. We propose here a process which is rather explicit and use simple analytic ingredients. Before delving into the details of the construction, let us first outline how it is performed by following the strategy used in [ST15, RT21] and how the twist parameter intervenes as in [Tey22]. Here again we perform the construction for 1 : 1 resonant saddles, the minor differences with the general $p : q$ case are being explained in Section 4.4.

First, we express the transition maps provided by an orbital necklace $({}^j\psi^\pm)$ as a collection of logarithmic data $({}^j\varphi^\pm)$ consisting in $2k$ germs of a holomorphic function near $0^{\pm 1}$ such that:

$$(4.1) \quad {}^j\psi^\pm(h) = \sigma h \exp {}^j\varphi^\pm(h), \quad {}^j\varphi^\pm(0^{\pm 1}) = 0.$$

Then the gluing conditions (3.2), that are written over ${}^j\mathcal{V}^\pm$ as:

$$\begin{aligned} {}^j_N\mathcal{H}^{+-}(u, y) &= \sigma \times {}^j_N\mathcal{H}^{-+}(u, y) \quad \text{on } {}^j\mathcal{V}^-, \\ {}^{j+1}_N\mathcal{H}^{-+}(u, y) &= \sigma \times {}^j_N\mathcal{H}^{+-}(u, y) \quad \text{on } {}^j\mathcal{V}^+, \end{aligned}$$

amounts to solving the nonlinear Cousin problem

$$(\star) \quad \begin{cases} jN^{+-} - jN^{-+} & = j\varphi^- \circ j_N \mathcal{H}^{-+} \\ j^{+1}N^{-+} - jN^{+-} & = j\varphi^+ \circ j_N \mathcal{H}^{+-} \end{cases}$$

where $N := (jN^\#) \in \mathcal{A}$ (Definition 3.5) is an unknown collection of sectorial functions. By solving (\star) we realize the abstract necklace dynamics as transitions between canonical first-integrals, yet still abstract at this stage. The fact that these sectorial first-integrals do come from a concrete holomorphic foliation is guaranteed by the following lemma.

Lemma 4.1. *Let a logarithmic data φ of an orbital necklace be given. Then there exists $c_O = c_O(k, \mu, \varphi) > 0$ such that for all $c > c_O$ the following assertions hold. Assume that N solves (\star) on the sectorial decomposition \mathcal{V} as in Definition 2.1.*

(1) *Each vector field*

$$jX^\# := X_0 - \frac{X_0 \cdot jN_*^\#}{1 + Y \cdot jN_*^\#} Y$$

is holomorphic on $j\mathcal{V}_^\#$ and admits $j_N \mathcal{H}_*^\#$ for first-integral.*

(2) *The collection $(jX^\#)$ is the restriction to sectors of a 1 : 1 resonant vector field X holomorphic on the domain \mathcal{V}_* .*

Once this vector field X is synthesized we recognize it is actually in the required normal form by bounding its growth as $u \rightarrow \infty$ for fixed y , see Corollary 4.7.

Proof. Define

$$jR^\# := -\frac{X_0 \cdot jN_*^\#}{1 + Y \cdot jN_*^\#}.$$

Taking c big enough ensures that $jR^\#$ is holomorphic on $j\mathcal{V}_*^\#$ by forcing $\|Y \cdot jN_*^\#\|_{j\mathcal{V}_*^\#} < 1$, so that $jX^\# = X_0 + jR^\# \times Y$ is holomorphic on $j\mathcal{V}_*^\#$ too. The claim is discussed in Corollary 4.7 1.

(1) This is the same proof as that of Lemma 3.4 2.

(2) Because of Riemann's removable singularity theorem, each $jR^\#$ (which is locally bounded near points of $\{xy = 0\}$, see Corollary 4.7 1) is the restriction of a function R holomorphic on \mathcal{V}_* if and only if $jR^{+-} = jR^{-+}$ on $j\mathcal{V}_*^-$ and $j^{+1}R^{-+} = jR^{+-}$ on $j\mathcal{V}_*^+$. On the one hand, since $j_N \mathcal{H}_*^{-+}$ is a first-integral of jX^{-+} , we have:

$$jX^{-+} \cdot j_N \mathcal{H}_*^{-+} = \sigma jX^{-+} \cdot \left(j_N \mathcal{H}_*^{-+} \exp j\varphi^- \circ j_N \mathcal{H}_*^{-+} \right) = 0,$$

while on the other hand we compute directly (after taking logarithmic derivatives):

$$jX^{-+} \cdot j_N \mathcal{H}_*^{-+} = j_N \mathcal{H}_*^{-+} \times \left(X_0 \cdot jN_*^{+-} + jR^{-+} \times (1 + Y \cdot jN_*^{+-}) \right),$$

as indeed $X_0 \cdot j_0 \mathcal{H}_*^\# = 0$ and $Y \cdot j_0 \mathcal{H}_*^\# = j_0 \mathcal{H}_*^\#$. Both identities considered together imply that $X_0 \cdot jN_*^{+-} + jR^{-+} \times (1 + Y \cdot jN_*^{+-}) = 0$, which can be rewritten as $jR^{+-} = jR^{-+}$ on $j\mathcal{V}_*^-$. The proof in the other intersection is similar.

□

Solving (\star) requires a refinement of the Cauchy-Heine transform in order to recover functions ${}^jN^\sharp$ whose pairwise difference in consecutive sectors is precisely ${}^j\varphi^\pm$. Yet this statement is imprecise since the collection N itself determines the first-integral ${}^j_N\mathcal{H}^\sharp$ that must be used to evaluate the right-hand side ${}^j\varphi^\pm \circ {}^j_N\mathcal{H}^\sharp$. The solution of the problem must therefore be obtained as a fixed-point.

In order to prove that this fixed-point method is well defined, and to ascertain its convergence, we need to control the size of the neighborhoods ${}^j_N\mathcal{H}^\sharp({}^j\mathcal{V}^\pm)$ of $0^{\pm 1}$ so that they fit within the disc of convergence of the corresponding ${}^j\varphi^\sharp$ and that everything takes place in a Banach space. This control is gained through the twist parameter $c > 0$, as follows from the estimates of Lemma 4.5 and Proposition 4.6. This fact can already be surmised from the bounds of Proposition 3.6 3.

The rest of the section regards giving precise proofs and statements leading to the resolution of (\star) , and Section 4.3 concludes this section by providing a proof of the orbital part of the Main Theorem.

4.1. Cauchy-Heine transform.

Definition 4.2. We name \mathcal{V} the collection of sectors $({}^j\mathcal{V}^\sharp)$ in the variables (u, y) as in Definition 2.1. Let $c > c(k, \mu)$ be given as in Lemma 3.2.

- (1) A collection $\Delta = ({}^j\Delta^\pm)$ of $2k$ star-shaped domains of $\mathbb{P}_1(\mathbb{C})$ will be called **admissible** if:
 - ${}^j\Delta^+$ is a star-shaped domain centered at 0,
 - ${}^j\Delta^-$ is a star-shaped domain centered at ∞ .
- (2) We say that $N = ({}^jN^\sharp) \in \mathcal{A}$ (Definition 3.5) is **adapted** to an admissible collection Δ if ${}^j\Delta^\pm$ contains a disk centered at $0^{\pm 1}$ of radius at least $\frac{2A}{e^{c/k}} \exp\|N\|_{\mathcal{V}}$.

