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# A FOUR-DIMENSIONAL COUSIN OF THE SEGRE CUBIC

LAURENT MANIVEL

ABSTRACT. This note is devoted to a special Fano fourfold defined by a four-dimensional space of skew-symmetric forms in five variables. This fourfold appears to be closely related with the classical Segre cubic and its Cremona-Richmond configuration of planes. Among other exceptional properties, it is infinitesimally rigid and has Picard number six. We show how to construct it by blow-up and contraction, starting from a configuration of five planes in a four-dimensional quadric, compatibly with the symmetry group  $\mathcal{S}_5$ . From this construction we are able to describe the Chow ring explicitly.

*Dedicated to the memory of Laurent Gruson*

## 1. INTRODUCTION

Fano threefolds were classified more than forty years ago, after some fifty years of efforts. The classification of Fano fourfolds is still elusive and will probably remain so for a long time. There are many ways to construct such manifolds, and a systematic study was launched a few years ago, of those that can be constructed from vector bundles on products of Grassmannians and more general flag manifolds [6]; a sample has already appeared in [5]. In this database, there is a unique fourfold with maximal Picard number, equal to six: the study of this fourfold is the object of this note.

This study turned out to be related with interesting questions at the intersection of algebraic geometry with Lie theory. Consider two complex vector spaces  $V_4$  and  $V_5$ , of dimension four and five respectively. The action of  $GL(V_4) \times GL(V_5)$  on  $V_4^\vee \otimes \wedge^2 V_5^\vee$  is known to be prehomogeneous, its open orbit being the complement of a degree 40 hypersurface [24, p.98]. It is in fact one of the most complicated prehomogeneous spaces, containing no less than 63 distinct orbits [23, 9]. An important literature has been devoted to this prehomogeneous space, including some in connections with quintic field extensions, in the spirit of Bhargava's work on higher reciprocity laws [16, 17, 7].

The Fano fourfold  $X_4$  we are interested in is defined by a generic element of the prehomogeneous space  $V_4^\vee \otimes \wedge^2 V_5^\vee$ . It has two natural projections to  $G(2, 4) \simeq \mathbb{Q}^4$  and to the six-dimensional  $G(3, 5)$  that we describe in some details in section 4. In particular we show it is a small resolution of a fourfold with ten singular points which appears to be a cousin, or a big brother of the Segre cubic primal; this small resolution contracts ten planes which can be seen as a special subcollection of the classical Cremona-Richmond configuration. We deduce:

**Theorem.** *Consider five general planes in one of the two families of projective planes in  $\mathbb{Q}^4$ . They intersect pairwise in ten points. Blow-up these ten points and then the strict transforms of the five planes. Then the strict transforms of the exceptional divisors of the first blowup can be contracted to a smooth Fano fourfold, which is precisely  $X_4$ .*

Then we show that the automorphism group is  $Aut(X_4) = \mathcal{S}_5$ , so that

$$Pic(X_4)^{\mathcal{S}_5} \simeq \mathbb{Z}^2$$

is generated by the pull-back of the hyperplane classes by the two projections. This suggests to construct the tensor that defines  $X_4$  by *reverse-engineering*, starting from the representation theory of  $\mathcal{S}_5$ ; we show how this leads to a normal form from this tensor. We then use the previous constructions to describe the Chow ring of  $X_4$  completely, including the action of  $\mathcal{S}_5$ . We also check that  $X_4$ , as expected, is infinitesimally rigid.

This study can be considered as a warm-up for the more mysterious case of  $U_5^\vee \otimes \wedge^2 V_5^\vee$ , directly related to  $E_8$ , which has infinitely many but well-described orbits for the action of  $GL(U_5) \times GL(V_5)$  (see [15] for a first approach). Among other nice geometric objects, this representation will give rise to an interesting family of special Fano sixfolds.

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## 2. MODELS

According to the classical Borel-Weil theorem, one can interpret the representation  $V_4^\vee \otimes \wedge^2 V_5^\vee$  as a space of global sections of an irreducible homogeneous vector bundle over a homogeneous space, and this in more than one way:

$$\begin{aligned} V_4^\vee \otimes \wedge^2 V_5^\vee &= \Gamma(G(2, V_4) \times \mathbb{P}(V_5), \mathcal{U}^\vee \boxtimes \mathcal{Q}^\vee(1)) \\ &= \Gamma(\mathbb{P}(V_4) \times \mathbb{P}(V_5^\vee), \mathcal{O}(1) \boxtimes \wedge^2 \mathcal{V}^\vee) \\ &= \Gamma(G(2, V_5), V_4^\vee \otimes \wedge^2 \mathcal{V}^\vee) \\ &= \Gamma(\mathbb{P}(V_4) \times \mathbb{P}(V_5), \mathcal{O}(1) \boxtimes \mathcal{Q}^\vee(1)) \\ &= \Gamma(G(2, V_4) \times G(3, V_5), \mathcal{U}^\vee \boxtimes \wedge^2 \mathcal{V}^\vee) \\ &= \Gamma(\mathbb{P}(V_4) \times G(3, V_5), \mathcal{O}(1) \boxtimes \wedge^2 \mathcal{V}^\vee) \\ &= \Gamma(\mathbb{P}(V_4) \times G(2, V_5), \mathcal{O}(1) \boxtimes \wedge^2 \mathcal{V}^\vee) \\ &= \Gamma(G(2, V_4) \times G(2, V_5), \mathcal{U}^\vee \boxtimes \wedge^2 \mathcal{V}^\vee). \end{aligned}$$

Here  $\mathcal{U}$  and  $\mathcal{V}$  denote tautological bundles on Grassmannians (with some abuse of notations since we use these symbols several times for distinct bundles on different Grassmannians). As a consequence, consider a general element  $\theta$  in  $V_4^\vee \otimes \wedge^2 V_5^\vee$ . Interpreting it as a global section of a vector bundle in these seven different ways, we obtain smooth subvarieties of codimensions equal to the ranks of the vector bundles in question, that we respectively denote as follows (the notation is such that  $X_d$  has dimension  $d$ ):

$$\begin{array}{ll} X_0 \subset G(2, V_4) \times \mathbb{P}(V_5), & X_1 \subset \mathbb{P}(V_4) \times \mathbb{P}(V_5^\vee), \\ X_2 \subset G(2, V_5), & X_3 \subset \mathbb{P}(V_4) \times \mathbb{P}(V_5), \\ X_4 \subset G(2, V_4) \times G(3, V_5), & X_6 \subset \mathbb{P}(V_4) \times G(3, V_5), \\ X_8 \subset \mathbb{P}(V_4) \times G(2, V_5), & X_8' \subset G(2, V_4) \times G(2, V_5). \end{array}$$

Another obvious thing to do is to consider  $\theta$  as a general morphism from  $V_4$  to  $\wedge^2 V_5^\vee$ . The image of  $\mathbb{P}(V_4)$  inside  $\mathbb{P}(\wedge^2 V_5^\vee)$  is then a generic projective three-plane, that has to meet the Grassmannian  $G(2, V_5^\vee)$  along a set  $Y_0$  of five reduced points (the degree of the Grassmannian being equal to five). Correspondingly, we get a

set  $P_0$  of five points in  $\mathbb{P}(V_4)$ , and a set  $\Pi_0$  of five planes in  $\mathbb{P}(V_5)$ , all in general position. Concretely, if we choose a basis  $e_1, \dots, e_4$  of  $V_4$ , with dual basis  $e_1^\vee, \dots, e_4^\vee$  of  $V_4^\vee$  and decompose  $\theta$  accordingly as

$$\theta = e_1^\vee \otimes \theta_1 + e_2^\vee \otimes \theta_2 + e_3^\vee \otimes \theta_3 + e_4^\vee \otimes \theta_4$$

then the contraction  $\theta(v) = v_1\theta_1 + v_2\theta_2 + v_3\theta_3 + v_4\theta_4$  has rank two when  $[v]$  belongs to  $P_0$ ; that is, it decomposes as  $f_1^\vee \wedge f_2^\vee$  for two linear forms  $f_1^\vee, f_2^\vee$  whose kernels intersect along the corresponding plane in  $\mathbb{P}(V_5)$ . We will denote the five two-forms of rank two (defined up to scalars) obtained by contracting  $\theta$  as  $\omega_1, \dots, \omega_5$ . It would be natural then to impose the normalization  $\omega_1 + \dots + \omega_5 = 0$ , and decompose  $\theta$  as

$$\theta = u_1^\vee \otimes \omega_1 + u_2^\vee \otimes \omega_2 + u_3^\vee \otimes \omega_3 + u_4^\vee \otimes \omega_4 + u_5^\vee \otimes \omega_5$$

for some linear forms  $u_1^\vee, \dots, u_5^\vee$  such that  $u_1^\vee + \dots + u_5^\vee = 0$ .

*Notations.*

$P_0 = \{p_1, \dots, p_5\}$  is a set of five points in  $\mathbb{P}(V_4)$ , in natural bijection with the set  $\{\omega_1, \dots, \omega_5\}$ , of five decomposable two-forms in  $\wedge^2 V_5^\vee$ , that define five points in  $G(2, V_5^\vee) \simeq G(3, V_5)$ , hence five planes  $P_1, \dots, P_5$  in  $\mathbb{P}(V_5)$ . They also define five planes  $\pi_1, \dots, \pi_5$  in  $G(2, V_4)$ , where  $\pi_k$  is the set of planes in  $V_4$  that contain  $p_k$ .

$L_0$  is the set of pairs of points in  $P_0$ . According to the previous identifications, it is in natural bijection with a set of ten lines in  $\mathbb{P}(V_4)$ , a set of ten points in  $\mathbb{P}(V_5)$ , and a set of ten points in  $G(2, V_4)$ .

### 3. SMALL DIMENSIONS

Most results in this section are classical. Our purpose is mainly to set up the scene for the main character, which will make its entry in the next section.