Here is the cornerstone of the construction.

Theorem 4.3. *There exists a constant $c_0 = c_0(k, \mu) \geq c(k, \mu)$ such that the upcoming statements hold for every fixed $c > c_0$. Let Δ be an admissible collection as well as some $N \in \mathcal{A}$ adapted to Δ . Take any collection $f = ({}^j f^\pm) \in \text{Holo}_{\mathfrak{b}}(\Delta)'$. There exists a constant $K = K(k, \mu) > 0$, as well as a unique collection $\Sigma(N, f) = ({}^j\Sigma^\sharp) \in \mathcal{A}$ such that the following properties hold.*

- (1) For all $j \in \mathbb{Z}/k\mathbb{Z}$ we have

$$\begin{aligned} {}^{j+1}\Sigma^{-+} - {}^j\Sigma^{+-} &= {}^j f^+ \circ {}^j_N\mathcal{H}^{+-} \text{ on } {}^j\mathcal{V}^+, \\ {}^j\Sigma^{+-} - {}^j\Sigma^{-+} &= {}^j f^- \circ {}^j_N\mathcal{H}^{-+} \text{ on } {}^j\mathcal{V}^-. \end{aligned}$$

- (2) $f \mapsto \Sigma(N, f)$ is a linear continuous map with

$$\|\Sigma(N, f)\|_{\mathcal{V}} \leq \frac{K}{c^2} \|f\|'_\Delta e^{\|N\|_{\mathcal{V}}};$$

we recall that \mathcal{V} is the collection of sectors $({}^j\mathcal{V}^\sharp)$.

- (3) ${}^j\Sigma^\sharp(u, y) = \mathcal{O}(\sqrt{u})$ as $u \rightarrow 0$.

(4) Moreover, with obvious notations:

$$\begin{aligned} \left\| u \frac{\partial \Sigma}{\partial u} \right\|_{\mathcal{V}} &\leq \frac{K}{c^2} \|f'\|_{\Delta} e^{\|N\|_{\mathcal{V}}} \left(1 + \left\| u \frac{\partial N}{\partial u} \right\|_{\mathcal{V}} \right), \\ \left\| y \frac{\partial \Sigma}{\partial y} \right\|_{\mathcal{V}} &\leq \frac{K}{c^2} \|f'\|_{\Delta} e^{\|N\|_{\mathcal{V}}} \left(1 + \left\| y \frac{\partial N}{\partial y} \right\|_{\mathcal{V}} \right), \end{aligned}$$

Proof.

(1) and 2. Here the y -variable plays the role of a parameter and is supposed to be fixed. The functions $(j\Sigma^{\sharp})$ are built by integrating $j f^{\pm} \circ_N j \mathcal{H}^{\sharp}$ against some kernel we describe below and along half-lines $j\Gamma_{\pm}^{\pm}$ bounding the sectors jV^{\pm} that are provided with the orientation $0 \rightarrow \infty$, as in Figure. 4.1. For the sake of simplicity we only deal with the case $k = 1$ (and drop the index j altogether), the general case resulting from an immediate adaptation of what has been done in [ST15, Theorem 2.5].

Being given $f = (f^+, f^-)$ meeting the hypothesis, we define

$$(4.2) \quad \begin{aligned} \Sigma^{+-}(u, y) &:= \frac{\sqrt{u}}{2i\pi} \int_{\Gamma^-} \frac{f^- \circ_N \mathcal{H}^{-+}(z, y)}{\sqrt{z}(z-u)} dz \\ &\quad - \frac{\sqrt{u}}{2i\pi} \int_{\Gamma^+} \frac{f^+ \circ_N \mathcal{H}^{+-}(z, y)}{\sqrt{z}(z-u)} dz \end{aligned}$$

$$(4.3) \quad \begin{aligned} \Sigma^{-+}(u, y) &:= \frac{\sqrt{u}}{2i\pi} \int_{\Gamma^+} \frac{f^- \circ_N \mathcal{H}^{-+}(z, y)}{\sqrt{z}(z-u)} dz \\ &\quad - \frac{\sqrt{u}}{2i\pi} \int_{\Gamma^-} \frac{f^+ \circ_N \mathcal{H}^{+-}(z, y)}{\sqrt{z}(z-u)} dz. \end{aligned}$$

Clearly these integrals are:

- well defined since N is adapted to Δ and $\left| j \mathcal{H}^{\sharp}(z, y) \right|^{\pm 1} \leq \frac{A}{c^{\epsilon}} \exp \|N\|$ if $z \in V^{\pm}$;
- absolutely convergent because of the flatness of the exponential term coming from $f^{\pm} \circ_N \mathcal{H}^{\sharp}$, since $f^{\pm}(h) = O(h)$.

Properties 1. and 2. have been established in [Tey22, Proposition 4.11]. The main point of the argument is the following: if $(u, y) \in \mathcal{V}^+$, then the Cauchy formula yields

$$\begin{aligned} \Sigma^{+-}(u, y) - \Sigma^{-+}(u, y) &= \frac{\sqrt{u}}{2i\pi} \oint_{\Gamma^- - \Gamma^+} \frac{f^- \circ_N \mathcal{H}^{-+}(z, y)}{\sqrt{z}(z-u)} dz \\ &\quad + \frac{\sqrt{u}}{2i\pi} \oint_{\Gamma^+ - \Gamma^-} \frac{f^+ \circ_N \mathcal{H}^{+-}(z, y)}{\sqrt{z}(z-u)} dz \\ &= 0 + 2i\pi \times \frac{\sqrt{u}}{2i\pi} \frac{f^+ \circ_N \mathcal{H}^{+-}(z, y)}{\sqrt{u}} \\ &= f^+ \circ_N \mathcal{H}^{+-}(u, y), \end{aligned}$$

with a similar identity involving f^- when $u \in V^-$. Of course one must apply the Cauchy formula on a compact contour and take a limit, but the flatness of the integrand ensures the process actually works out nicely.

Analogous details are dealt with in [ST15, Theorem 2.5].

3. Because ${}^j_N\mathcal{H}^\sharp$ is 1-flat at 0 and ∞ , the contribution of $\frac{1}{\sqrt{z}}$ to the integral is irrelevant: we would have shown that the integral without that term is bounded. Hence Σ is $O(\sqrt{u})$.

We deduce from this asymptotic bound the fact that Σ is unique. Indeed if $\tilde{\Sigma} \in \mathcal{A}$ is another collection satisfying 1. and $\tilde{\Sigma} = O(\sqrt{u})$, then for fixed y the sectorial functions ${}^jC^\sharp : u \mapsto {}^j\Sigma^\sharp(u, y) - {}^j\tilde{\Sigma}^\sharp(u, y)$ coincide on consecutive intersections, hence are sectorial restrictions of a holomorphic function C on \mathbb{C}^\times . Since C is bounded it extends to an entire function thanks to Riemann's theorem on removable singularity, thus a constant according to Liouville's theorem. But this constant vanishes because Σ and $\tilde{\Sigma}$ are $O(\sqrt{u})$, therefore $\Sigma = \tilde{\Sigma}$.

4. These estimates follow in exactly the same manner as their counterpart in [ST15, Theorem 2.5].

□

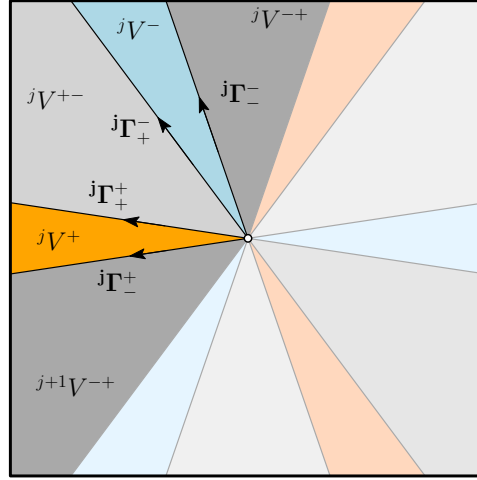


FIGURE 4.1. Contours used for the Cauchy-Heine transform.

Remark 4.4. The choice of the normalizing factor $\frac{\sqrt{u}}{\sqrt{z}}$ in the definition (4.2) of ${}^j\Sigma^\sharp$ has been done to ensure a “nice” behavior under the involution $\tau : u \mapsto \frac{1}{u}$. This has been discussed in [Tey22, Section 4.3, Proposition 4.11], and we get back to that fact in Section 6.

4.2. The fixed-point method. Let $\varphi := ({}^j\varphi^\sharp)$ be the logarithmic data associated to an orbital necklace and let Δ be an admissible collection of open discs centered at $0^{\pm 1}$ such that ${}^j\Delta^\sharp$ lies in the domain of convergence of ${}^j\varphi^\sharp$. Consider the partial map built from Theorem 4.3:

$$\Lambda := \Sigma(\bullet, \varphi) : N \mapsto \Sigma(N, \varphi) ,$$

which is defined on the space of all functions N adapted to Δ . Starting from the collection $N_0 := (0) \in \mathcal{A}$, which is of course adapted to Δ if c is large enough, we can compute $N_1 := \Lambda(N_0) \in \mathcal{A}$. We wish to iterate that map to obtain a sequence $(N_n)_n$ lying in the Banach space \mathcal{A} .

Lemma 4.5. *Let \mathcal{B} be the unit ball of \mathcal{A} . There exists $c_1 = c_1(k, \mu, \varphi) \geq c_0(k, \mu)$ such that for all $c > c_1$, every element of \mathcal{B} is adapted to Δ and Λ is a self-map of \mathcal{B} .*

This lemma ensures that the sequence $(N_n)_n$ is well defined and lies in \mathcal{B} .

Proof. Let $0 < \rho$ be less than the minimum of all the radii of convergence of $j\varphi^\sharp$. We need to ensure that, for all c and all $N \in \mathcal{B}$:

$$\frac{2A}{e^{c/k}} \exp \|N\|_{\mathcal{V}} \leq \rho,$$

where \mathcal{V} is the collection of fibered sectors $(j\varphi^\sharp)$. Assume that

$$\|N\|_{\mathcal{V}} \leq 1,$$

so that N is adapted to Δ whenever

$$c > k \ln \left(\frac{2A}{e\rho} \right).$$

Because of Theorem 4.3 2. one has

$$\|\Lambda(N)\|_{\mathcal{V}} \leq \frac{Ke}{c^2} \|\varphi\|'_{\Delta}.$$

By taking $c > \sqrt{Ke\|\varphi\|'_{\Delta}}$ we therefore ensure that $\|\Lambda(N)\|_{\mathcal{V}} \leq 1$, as expected. Hence we may define $c_1 := \max \left\{ \sqrt{Ke\|\varphi\|'_{\Delta}}, k \ln \left(\frac{2A}{e\rho} \right) \right\}$. \square

Proposition 4.6. *There exists $c_2 = c_2(k, \mu, \varphi) \geq c_1$ such that, for all $c > c_2$ the mapping Λ is a $\frac{1}{2}$ -contracting self-map of \mathcal{B} . In particular, the sequence $(N_n)_n$ converges towards the unique fixed point N of $\Lambda|_{\mathcal{B}}$.*

Proof. The existence of this bound follows immediately from [Tey22, Proposition 4.13]. Although the cited result is for the nonparametric version of Theorem 4.3 (i.e. for $\nu := 1$), it gives the conclusion here too since $|y|$ takes values bounded by 2. \square

4.3. Proof of the orbital realization. Let us summarize what we have done so far. Starting from an orbital necklace in a formal orbital class (k, μ) , we consider its logarithmic nonlinear part $\varphi = (j\varphi^\sharp)$. By taking $c > c_2$ as in Proposition 4.6, the sequence $(\Lambda^{on}(0))_{n \geq 0}$ converges towards a fixed-point N of Λ , i.e. a solution in \mathcal{A} of (\star) . In order to complete the orbital realization and recover X_R in the expected class, we wish to apply Lemma 4.1 and next to prove that $R \in \mathcal{C}_*$, where

$$\mathcal{C}_* := \left\{ F : F(x, y) = y \sum_{n=1}^{2k} u_*^n f_n(y), f_n \in \text{Hol}_{\text{ob}}(\{|y| < 2\}) \right\}.$$

For this we need the following corollary, which settles the orbital realization for 1 : 1 resonant saddles.

Corollary 4.7. *There exists $c_O = c_O(k, \mu, \varphi) \geq c_2$ such that for all $c > c_O$ the following assertions hold.*

- (1) *Each function ${}^jR^\sharp := -\frac{X_0 \cdot jN_*^\sharp}{1 + Y \cdot jN_*^\sharp}$ belongs to $\text{Holo}({}^j\mathcal{V}_*^\sharp)$. Moreover this function is locally bounded near $\{xy = 0\}$.*
- (2) *These functions are actually the restrictions to sectors of a function $R \in \mathcal{C}_*$.*

Proof. First notice that by construction ${}^jN^\sharp \in \text{Holo}_b({}^jV^\sharp \times D^\sharp)$, where $D^{+-} = \{y : |y| < 2\}$ and $D^{-+} = \{y : 1 < |y| < 2\}$. The intersection of the latter domains is the annulus

$$C := \{1 < |y| < 2\}.$$

For the sake of readability let us drop all indices. Observe that

$$\begin{aligned} x \frac{\partial N_*}{\partial x}(x, y) &= u \frac{\partial N}{\partial u}(u(x, y), y) \\ y \frac{\partial N_*}{\partial y}(x, y) &= u \frac{\partial N}{\partial u}(u(x, y), y) + \frac{\partial N}{\partial y}(u(x, y), y). \end{aligned}$$

In particular $Y \cdot N_* = \left(y \frac{\partial N}{\partial y}\right)_*$ and $X_0 \cdot N_* = \left(u^{k+1} \frac{\partial N}{\partial u} + y(1 + \mu u^k) \frac{\partial N}{\partial y}\right)_*$.

- (1) Since $\|N\|_{\mathcal{V}} \leq 1$ (Lemma 4.5), the estimates of Theorem 4.3 4. imply that

$$\frac{\left\|y \frac{\partial N}{\partial y}\right\|_{\mathcal{V}}}{1 + \left\|y \frac{\partial N}{\partial y}\right\|_{\mathcal{V}}} \leq \frac{Ke}{c^2} \|\varphi'\|_{\Delta}.$$

If c is large enough, then the right-hand side is smaller than $\frac{1}{3}$ so that

$$\left\|y \frac{\partial N}{\partial y}\right\| \leq \frac{1}{2}.$$

Therefore $Y \cdot N_*$ has a \mathcal{V}_* -norm less than $\frac{1}{2}$ and $1 + Y \cdot N_*$ cannot vanish on ${}^j\mathcal{V}_*^\sharp$. Moreover the latter modulus is bounded from below by $\frac{1}{2}$, which gives the conclusion.