**Proposition 3.1.**  *$X_0$  consists in 10 points of  $G(2, V_4) \times \mathbb{P}(V_5)$ , in natural bijection with  $L_0$ .*

*Proof.* By definition, a point  $(A_2, B_1)$  belongs to  $X_0$  if and only if we can decompose  $\theta$  in such a way that  $A_2$  is cut out by the linear forms  $e_3^\vee, e_4^\vee$  and the skew-symmetric forms  $\theta_1, \theta_2$  have the same kernel  $B_1 \subset V_5$ . Otherwise said,  $\theta_1$  and  $\theta_2$  belong to  $\wedge^2 B_1^\perp$ . Since in the latter space, decomposable tensors are parametrized by a quadric, we can make a change of basis in  $A_2^\perp$  and suppose that  $\theta_1$  and  $\theta_2$  are indeed decomposable. Concretely, this means that we can write  $\theta$  in the form

$$\theta = e_1^\vee \otimes f_1^\vee \wedge f_2^\vee + e_2^\vee \otimes f_3^\vee \wedge f_4^\vee + e_3^\vee \otimes \theta_3 + e_4^\vee \otimes \theta_4.$$

Then  $[e_1]$  belongs to  $P_0$ , the associated plane in  $\mathbb{P}(V_5)$  being  $P_1 = \langle f_1, f_2 \rangle^\perp$ , and also  $[e_2]$  belongs to  $P_0$ , the associated plane being  $P_2 = \langle f_3, f_4 \rangle^\perp$ . In particular  $A_2 = \langle e_1, e_2 \rangle$  and  $B_1 = P_1 \cap P_2$ , as claimed.  $\square$

**Proposition 3.2.**  *$X_1$  is the union of five disjoint lines, in natural bijection with  $P_0$ .*

*Proof.* By definition, a point in  $X_1$  is a pair  $(A_1, B_4)$  such that  $\theta(v)$  vanishes on  $B_4$  when  $v$  generates  $A_1$ . But then  $\theta(v)$  must have rank two, of the form  $f_1^\vee \wedge f_2^\vee$ . In particular  $A_1$  must correspond to one of the five points of  $P_0$ , and the hyperplane  $B_4$  can move in the pencil  $\langle f_1^\vee, f_2^\vee \rangle$ .  $\square$

**Proposition 3.3.**  *$X_2 \subset G(2, V_5)$  is a del Pezzo surface of degree five.*

*Proof.* Obvious.  $\square$

Recall that the del Pezzo surface of degree five contains 10 lines. Since the embedding in  $G(2, V_5)$  is anticanonical, this means in our setting that there exists ten flags  $A_1 \subset A_3 \subset V_5$  such that  $\theta(v, w) = 0$  for any  $v \in A_1, w \in A_3$ . It is easy to see that these ten flags are in natural bijection with  $L_0$ , the ten points  $[A_1]$  in  $\mathbb{P}(V_5)$  being exactly the intersections of the planes  $P_1, \dots, P_5$ .

**Proposition 3.4.** *The projection of  $X_3$  to  $\mathbb{P}(V_4)$  is the blow-up of the five points of  $P_0$ . The projection to  $\mathbb{P}(V_5)$  is a small resolution of a Segre cubic primal  $C_3$ , ten lines being contracted to the ten singular points of  $C_3$  defined by  $L_0$ .*

*Proof.* (Well known.) By definition,  $X_3$  parametrizes the pairs  $(A_1 = [v], B_1)$  such that  $B_1$  is contained in the kernel of  $\theta(v)$ . Generically this two-form has rank four and the kernel is one-dimensional, which implies that  $X_3$  projects birationally to  $\mathbb{P}(V_4)$ . The projection has non trivial fibers when the rank of  $\theta(v)$  drops, that is, over one of the five points in  $P_0$ . Then the kernel has dimension three and the fiber is a projective plane, as it has to be.

Now we turn to the projection to  $\mathbb{P}(V_5)$ . By definition, the fibers are linear subspaces defined by the image of the morphism  $Q(-1) \rightarrow V_4^\vee \otimes \mathcal{O}_{\mathbb{P}(V_5)}$  induced by  $\theta$ . In particular the fibers are non trivial over the corresponding determinantal locus  $C_3$ , which is a cubic threefold since  $\det(Q(-1)) = \mathcal{O}_{\mathbb{P}(V_5)}(-3)$ . This threefold becomes singular exactly when the rank drops to two. If  $w \in V_5$  generates  $B_1$ , this means that the morphism from  $V_4$  to  $V_5^\vee$  sending  $e_i$  to  $\theta_i(w, \bullet)$  has rank two. So we may suppose after a change of basis that  $\theta_1(w, \bullet) = \theta_2(w, \bullet) = 0$ . In other words,  $\theta_1$  and  $\theta_2$  have the same kernel  $B_1$ , and after another change of basis if necessary we have already seen that we can suppose they are decomposable. So they define two points in  $P_0$ , in such a way that  $B_1$  is the point obtained as the intersection of the corresponding planes in  $\mathbb{P}(V_5)$ , while the line contracted to this point is the span of the corresponding points in  $\mathbb{P}(V_4)$ .  $\square$

As a result,  $C_3$  is a cubic threefold with 10 nodes. (In fact  $C_3$  is the image of the rational map from  $\mathbb{P}(V_4)$  to  $\mathbb{P}(V_5)$  sending  $[v]$  to the kernel of the two-form  $\theta(v)$ , and essentially by definition this is a Segre cubic primal [11].)