- (2) The fact that the sectorial functions ${}^jR^\sharp$ glue to some $R \in \text{Holo}(\mathcal{V}_*)$ has been explained in Lemma 4.1. The construction makes clear that $R = Q_*$ where

$$Q := \frac{u^{k+1} \frac{\partial N}{\partial u} + (c(1 - u^{2k}) + \mu u^k) y \frac{\partial N}{\partial y}}{1 + y \frac{\partial N}{\partial y}} \in \text{Holo}(\mathbb{C} \times C).$$

Let us bound the growth of $u \mapsto Q(u, y)$ as $u \rightarrow \infty$ with $y \in C$ fixed. We have:

$$\begin{aligned} |Q(u, y)| &\leq 2|u|^k \left|u \frac{\partial N}{\partial u}(u, y)\right| + 2(c|1 - u^{2k}| + \mu|u|^k) \left|y \frac{\partial N}{\partial y}(u, y)\right| \\ &\leq |u|^k \frac{2Ke}{c^2} \|\varphi'\|_{\Delta} + c(1 + |u|^{2k}) + |\mu| |u|^k. \end{aligned}$$

As a conclusion $u \mapsto Q(u, y)$ is $\mathcal{O}(u^{2k})$, hence $Q \in \text{Holo}(C)[u]_{\leq 2k}$.

So far we have reached the point where $R(x, y) = \sum_{n=0}^{2k} f_n(y)(xy)^n$ for some functions $f_n \in \text{Holo}_b(C)$. But the construction actually yields a function

R which is holomorphic on \mathcal{V}_* , and this domain contains in addition to $\mathbb{C} \times \mathbb{C}$ a part corresponding to $\{(x, y) \in \mathcal{V}_* : |y| < 2 \text{ and } u \in {}^j V^{+-}\}$. For instance if we fix $x := 1$ then $R(1, y)$ is holomorphic on the sector $V_0 := \{0 < |y| < 2, \left|k \arg y - \pi \frac{2j+1}{2}\right| < \frac{5\pi}{8}\}$. In particular each f_n is holomorphic and bounded on $C \cup V_0$. But as $x := e^{i\theta}$ for $\theta \in [0, 2\pi]$ moves along the unit circle, we deduce that $y \mapsto R(e^{i\theta}, y)$, and thus each f_n , is holomorphic on every sector $V_\theta := \{0 < |y| < 2, \left|k \arg y + k\theta - \pi \frac{2j+1}{2}\right| < \frac{5\pi}{8}\}$, whose union covers $\{0 < |y| < 2\}$. As a conclusion f_n is holomorphic and bounded on the whole punctured disc $\{0 < |y| < 2\}$, therefore $f_n \in \text{Hol}_b(\{|y| < 2\})$.

Because $R = -X_R \cdot N_*$ (Lemma 3.4) and since $\frac{\partial N}{\partial u}(u, y) = O(y)$ in the sector ${}^0 V^{+-} \times D^{+-}$ (see 4.2), we may take $y := 0$ for fixed u to conclude $Q(u, 0) = 0$, which shows $f_n(0) = 0$.

In a similar fashion we wish to evaluate $Q(0, y) = c \frac{y \frac{\partial N}{\partial y}}{1 + y \frac{\partial N}{\partial y}}(0, y)$. But from Theorem 4.3 3. we have $y \frac{\partial N}{\partial y}(u, y) = O(\sqrt{u})$ as $u \rightarrow 0$, which finally gives $Q(0, y) = 0$. This completes the proof. \square

4.4. The case of a $p : q$ resonant saddle. The general case of a $p : q$ saddle foliation follows exactly the same steps as for $1 : 1$ saddles, and most of the necessary results hold *verbatim*. The main modification is the necessity to use a convenient version of the model first-integral, namely:

$$H(u, y) = y^{1/q} u^{-p/q} \exp \frac{c(u^{-k} + u^k)}{qk}$$

$$u_* = x^q y^p.$$

The fact that H is multivalued in the y -variable is not an obstacle since the pull-back sectors ${}^j \mathcal{V}_*^\sharp$ are simply connected.

All subsequent arguments work in the same way.

5. PERIOD OPERATOR AND ITS NATURAL SECTION

This section is devoted to the proof of the Cohomological Theorem. From now on we work within the domain $\mathcal{U}_* := \mathbb{C} \times \{y : |y| < 2\}$ on which we consider a holomorphic vector field in normal form

$$X_R := xu_*^k \frac{\partial}{\partial x} + (1 + \mu u_* + R) Y$$

build in the previous section (Corollary 4.7). We denote by

$${}^j \mathcal{U}^\sharp := \{(x, y) \in \mathcal{U}_* : u(x, y) \in {}^j V^\sharp, |y| < 2\}$$

the corresponding sectors and their pairwise intersections ${}^j \mathcal{U}^\pm$. We also let \mathcal{F}_R stand for the induced holomorphic foliation on \mathcal{U}_* .

Being given a germ G of a holomorphic function at $(0,0)$, that is $G \in \mathbb{C}\{x,y\}$, we outline how to solve

$$(5.1) \quad X_R \cdot F = G$$

in order to provide the sectorial normalization of Lemma 3.4 with a more geometrical flavor, since the functions ${}^jN^\sharp$ involved in the sectorial normalization of X_R are solutions of the equation $X_R \cdot N = -R$.

Theorem 5.1. *Consider the cohomological equation (5.1) with $G \in \mathbb{C}\{x,y\}$.*

- (1) *There exists a formal solution $F \in \mathbb{C}[[x,y]]$ if and only if the Taylor expansion of G at $(0,0)$ does not contain terms u^n for $n \in \{0, \dots, k\}$. We write $\mathbb{C}\{x,y\}_{>k}$ the space of all such germs.*
- (2) *There exists a neighborhood Ω of $(0,0)$ on which G is holomorphic and bounded, such that to each $(x,y) \in \Omega \cap {}^j\mathcal{U}^\sharp$ one can attach a path*

$${}^j\gamma^\sharp(x,y) : [0, +\infty[\rightarrow \Omega \cap {}^j\mathcal{U}^\sharp$$

tangent to X_R that starts at (x,y) and accumulates on $(0,0)$.

- (3) *The parametric integral ${}^jF^\sharp := \int_{{}^j\gamma^\sharp} G \frac{du}{u^{k+1}}$ is convergent and defines a holomorphic function ${}^jF^\sharp \in \text{Holo}_b(\Omega \cap {}^j\mathcal{U}^\sharp)$ solving (5.1) if and only if $G \in \mathbb{C}\{x,y\}_{>k}$.*

The proof of this result is given in Section 5.1.

Remark 5.2. We wish to underline that 3. is another instance of a phenomenon observed in [Tey04b] for saddle-node vector fields: if η is a meromorphic time-form of a vector field X (that is $\eta(X) = 1$), then integrals of $G\eta$ along asymptotic paths converge if and only if $X \cdot F = G$ admits a formal solution F . We do not know if it is true when X is a quasi-resonant saddle (irrational eigenratio) or has a nonreduced singularity at $(0,0)$.

The difference of two consecutive sectorial solutions ${}^jF^\sharp$ is a first-integral of X_R over the pairwise intersections of the corresponding sectors, and the fact that they do not agree measure how far they are from being the restriction of a holomorphic function on Ω . (Indeed if those bounded functions were to agree on all intersections, then we would apply Riemann's theorem on removable singularity.) Besides, each such pairwise difference factors through ${}^j_N\mathcal{H}_*^\sharp$, as explained below.

Lemma 5.3. *Any first-integral $\phi \in \text{Holo}(\Omega \cap {}^j\mathcal{U}^\pm)$ of X_R factors through ${}^j_N\mathcal{H}_*^{\pm\mp}$: there exists $f \in \text{Holo}({}^j\Delta^\pm)$ such that*

$$\phi = f \circ {}^j_N\mathcal{H}_*^{\pm\mp},$$

where ${}^j\Delta^\pm := {}^j_N\mathcal{H}_^{\pm\mp}(\Omega \cap {}^j\mathcal{U}^\pm)$. Moreover if ϕ is bounded, then f also is.*

Proof. The function ϕ is constant on the leaves of \mathcal{F}_R and therefore defines a holomorphic function on its space of leaves. The conclusion follows from the fact that ${}^j_N\mathcal{H}_*^{\pm\mp}$ is a coordinate on the corresponding sectorial leaf space, as guaranteed by Lemma 3.4 (maybe at the expense of reducing the size of Ω). \square

Therefore one can build a linear operator

$$\tilde{\mathcal{T}}_R : G \in \mathbb{C}\{x,y\}_{>k} \mapsto ({}^j f^\pm)$$

such that

$$\begin{aligned} j_{F^{+-}} - j_{F^{-+}} &= j_{f^-} \circ j_N \mathcal{H}_*^{-+} \\ j^{+1} F^{-+} - j_{F^{+-}} &= j_{f^+} \circ j_N \mathcal{H}_*^{+-}. \end{aligned}$$

Definition 5.4. The previous operator \mathfrak{T}_R is called the **period operator** associated to X_R .

Remark 5.5.

- (1) The value of the period $j\mathfrak{T}_R^\pm(G)(h)$ is obtained by computing the integral $\int_{j\gamma^\pm(h)} \frac{G}{x} \frac{dx}{x}$, where $j\gamma^\pm(h)$ is an asymptotic cycle obtained by the concatenation of the asymptotic tangent paths described in Theorem 5.1 2. passing through $(x, y) \in j\mathcal{U}^\pm$ and such that $h = j_N \mathcal{H}^{\pm\mp}(x, y)$.
- (2) As it has been already pointed out above, the formal solution of $X_R \cdot F = G$ converges if and only if $\mathfrak{T}_R(G) = 0$.
- (3) When $y := 0$ resp. $x := 0$, the formal solution $F(x, 0)$ resp. $F(0, y)$ is simply given by integrating $-pcx \frac{\partial}{\partial x} F(x, 0) = G(x, 0)$ resp. $qcy \frac{\partial}{\partial y} F(0, y) = G(0, y)$, hence it is analytic on $\{xy = 0\}$. As a consequence $j_{f^\pm}(0^{\pm 1}) = 0$.

Corollary 5.6. Let a formal class (k, μ, P) be given. The complete modulus $(j\psi^\pm, jf^\pm)$ of $Z_{G,R}$ can be expressed as periods along X_R , namely:

$$\begin{aligned} (j\varphi^\pm) &= \mathfrak{T}_R(-R) \\ (jf^\pm) &= \mathfrak{T}_R\left(\frac{1}{G} - \frac{1}{P_*}\right) \end{aligned}$$

where $j\psi^\pm = \sigma \text{Id} \exp j\varphi^\sharp$.

Proof. This is a direct consequence of what has been explained previously. \square

The technique used to carry out the realization of an orbital necklace in Section 4 actually allows us to show that \mathfrak{T}_R admits an explicit section. This is done by reusing the refined Cauchy-Heine transform presented in Theorem 4.3.

Theorem 5.7. Consider the admissible collection Δ defined by $\left(j_N \mathcal{H}^{\pm\mp}(j\mathcal{U}^\pm)\right)$ and let a collection $f = (jf^\pm) \in \text{Holo}_b(\Delta)'$ be given. Then there exists $\mathfrak{S}_R(f) \in \mathcal{C}_*$ (the function space is defined in Section 4.3) such that

$$\mathfrak{T}_R(\mathfrak{S}_R(f)) = f.$$

The proof of this result is postponed until Section 5.2.

Remark 5.8. In particular $\mathfrak{T}_R : \mathcal{C}_* \longrightarrow \text{Holo}_b(\Delta)'$ is an isomorphism of Banach spaces, with inverse its **natural section** \mathfrak{S}_R . The estimates given in Theorem 4.3 allow to give explicit bound on their norms.

We finally are able to establish the temporal realization of the Main Theorem.

Corollary 5.9. Let U be a holomorphic unit. Every vector field UX_R is analytically conjugate to a unique $Z_{G,R}$ with $G \in \mathcal{C}_*$.

Proof. Let U be given and consider the temporal normalizing equation $X_R \cdot \widehat{T} = \frac{1}{U} - \frac{1}{P}$, which admits a formal solution if P_* is given by the projection of the Taylor series of U on the space $\mathbb{C}[u_*]_{\leq k}$. According to the previous theorem, we can find $G \in \mathcal{C}_*$ such that $\mathcal{T}_R(G) = \mathcal{T}_R\left(\frac{1}{U} - \frac{1}{P_*}\right)$. Then the cohomological equation $X_R \cdot T = \frac{1}{U} - \frac{1}{V}$ admits an analytic solution T , where $\frac{1}{V} := \frac{1}{P_*} + G$, and that implies UX_R is analytically conjugate to $VX_R = Z_{G,R}$. \square

5.1. Cohomological equations and their sectorial solutions: proof of Theorem 5.1.

Let us consider equation (5.1).

- (1) Because $Y \cdot (x^a y^b) = (bq - ap)x^a y^b$ and $X_R = u^k x \frac{\partial}{\partial x} + (1 + \mu u^k + R)Y$, no terms u^n may belong to the image of X_R if $n \leq k$. Conversely, the coefficients of F can be computed recursively by looking at larger and larger homogeneous degrees $a + b$ of monomials $x^a y^b$ appearing in the Taylor expansion of G . Details are left to the reader.
- (2) Let (x_*, y_*) be fixed in a sector ${}^j\mathcal{U}^\sharp$. It is well known that the holonomy of \mathcal{F}_{X_R} , computed on $\{y = y_*\}$ and obtained by lifting the path $(0, y(t)) = (0, y_* e^{it})$ into the leaves of the foliation, is a nonlinearizable parabolic germ tangent to $x \mapsto e^{2i\pi p/q} x$. Starting close enough to 0, its forward or backwards orbit (the direction $t \rightarrow \pm\infty$ depending on the sector ${}^j\mathcal{U}^{+-}$ or ${}^j\mathcal{U}^{-+}$) accumulates on 0. The tangent path that is obtained this way can be deformed within its supporting leaf \mathcal{L} to land on $(0, 0)$. Indeed on a sufficiently small domain $\Omega \ni (0, 0)$, the leaf $\mathcal{L} \cap \Omega$ is very close to the set of level $h := y_* \exp \frac{cx_*^{-kq} y_*^{-kp}}{k}$ of $y \exp \frac{cu^{-k}}{k}$, and that implicit relation can be inverted as

$$x(t) = \frac{y(t)^{-p/q}}{\left(\frac{c}{k} \log \frac{h}{y(t)}\right)^{1/kq}}.$$