**Reminder on the Segre cubic primal.** Recall that the Segre primal can be defined, if  $x_0, \dots, x_5$  are homogeneous coordinates on  $\mathbb{P}^5$ , by the two equations

$$x_0 + \dots + x_5 = 0, \quad x_0^3 + \dots + x_5^3 = 0.$$

This presentation exhibits an  $\mathcal{S}_6$  symmetry, and it is known that  $\text{Aut}(C_3) = \mathcal{S}_6$ . Classically, the Segre primal contains 15 planes. (See [10, Chapter 9] for much more information.)

The Segre cubic primal admits a classical modular interpretation, according to which  $C_3 \simeq (\mathbb{P}^1)^6 // SL_2$ . Moreover  $\bar{M}_{0,6}$  is a resolution of its singularities (that just blows-up the singular points) and according to Kapranov it can be constructed by blowing-up five general points in  $\mathbb{P}^3$ , plus the strict transforms of the ten lines that join them [18]. (Note also that  $\bar{M}_{0,6}$  compactifies the moduli space of genus 2 curves.)

Note also that  $C_3$  is known to be  $G$ -birationally rigid, and even  $G$ -birationally superrigid, when  $\mathcal{A}_5 \subset G \subset \mathcal{S}_6$  [1].

Blowing-up the ten singular points in  $C_3$  we get ten exceptional divisors isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ , each of which is contracted to  $\mathbb{P}^1$  in  $X_3$ . According to [14] any of

the rulings of  $\mathbb{P}^1 \times \mathbb{P}^1$  can in fact be contracted, yielding  $2^{10} = 1024$  small resolutions of the singularities of  $C_3$ , falling into 13 orbits of  $\mathcal{S}_6$ , including 6 for which the resolution is projective. Homological Projective Duality for the Segre cubic is discussed in [3].

**On the planes in the Segre cubic.** In coordinates, the 15 planes on the Segre cubic are given by three equations

$$x_a + x_b = x_c + x_d = x_e + x_f = 0,$$

for  $(abcdef)$  a permutation of  $(123456)$ ; we denote such a plane by  $(ab|cd|ef)$ . Together with the 15 points in the hyperplane  $x_0 + \dots + x_5 = 0$  with four coordinates equal to zero, they form a  $(15_3, 15_3)$  configuration classically known as the Cremona-Richmond configuration: each plane contains three of the 15 points and each of those points belongs to three planes of the configuration. But beware that two planes may meet along a single point, or a projective line; the second possibility occurs when their symbols have a common pair.

**Proposition 3.5.** *There are exactly 6 collections of five planes among the fifteen planes in  $C_3$ , meeting pairwise along single points. These collections are exchanged transitively by the action of  $\mathcal{S}_6$ . Each one has for stabilizer a copy of  $\mathcal{S}_5$ , embedded in  $\mathcal{S}_6$  in a non standard way.*

To understand the last sentence, recall that  $\mathcal{S}_6$  has the exceptional property that its outer automorphism group is non trivial: there exists a unique outer automorphism, and a non standard embedding of  $\mathcal{S}_5$  in  $\mathcal{S}_6$  is the composition of a standard embedding by such an outer automorphism. Note that this outer automorphism of  $\mathcal{S}_6$  exchanges the two conjugacy classes consisting of transpositions on one hand, and products of three disjoint transpositions on the other hand; the former corresponds to points, the latter to planes in the Cremona-Richmond configuration, which is for this reason self-dual.

*Proof.* Suppose given a collection of five planes, any two of which meet at a single point. This means that each plane is represented by three pairs, none of which being shared with another plane. So we have a total amount of 15 distinct pairs; necessarily, all the 15 pairs of integers from 1 to 6 must appear exactly once.

Up to permutation, we can assume that one of our planes is  $(12|34|56)$ . Then the plane containing  $(13)$  is either  $(13|25|46)$  or  $(13|26|45)$  and up to permuting 5 and 6 we can suppose it is the first one. Then the other planes are determined. For example, for the one containing  $(14)$ , we must split  $(2356)$  into two pairs, and since  $(25)$  and  $(56)$  have already been used the only possibility is  $(14|26|35)$ . This also shows that we have three choices for the plane containing  $(12)$ , then two choices for the plane containing  $(13)$ , and then no more choices; this means there are exactly six possibilities. Explicitely, they are the following:

$$\begin{array}{cccccc} (12|34|56) & (12|34|56) & (12|35|46) & (12|35|46) & (12|36|45) & (12|36|45) \\ (13|25|46) & (13|26|45) & (13|24|56) & (13|26|45) & (13|25|46) & (13|24|56) \\ (14|26|35) & (14|25|36) & (14|25|36) & (14|23|56) & (14|23|56) & (14|26|35) \\ (15|24|36) & (15|23|46) & (15|26|34) & (15|24|36) & (15|26|34) & (15|23|46) \\ (16|23|45) & (16|24|35) & (16|23|45) & (16|25|34) & (16|24|35) & (16|25|34) \end{array}$$

Let us denote these six configurations by  $ABCDEF$ . The action of  $\mathcal{S}_6$  on them induces a morphism  $\mathcal{S}_6 \rightarrow \mathcal{S}_6$ , and a direct examination shows that it sends the transposition (12) to the permutation  $(AB)(CD)(EF)$ . So it has to correspond to the outer automorphism of  $\mathcal{S}_6$ , and our final claim follows.  $\square$

*Question.* Is there an interpretation in terms of the root system  $E_7$ ? In fact the Lie algebra  $\mathfrak{e}_7$  admits a  $\mathbb{Z}_3$ -grading of the form

$$\mathfrak{e}_7 = \mathfrak{sl}_3 \times \mathfrak{sl}_6 \oplus (\mathbb{C}^3 \otimes \wedge^2 \mathbb{C}^6) \oplus (\mathbb{C}^3 \otimes \wedge^2 \mathbb{C}^6)^\vee,$$

and roots of  $\mathfrak{e}_7$  defined by weights of  $\mathbb{C}^3 \otimes \wedge^2 \mathbb{C}^6$  can be interpreted as triples of pairs [21]. Note that roots of  $\mathfrak{e}_7$  are classically connected with the 28 bitangents of a plane quartic.