Each time $y(t)$ makes a turn around 0, the amplitude $|x(t)|$ is multiplied with a factor of order about $\frac{1}{2\pi^{1/kq}}$. By letting $|y(t)|$ goes to 0 in a slower fashion than $y(t)$ winds around 0, we obtain a path that accumulates on $(0, 0)$, e.g. by choosing $0 < \alpha < \frac{1}{pk}$ and setting

$$y(t) := y_* (1+t)^\alpha e^{it}.$$

Hence

$$\begin{aligned} x(t) &\sim_{t \rightarrow \infty} \text{cst} \times t^{-\alpha p/q - 1/kq} e^{-it p/q} \\ u(t) &\sim_{t \rightarrow \infty} \text{cst} \times t^{-1/k}. \end{aligned}$$

One easily checks that the image of the lifted path γ remains in the given u -sector since

$$\arg u(t) = -\frac{1}{kq} \arg(\text{cst} + \alpha \ln(1+t) - it) \longrightarrow_{t \rightarrow \infty} \pm \frac{\pi}{2kq}.$$

- (3) According to the previous computations, we know that along the tangent path γ we have

$$\gamma^* \frac{du}{u^{k+1}} \sim_{t \rightarrow \infty} \text{cst} \times dt.$$

Integrating a monomial $x^a y^b$ along this path therefore yields the estimate

$$\gamma^* \left(x^a y^b \frac{du}{u^{k+1}} \right) \sim_{t \rightarrow \infty} \text{cst} \times t^{\frac{\alpha}{q}(ap-bq) - \frac{a}{kq}} \times e^{-iqt(ap-bq)} dt.$$

There are two cases to consider.

- Either $ap = bq$, that is $x^a y^b = u^n$, and $\gamma^* \left(x^a y^b \frac{du}{u^{k+1}} \right) \sim_{\infty} t^{-n/k}$ which is integrable if and only if $n > k$. In that case the integral is absolutely convergent.
- Or $ap \neq bq$ and $\gamma^* \left(x^a y^b \frac{du}{u^{k+1}} \right)$ is conditionally integrable by Dirichlet's test.

Therefore ${}^j F^\sharp(x_*, y_*)$ is a convergent integral if and only if $G \in \mathbb{C}\{x, y\}_{>k}$. It is clearly locally analytic in (x_*, y_*) . The fact that it is holomorphic on $\Omega \cap {}^j \mathcal{U}^\sharp$ comes from the fact that if Ω is small enough, then every leaf of $\mathcal{F}_{X_R}|_{{}^j \mathcal{U}^\sharp}$ is simply connected according to the incompressibility result of e.g. [Tey15, Proposition 3.1]. \square

5.2. Natural section of the period operator: proof of Theorem 5.7. We use Theorem 4.3 by taking N as the sectorial normalizations of X_R and $f = ({}^j f^\sharp) \in \mathcal{A}$: we obtain sectorial functions $({}^j F^\sharp) := \Sigma(N, f)$ solving the period Cousin problem. Define now

$$G := X_R \cdot {}^j F^\sharp.$$

We conclude the proof by invoking the same arguments as in the proof of Corollary 4.7 2.

- Because by construction the difference of consecutive ${}^j F^\sharp$ is a first integral of X_R , the function G is independent on the sector.
- It is moreover locally bounded near $\{xy = 0\}$, so that it extends to a function holomorphic on \mathcal{V}_* by Riemann's removable singularity theorem.
- Then for every fixed $y \in \{1 < |y| < 2\}$ we have $G(x, y) = O(u^{2k})$ as $u \rightarrow \infty$, i.e. $G(x, y) = \sum_{n=0}^{2k} u^n f_n(y)$.
- Moreover $f_0(y) = 0$ because ${}^j F^\sharp$ is $O(\sqrt{u})$ as $u \rightarrow 0$.
- Each function f_n extends holomorphically to the disc $\{|y| < 2\}$ because of the shape of \mathcal{V}_* .

Therefore $G \in \mathcal{C}_*$ as expected. \square

6. ISOTROPY OF RESONANT FOLIATIONS AND UNIQUENESS OF THE NORMAL FORM

As an application of the material introduced previously, we explain how we deduce from the description of the cokernel of the period operator that the vector fields obtained in the Normalization Theorem are essentially unique in a given analytical class. Said differently, we wish to prove that the automorphisms of the versal family $\{Z_{G,R} : G, R \in \mathcal{C}_*\}$ are only given by the linear mappings $(x, y) \mapsto (\alpha x, \beta y)$ with $(\alpha^q \beta^p)^k = 1$. We mainly rely on the following structure theorem, which describes the isotropy group of resonant foliations.

Theorem 6.1. *Take two vector fields $Z = U(X_0 + RY)$ and $\tilde{Z} = \tilde{U}(X_0 + \tilde{R}Y)$, not necessarily in normal form. If Ψ is a conjugacy between Z and \tilde{Z} , say $\Psi^* Z = \tilde{Z}$, then it*

can be factored as

$$\Psi = \Lambda \circ \mathcal{N} \circ \mathcal{T}$$

where:

- $\Lambda : (x, y) \mapsto (\alpha x, \beta y)$ with $(\alpha^q \beta^p)^k = 1$;
- the diffeomorphism $\mathcal{N} \in \text{Diff}(\mathbb{C}^2, 0)$ preserves the resonant monomial, that is $\mathcal{N} = \Phi_Y^N = (xe^{-pN}, ye^{qN})$ for some germ N at $(0, 0)$ of a holomorphic function;
- the diffeomorphism $\mathcal{T} \in \text{Diff}(\mathbb{C}^2, 0)$ sends a leaf of $\mathcal{F}_{\bar{Z}}$ within itself (it is a tangential isotropy of $\mathcal{F}_{\bar{Z}}$), that is $\mathcal{T} = \Phi_Z^T$ for some holomorphic germ T .

We prove this result in Section 6.1 below.

Remark 6.2.

- (1) Even without applying the Normalization Theorem, it is known since Dulac's works [Dul09, Dul04] that any resonant saddle vector field can be written $U(X_0 + RY)$ in a convenient local analytic chart about $(0, 0)$.
- (2) A similar result was obtained in [BCM99], although the foliations were presented in a different fashion and the fixed fibration was $\{x = \text{cst}\}$ instead of the singular fibration $\{u = \text{cst}\}$ considered here.

In order to deduce the uniqueness statement of the Normalization Theorem from Theorem 6.1, we only need to show that $\mathcal{N} = \mathcal{T} = \text{Id}$ when $Z = Z_{G,R}$ and $\bar{Z} = Z_{\bar{G},\bar{R}}$ are in normal form. There exists U a holomorphic unit such that $X_R \circ \mathcal{N} = UX_{\bar{R}} \cdot \mathcal{N}$. Let us write this relation in the basis $(u^k x \frac{\partial}{\partial x}, Y)$:

$$\begin{cases} u^{k+1} & = U \times u^{k+1} \\ c(1 - u^{2k}) + \mu u^k + R(u, ye^{qN}) & = U \times (c(1 - u^{2k}) + \mu u^k + \bar{R}(u, y) + X_{\bar{R}} \cdot N) \end{cases}$$

We deduce from this system that $U = 1$ on the one hand, while on the other hand we have

$$X_{\bar{R}} \cdot N = R(u, ye^{qN}) - y\bar{R}(u, y) =: L.$$

Proposition 6.3. $N \in y\mathbb{C}\{y\}$.

We postpone the proof of this result till Section 6.2; in the meantime it tells us that $L \in \mathbb{C}[u]_{\leq 2k}\{y\}_{>0}$. Since N is holomorphic we can assert that $\mathcal{T}_R(L) = 0$ according to Cohomological Theorem 2.(a). But the item 2.(b) of the same theorem implies that $L = 0$, which in turn implies $N = \text{cst}$ and $R = \bar{R}$, as expected.

To conclude the uniqueness of the normal form, we apply again the previous argument: since $Z_{G,R}$ and $Z_{\bar{G},\bar{R}}$ are conjugate by \mathcal{T} , there exists a holomorphic germ T such that $X_R \cdot T = G - \bar{G} \in \mathbb{C}[u]_{\leq 2k}\{y\}_{>0}$, so that $G = \bar{G}$. This concludes the proof of the Normalization Theorem.

6.1. Proof of Theorem 6.1. Since Λ can be read in the linear part of Ψ (which must be diagonal thanks to the form of X_0), we may as well assume that Ψ is tangent to the identity. Let $(j\Psi^\#)$ resp. $(j\bar{\Psi}^\#)$ be the collection of sectorial normalizations of $X := X_0 + RY$ resp. $\bar{X} := X_0 + \bar{R}Y$ given in Lemma 4.3. By assumption Ψ induces the identity between the orbital necklaces of \mathcal{F}_Z and $\mathcal{F}_{\bar{Z}}$, hence $(j\bar{\Psi}^\#)^{\circ-1} \circ j\Psi^\# =: \mathcal{N}$ does not depend on the sector, and therefore extend as an

element $\mathcal{N} \in \text{Diff}(\mathbb{C}^2, 0)$ such that $\mathcal{N}^*X = \widetilde{X}$. But by construction \mathcal{N} fixes the monomial u .

Consider now $\mathcal{T} := \mathcal{N}^{\circ-1} \circ \Psi$, which is a conjugacy between $(U \circ \mathcal{N})X_{\overline{R}}$ and $\widetilde{U}X_{\overline{R}}$, that sends a leaf of $\mathcal{F}_{\overline{X}}$ into itself. Following the arguments of [BCM99] we deduce that \mathcal{T} has the expected form. \square

6.2. Proof of Proposition 6.3. The argument takes place in the variables (u, y) . Indeed, as it has been remarked in Section 1.5, the foliations \mathcal{F}_{X_R} and $\mathcal{F}_{X_{\overline{R}}}$ correspond to foliations \mathcal{F} and $\widetilde{\mathcal{F}}$ in (u, y) -space given by the differential 1-forms ω_R and $\omega_{\overline{R}}$ defined in (1.5). Because \mathcal{N} fixes the resonant monomial, it induces an orbital conjugacy $\Phi : (u, y) \mapsto (u, y\phi(u, y))$ between ω_R and $\omega_{\overline{R}}$. Using a construction *à la* Mattei-Moussu (path-lifting technique), Φ extends holomorphically to a neighborhood of $\{y = 0\} \simeq \mathbb{C}$. The key point is to extend it to a neighborhood of $\mathbb{P}_1(\mathbb{C})$ by using a suitable compactification.

The foliations \mathcal{F} and $\widetilde{\mathcal{F}}$ can be extended to $\mathbb{P}_1(\mathbb{C}) \times D$, where D is the disc $\{|y| < 2\}$. In the chart $(z, y) = (\frac{1}{u}, y)$, the 1-form ω_R is written

$$z^{-k-1} dy - y \left(c(1 - z^{-2k}) + \mu z^{-k} + R(z^{-1}, y) \right) z^{-2} dz,$$

which becomes holomorphic after multiplication with z^{2k+2} :

$$\omega_{R^*}(z, y) = z^{k+1} dy + y \left(c(1 - z^{2k}) - \mu z^k + R^*(z, y) \right) dz$$

where

$$R^*(z, y) = -z^{2k} R(z^{-1}, y) = -y \sum_{n=1}^{2k} f_n(y) z^{2k-n}.$$

The singularity of \mathcal{F} and $\widetilde{\mathcal{F}}$ at $(\infty, 0)$ is again a saddle-node, with formal orbital class $(k, -\mu)$. The diffeomorphism Φ induces a conjugacy between their weak holonomies computed on a transversal $\{u = \text{cst}\}$ close to $\{\infty\} \times D$, but unlike for the case of resonant saddles this does not automatically imply that \mathcal{F} and $\widetilde{\mathcal{F}}$ are Φ -conjugate on a full neighborhood of $(\infty, 0)$. Fortunately that fact is guaranteed by the construction of the normal form, as already hinted at in Remark 4.4.

Lemma 6.4. *Consider the involution τ of $\mathbb{Z}/k\mathbb{Z} \times \{+, -\}$ defined by $\tau(j, \pm) := (k-1-j, \mp)$, and let it act on collections $(j f^\pm)_{j \in \mathbb{Z}/k\mathbb{Z}}$ in the natural way. Then we have*

$$\mathfrak{I}_{R^*}(R^*) = -\tau^* \mathfrak{I}_R(R).$$

Proof. The action of τ on the indices (j, \pm) corresponds to the action of $\tau : u \mapsto \frac{1}{u}$ on the sectors in the u -variable, i.e. the sector ${}^j V^{\pm\mp}$ is sent to ${}^{k-1-j} V^{\mp\pm} =: \tau^* {}^j V^\sharp$ by the transform. For every component Γ of $\partial^j V^\sharp$ we compute:

$$I_\Gamma\left(\frac{1}{u}, y\right) = \frac{\sqrt{\frac{1}{u}}}{2i\pi} \int_\Gamma \frac{{}^j f^\pm \circ {}^j_N \mathcal{H}^{\pm\mp}(z, y)}{\sqrt{z}\left(z - \frac{1}{u}\right)} dz = \frac{\sqrt{u}}{2i\pi} \int_\Gamma \frac{{}^j f^\pm \circ {}^j_N \mathcal{H}^{\pm\mp}(z, y)}{\sqrt{z}(uz - 1)} dz$$

and perform the change of variable $w := \tau(z)$. This change of variable reverses the orientation of the image half-line $\tau^*\Gamma \subset \partial\tau^*jV^\sharp$, therefore:

$$\begin{aligned} I_\Gamma\left(\frac{1}{u}, y\right) &= -\frac{\sqrt{u}}{2i\pi} \int_{-\tau^*\Gamma} \frac{jf^\pm \circ j_N \mathcal{H}^{\pm\mp}\left(\frac{1}{w}, y\right) dw}{\sqrt{\frac{1}{w}}\left(\frac{u}{w} - 1\right) w^2} \\ &= -\frac{\sqrt{u}}{2i\pi} \int_{\tau^*\Gamma} \frac{jf^\pm \circ j_N \mathcal{H}^{\pm\mp}\left(\frac{1}{w}, y\right)}{\sqrt{w}(w-u)} dw. \end{aligned}$$