#### 4. THE FANO FOURFOLD

Recall that our main character  $X_4 \subset G(2, V_4) \times G(3, V_5)$  is defined by  $\theta$  a general element in  $V_4^\vee \otimes \wedge^2 V_5^\vee$ , considered as a general section of the vector bundle  $\mathcal{U}^\vee \boxtimes \wedge^2 \mathcal{V}^\vee$ . Here  $\mathcal{U}$  denotes the tautological rank two bundle on  $G(2, V_4)$ , while  $\mathcal{V}$  denotes the tautological rank three bundle on  $G(3, V_5)$ .

In this section we describe the geometry of  $X_4$  by blowups and contractions.

**4.1. The main invariants.** We start by computing the main numerical invariants of  $X_4$ , including its Hodge numbers.

**Proposition 4.1.**  *$X_4$  is a rational Fano fourfold of index one.*

*Its cohomology is pure, with  $h^{1,1} = 6$  and  $h^{2,2} = 17$ .*

*Moreover  $h^0(-K_{X_4}) = 40$  and  $K_{X_4}^4 = 172$ .*

*Proof.* The first assertion is an immediate consequence of the adjunction formula.

The Hodge numbers and invariants can be computed using exact sequences, along the lines explained in [5]. (They could also be deduced from the geometric descriptions that will follow.) Since  $172 = 4 \times 43$  is not divisible by any fourth power, the index must be one.  $\square$

Note that  $h^0(-K_{X_4}) = 40 < \dim(\wedge^2 V_4 \otimes \wedge^3 V_5) = 60$ , which means that  $X_4$  is linearly degenerate inside  $G := G(2, V_4) \times G(3, V_5)$ . This can be checked by considering the twisted Koszul complex

$$0 \longrightarrow \wedge^6 E^\vee(1, 1) \longrightarrow \cdots \longrightarrow E^\vee(1, 1) \longrightarrow \mathcal{O}_G(1, 1) \longrightarrow \mathcal{O}_{X_4}(1) \longrightarrow 0.$$

Indeed  $H^0(E^\vee(1, 1)) \simeq V_4 \otimes V_5$  has dimension 20, while it can be checked that  $H^0(\wedge^k E^\vee(1, 1)) = 0$  for  $k > 1$ .

We will describe in some details the two projections  $p_1, p_2$ :

$$\begin{array}{ccc} & X_4 & \\ p_1 \swarrow & & \searrow p_2 \\ G(2, V_4) & & G(3, V_5). \end{array}$$

We start with the second one.

#### 4.2. The second projection and the Cremona-Richmond configuration.

We start with the projection to  $G(3, V_5)$ , which is very similar to the resolution of singularities of the Segre cubic primal.

**Proposition 4.2.** *The projection of  $X_4$  to  $G(3, V_5)$  is a small resolution of a codimension two subvariety  $C_4$  of degree 12, contracting ten planes to ten singular points in natural bijection with  $L_0$ .*

*Proof.* The fiber of  $p_2 : X_4 \rightarrow G(3, V_5)$  over a point  $[V] \in G(3, V_5)$  is defined by the morphism  $\theta_V : \wedge^2 V \rightarrow V_4^\vee$  induced by  $\theta$ . In particular the fibers are non trivial when the rank is at most two, which happens in codimension two. We conclude that the image  $C_4$  of  $X_4$  is a determinantal fourfold. Its structure sheaf admits a resolution by the Lascoux complex [20]

$$(1) \quad 0 \rightarrow \mathcal{V}^\vee(-3) \rightarrow V_4 \otimes \mathcal{O}_{G(2, V_5)}(-2) \rightarrow \mathcal{O}_{G(2, V_5)} \rightarrow \mathcal{O}_{C_4} \rightarrow 0,$$

where  $\mathcal{V}$  denotes the rank three tautological bundle. We deduce in particular that the class of  $C_4$  in the Chow ring of the Grassmannian  $G(3, V_5)$  is  $3\sigma_{11} + 2\sigma_2$ , so that its degree is  $3 \times 2 + 2 \times 3 = 12$ .

The rank of  $\theta_V$  drops to one on the singular locus of  $C_4$ , which must have codimension 6, hence be a finite set, over which the fibers are projective lines. The fact that  $\theta_V$  has a two dimensional kernel means that we can find a basis  $v_1, v_2, v_3$  of  $V$  such that  $\theta_i(v_1, v_2) = \theta_i(v_1, v_3) = 0$  for all  $i$ . Completing with two vectors  $v_4, v_5$  and taking the dual basis, we conclude that every  $\theta_i$  belongs to the space of forms generated by  $v_1^\vee \wedge v_4^\vee, v_1^\vee \wedge v_5^\vee$  and  $\wedge^2(v_1^\perp)$ . In particular  $\langle \theta_1, \theta_2, \theta_3, \theta_4 \rangle$  has to meet  $\wedge^2(v_1^\perp)$  in dimension at least two, which means that  $V$  defines a pair of planes  $\pi_p, \pi_q$  in  $P_0$ , whose intersection point is a line in  $V$ . Finally,  $V$  defines a hyperplane  $H_{pq}$  of  $V_4$ , and the corresponding fiber is the set  $G(2, H_{pq}) \simeq \mathbb{P}^2$  of planes in  $H_{pq}$ .

Conversely, such a pair of planes being given, we can decompose  $\theta$  in an adapted basis as

$$\theta = e_1^\vee \otimes f_1^\vee \wedge f_2^\vee + e_2^\vee \otimes f_3^\vee \wedge f_4^\vee + e_3^\vee \otimes \theta_3 + e_4^\vee \otimes \theta_4,$$

and then the conditions  $\theta_3(f_5, \bullet) = \theta_4(f_5, \bullet) = 0$  define a 3-plane  $V$  containing  $f_5$ . This exactly means that the singular locus of  $C_4$  consists in ten points, in natural bijection with  $L_0$ .  $\square$

**Proposition 4.3.** *Each singular point of  $C_4$  defines a plane in the Segre cubic primal  $C_3$ . The five remaining planes are the projectivized kernels of the five singular form  $\omega_1, \dots, \omega_5$ .*

*Proof.* By definition, a point  $[v] \in \mathbb{P}(V_5)$  belongs to  $C_3$  when the four linear forms  $\theta_i(v, \bullet)$  on  $V_5$  are linearly dependent. In the proof above, we have seen that a singular point in  $C_4$  corresponds to a three-plane  $V = \langle v_1, v_2, v_3 \rangle$  in  $V_5$  with  $\theta(v_1, v_2) = \theta(v_1, v_3) = 0$ . So for any  $v \in V$ , the linear forms  $\theta_i(v, \bullet)$  vanish on  $v_1$ , and also on  $v$  by skew-symmetry. When  $v$  and  $v_1$  are independent, the four linear forms  $\theta_i(v, \bullet)$  therefore belong to the three-dimensional space  $\langle v, v_1 \rangle^\perp \subset V_5^\vee$ , so they must be linearly dependent. Hence  $\mathbb{P}(V) \subset C_3$ .

That the projectivized kernels  $\mathbb{P}(K_j)$  of the five singular skew forms  $\theta_j$  are contained in  $C_3$  is obvious, since  $\theta_j(v, \bullet) = 0$  for  $v \in K_j$  is a linear dependence relation between the  $\theta_i(v, \bullet)$ .  $\square$

Note that we also have a special point  $[v_1]$  in each of the ten planes  $\mathbb{P}(V)$ . Moreover the five planes  $\mathbb{P}(K_1), \dots, \mathbb{P}(K_5)$  meet pairwise at a single point. In



particular, they provide one of the special subcollections of the Cremona-Richmond configuration described in Proposition 3.5.

Also observe that a form  $\omega$  which is as above in the span of  $v_1^\vee \wedge v_4^\vee$ ,  $v_1^\vee \wedge v_5^\vee$  and  $\wedge^2(v_1^\perp)$ , but does not belong to  $\wedge^2(v_1^\perp)$ , can be written as  $v_1^\vee \wedge w^\vee + \gamma$  with  $\gamma \in \wedge^2(v_1^\perp)$  and  $w^\vee$  a combination of  $v_4^\vee$  and  $v_5^\vee$ . It has rank two when  $\gamma$  has rank (at most) two and  $w^\vee \wedge \gamma = 0$ , which means if  $w^\vee \neq 0$  that  $\gamma$  is divisible by  $w^\vee$ . But then  $\omega$  itself is divisible by  $w^\vee$ , and since  $w^\vee$  is a combination of  $v_4^\vee$  and  $v_5^\vee$  this implies that  $\omega(v_2, v_3) = 0$ . In other words, the linear form that defines  $H_{pq} \subset V_4$  vanishes at the point that corresponds to  $\omega$ . This exactly means that

$$p_i \in H_{jk} \quad i \neq j, k.$$

We thus get in  $G(2, V_4)$  a collection of  $5 + 10$  planes, such that each plane of the second type meets exactly three planes of the first type. Hence a configuration  $(10_3, 5_6)$ . The condition that  $(jk)$  be disjoint from  $(lm)$ , so that the two hyperplanes meet in  $p_n$ , defines a copy of the Petersen graph.