But:

$$j_N \mathcal{H}^{\pm\mp}\left(\frac{1}{w}, y\right) = \widehat{j_0 \mathcal{H}^{\pm\mp}}(w, y) \exp jN^{\pm\mp}\left(\frac{1}{w}, y\right) = \widehat{k^{-1-j} \tau^* N^{\mp\pm}}(w, y),$$

where τ^*N is the collection defined by $k^{-1-j}N^{\mp\pm}(u, y) := jN^{\pm\mp}\left(\frac{1}{u}, y\right)$ and $\widehat{j_0 \mathcal{H}^\sharp}$ is the model first-integral where μ is replaced by $-\mu$. Finally:

$$I_\Gamma\left(\frac{1}{u}, y\right) = -\widehat{I_{\tau^*\Gamma}}(u, y)$$

where in the right-hand side we have replaced μ by $-\mu$. The conclusion follows from 4.2 as the collection N is obtained as a fixed-point of the Cauchy-Heine operator built from a linear combination of terms I_Γ . \square

According to the discussion performed in Section 1.4, more precisely the period presentation of the orbital modulus laid out in Corollary 5.6, we deduce that the following chain of identities holds:

$$\begin{aligned} \widetilde{\Upsilon}_{\widetilde{R}^*}(\widetilde{R}^*) &= -\tau^* \widetilde{\Upsilon}_{\widetilde{R}}(\widetilde{R}) \\ &= -\tau^* \Upsilon_R(R) = \Upsilon_{R^*}(R^*) \end{aligned}$$

since \mathcal{F} and $\widetilde{\mathcal{F}}$ are locally conjugate near $(0,0)$. Hence, \mathcal{F} and $\widetilde{\mathcal{F}}$ have same orbital modulus at $(\infty,0)$ and are conversely locally conjugate near $(\infty,0)$. It is well known that there exists a unique conjugacy which is fibered in the u -variable (see e.g. [MR82, Tey04a]), therefore Φ extends as a fibered diffeomorphism $(u, y) \mapsto \Phi(u, y)$ on the whole $\mathbb{P}_1(\mathbb{C}) \times D$. Because $\Phi(u, y) = \left(u, y \sum_{n \geq 0} \phi_n(u) y^n\right)$ this means that each function ϕ_n extends holomorphically as an entire and bounded function of u , hence a constant. This completes the proof of Proposition 6.3. \square

REFERENCES

- [BCM99] M. Berthier, D. Cerveau, and R. Meziani. Transformations isotropes des germes de feuilletages holomorphes. *J. Math. Pures Appl.* (9), 78(7):701–722, 1999.
- [BL97] Michel Berthier and Frank Loray. Cohomologie relative des formes résonnantes non dégénérées. *Asymptot. Anal.*, 15(1):41–54, 1997.
- [Bru83] A. Bruno. Analytic invariants of a differential equation. *Soviet Math. Dokl.*, 28(3):691–695, 1983.
- [Bru89] A. Bruno. *Local methods in nonlinear differential equations*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1989. Part I. The local method of nonlinear analysis of differential equations. Part II. The sets of analyticity of a normalizing transformation, Translated from the Russian by William Hovingh and Courtney S. Coleman, With an introduction by Stephen Wiggins.
- [BT99] M. Berthier and F. Touzet. Sur l'intégration des équations différentielles holomorphes réduites en dimension deux. *Bol. Soc. Brasil. Mat. (N.S.)*, 30(3):247–286, 1999.

- [Dul04] H. Dulac. Recherches sur les points singuliers des équations différentielles. *Journal de l'École Polytechnique*, 1904.
- [Dul09] H. Dulac. Sur les points singuliers d'une équation différentielle. *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys.* (3), 1:329–379, 1909.
- [Éca85] J. Écalle. *Les fonctions résurgentes. Tome III*, volume 85 of *Publications Mathématiques d'Orsay [Mathematical Publications of Orsay]*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1985. L'équation du pont et la classification analytique des objets locaux. [The bridge equation and analytic classification of local objects].
- [Eca05] Jean Ecalle. Twisted resurgence monomials and canonical-spherical synthesis of local objects. In *Analyzable functions and applications*, volume 373 of *Contemp. Math.*, pages 207–315. Amer. Math. Soc., Providence, RI, 2005.
- [Eli93] P. Elizarov. Tangents to moduli maps. In *Nonlinear Stokes phenomena*, volume 14 of *Adv. Soviet Math.*, pages 107–138. Amer. Math. Soc., Providence, RI, 1993.
- [Lor04] F. Loray. Versal deformation of the analytic saddle-node. *Astérisque*, (297):167–187, 2004. Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes. II.
- [LR16] Michèle Loday-Richaud. *Divergent series, summability and resurgence. II*, volume 2154 of *Lecture Notes in Mathematics*. Springer, [Cham], 2016. Simple and multiple summability, With prefaces by Jean-Pierre Ramis, Éric Delabaere, Claude Mitschi and David Sauzin.
- [MR82] J. Martinet and J.-P. Ramis. Problèmes de modules pour des équations différentielles non linéaires du premier ordre. *Inst. Hautes Études Sci. Publ. Math.*, (55):63–164, 1982.
- [MR83] J. Martinet and J.-P. Ramis. Classification analytique des équations différentielles non linéaires résonnantes du premier ordre. *Ann. Sci. École Norm. Sup.* (4), 16(4):571–621, 1983.
- [RT21] Christiane Rousseau and L Teyssier. Analytic normal forms and inverse problems for unfoldings of 2-dimensional saddle-nodes with analytic center manifold. *Annales Scientifiques de l'ENS*, 54(1):133–233, 2021.
- [Sau09] D. Sauzin. Mould expansions for the saddle-node and resurgence monomials. In *Renormalization and Galois theories*, volume 15 of *IRMA Lect. Math. Theor. Phys.*, pages 83–163. Eur. Math. Soc., Zürich, 2009.
- [ST15] Reinhard Schäfke and L Teyssier. Analytic normal forms for convergent saddle-node vector fields. *Annales de l'Institut Fourier*, 65(3):933–974, 2015.
- [Tey04a] L. Teyssier. Analytical classification of singular saddle-node vector fields. *J. Dynam. Control Systems*, 10(4):577–605, 2004.
- [Tey04b] L. Teyssier. Équation homologique et cycles asymptotiques d'une singularité nœud-col. *Bull. Sci. Math.*, 128(3):167–187, 2004.
- [Tey04c] L. Teyssier. Examples of non-conjugated holomorphic vector fields and foliations. *J. Differential Equations*, 205(2):390–407, 2004.
- [Tey15] L Teyssier. Germes de feuilletages présentables du plan complexe. *Bulletin of Brazilian Mathematical Society*, 46(2):275–329, 2015.
- [Tey22] L Teyssier. Spherical normal forms for germs of parabolic line biholomorphisms. 2022.
- [VG96] S. Voronin and A. Grintchy. An analytic classification of saddle resonant singular points of holomorphic vector fields in the complex plane. *J. Dynam. Control Systems*, 2(1):21–53, 1996.
- [VM03] S. Voronin and Y. Meshcheryakova. Analytic classification of germs of holomorphic vector fields with a degenerate elementary singular point. *Vestnik Chelyab. Univ. Ser. 3 Mat. Mekh. Inform.*, (3(9)):16–41, 2003.