Being a degeneracy locus of a morphism between vector bundles,  $C_4$  admits two natural resolutions of singularities;  $X_4$  is one of them. For the other one, we need to impose a rank one kernel in the source of the morphism  $\wedge^2 V \rightarrow V_2^\vee$ ; note that a rank one subspace of  $\wedge^2 V$  is always of the form  $\wedge^2 W$  for  $W \subset V$  a rank two subspace. But then the composition  $\wedge^2 W \rightarrow V_2^\vee$  vanishes exactly when  $W$  defines a point in the del Pezzo surface  $X_2 \subset G(2, V_5)$ . Our second resolution of singularities is thus simply  $\mathbb{P}_{X_2}(Q)$ , the projectivisation of the quotient bundle of  $G(2, V_5)$ , restricted to  $X_2$ . The two resolutions are dominated by  $\tilde{X}_4$ , the set of triples  $(U_2, V_3 \supset W_2)$  such that  $(U_2, V_3)$  belongs to  $X_4$  and  $W_2$  belongs to  $X_2$ . We get a diagram:

$$\begin{array}{ccccc}
 & & \tilde{X}_4 & & \\
 & \swarrow \alpha & \downarrow \pi & \searrow \beta & \\
 X_4 & & & & \mathbb{P}_{X_2}(Q) \\
 \swarrow p_1 & & \downarrow p_2 & \swarrow q_2 & \searrow q_1 \\
 G(2, V_4) & & C_4 & & X_2.
 \end{array}$$

**Proposition 4.4.** *The morphism  $q_2 : \mathbb{P}_{X_2}(Q) \rightarrow C_4$  is a small resolution of singularities, contracting ten lines to the ten singular points of  $C_4$ . These ten lines are mapped by  $q_1$  to the ten lines in the del Pezzo surface  $X_2$ .*

*The morphism  $\beta$  is the blow-up of the ten exceptional lines of  $q_2$ , as well as  $\alpha$  is the blow-up of the ten exceptional planes of  $p_2$ .*

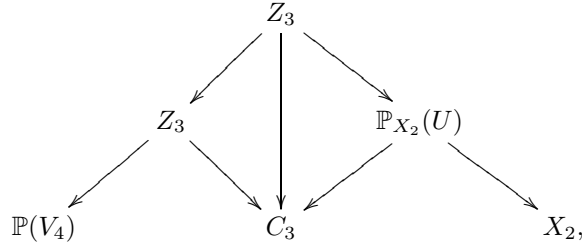
*Finally,  $\pi$  is the blow-up of the ten singular points of  $C_4$ , its exceptional divisor being the disjoint union of ten copies of  $\mathbb{P}^2 \times \mathbb{P}^1$ .*

*Remark.* Contrary to  $X_4$ , the fourfold  $X'_4 = \mathbb{P}_{X_2}(Q)$  is not Fano but only weak Fano. Indeed, the canonical bundle of  $X_2$  is  $\det(Q^\vee)$ , so the canonical bundle of  $X'_4$  is  $\mathcal{O}_{X'_4}(-3)$ . The quotient bundle  $Q$  is obviously not ample on  $G(2, V_5)$ , and neither is it when restricted to  $X_2$  since the morphism defined by  $\mathcal{O}_{X'_4}(1)$  is precisely  $q_2$

and has non trivial fibers. But of course  $Q$  is obviously nef, and it is also big since

$$\int_{X'_4} \mathcal{O}_{X'_4}(1)^4 = \int_{X_2} s_2(Q) = \int_{G(2, V_5)} s_2(Q) \sigma_1^4 = 2 > 0.$$

Note also the striking similarity with the two main projective resolutions of the Segre cubic, which can be encapsulated in a similar diagram



where  $Z_3$  is the blowup of  $\mathbb{P}(V_4) = \mathbb{P}^3$  at five points. Two important differences:  $Z_3$ , contrary to  $X_4$ , is only weak Fano;  $Z_3$  and  $Z'_3 = \mathbb{P}_{X_2}(U)$ , contrary to  $X_4$  and  $X'_4$ , are related by flops and therefore derived-equivalent. Instead of that, we have:

**Proposition 4.5.** *The birational map  $q_2 \circ p_2^{-1} : X_4 \dashrightarrow X'_4$  is a flip.*

*Proof.* Since  $X_4$  is Fano, we need only to check that the canonical bundle of  $X'_4$  is nef on the non trivial fibers of the projection to  $C_4$ . But we have seen that  $K_{X'_4} = \mathcal{O}_{X'_4}(-3)$ , the fiber of  $\mathcal{O}_{X'_4}(-1)$  at a point defined by a flag  $U_2 \subset U_3$  being  $U_3/U_2$ . On a fiber  $F$  of the projection to  $C_4$ , by definition  $U_3$  is fixed, so  $\mathcal{O}_{X'_4}(-1)|_F$  is base point free, hence also  $K_{X'_4}|_F$ .  $\square$

According to the Bondal-Orlov conjecture, there should therefore exist a fully faithful functor  $D^b(X'_4) \rightarrow D^b(X_4)$  that would be interesting to describe explicitly.

**4.3. Pencils of skew-forms and the first projection.** In order to describe the projection to  $G(2, V_4)$ , we first note that a plane in  $V_4$  defines through  $\theta$  a pencil of skew-symmetric forms in five variables, and that such pencils have been classified. In fact, for a two dimensional vector space  $V_2$ , the action of  $GL(V_2) \times GL(V_5)$  on  $V_2^\vee \otimes \wedge^2 V_5^\vee$  has finitely many orbits, which are described in [19]. Let us only mention that there are exactly eight orbits: the open orbit  $\mathcal{O}_7$ , an orbit  $\mathcal{O}_6$  of codimension two and another  $\mathcal{O}_5$  of codimension four, and then all the other orbits have bigger codimension.

The orbit  $\mathcal{O}_5$  (or rather its closure) is characterized as consisting of tensors of rank at most four, in the sense that they belong to  $V_2^\vee \otimes \wedge^2 V_4$  for some hyperplane  $V_4 \subset V_5^\vee$ . The orbit  $\mathcal{O}_6$  (or rather its closure) is characterized as consisting of those pencils in  $\wedge^2 V_5^\vee$  admitting a rank two element. So the open orbit  $\mathcal{O}_7$  parametrizes pencils of forms of constant rank four. By [22, Proposition 2], given such a pencil one can find a basis of  $V_5$  for which the two skew-forms

$$\omega_1 = f_1^\vee \wedge f_3^\vee + f_2^\vee \wedge f_4^\vee, \quad \omega_2 = f_1^\vee \wedge f_4^\vee + f_2^\vee \wedge f_5^\vee$$

are generators. The projective line  $\langle f_1^\vee, f_2^\vee \rangle$  is the *pivot* of the pencil. Now, observe that if a three-plane  $V \subset V_5$  is isotropic with respect to any skew-form  $s\omega_1 + t\omega_2$  of the pencil, it has to contain its kernel, which is generated by  $s^2 f_3 - st f_4 + t^2 f_5$ . So necessarily  $V = \langle f_3, f_4, f_5 \rangle$ , the orthogonal to the pivot.

**Proposition 4.6.** *The projection of  $X_4$  to  $G(2, V_4)$  is birational. The exceptional locus in  $G(2, V_4)$  is the union of five planes, intersecting in the ten points of  $L_0$ , whose fibers are quadratic surfaces.*

*Proof.* The fiber of the projection  $p_1 : X_4 \rightarrow G(3, V_5)$  over the point  $[U] \in G(2, V_4)$  is defined by the morphism  $\theta_U : U \rightarrow \wedge^2 V_5^\vee$ . This morphism is injective and we thus get a pencil of skew-symmetric forms. If this pencil is generic, which means that it has constant rank, then we have just seen that there is a unique three-plane in  $V_5$  which is isotropic with respect to any skew-form in the pencil. This three-plane is the image of the induced map  $\theta_U^{(2)} : S^2 U \rightarrow \wedge^4 V_5^\vee \simeq V_5$ . In particular,  $p_1$  is birational.

Special fibers will occur when the pencil  $Im(\theta_U)$  becomes special in some way. By the usual arguments for orbital degeneracy locus [4], we need to take into account, in the space of pencils, only those orbits of codimension smaller than five, which apart from the open orbit are the orbits  $\mathcal{O}_5$  and  $\mathcal{O}_6$  we have described above.

Pencils in  $\mathcal{O}_5$  contain two skew-forms of rank two. In our case, they must be two of the skew-forms  $\omega_1, \dots, \omega_5$ , say  $\theta_1$  and  $\theta_2$ . Choose an adapted basis such that  $\theta_1 = f_1^\vee \wedge f_2^\vee$  and  $\theta_2 = f_3^\vee \wedge f_4^\vee$ , so that

$$\theta_U = e_1^\vee \otimes f_1^\vee \wedge f_2^\vee + e_2^\vee \otimes f_3^\vee \wedge f_4^\vee.$$

It is straightforward to check that the three-planes that are isotropic with respect to any skew-form in the pencil are those generated by  $f_5$ , a vector in  $\langle f_1, f_2 \rangle$ , and a vector in  $\langle f_3, f_4 \rangle$ . We thus get for fiber a copy of  $\mathbb{P}^1 \times \mathbb{P}^1$ .

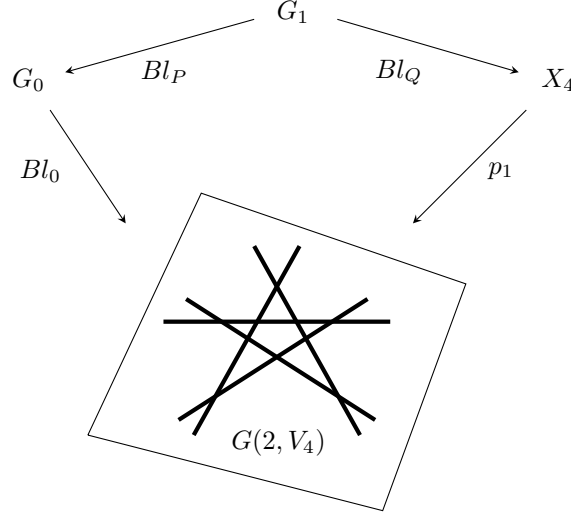
Finally, pencils in  $\mathcal{O}_6$  contain exactly one skew-form of rank two, say  $\theta_1$ . To describe the corresponding fiber we must understand the 3-planes isotropic with respect to both the generic form  $\theta_2$  and the degenerate form  $\theta_1 = f_1^\vee \wedge f_2^\vee$ . Such a 3-plane must contain the kernel of  $\theta_2$ ; let us choose a generator  $f_5$  and a hyperplane  $H_4$  in  $V_5$  not containing  $f_5$ . We may suppose that  $f_2^\vee$  vanishes on  $f_5$ . The 3-planes we are looking for are in correspondence with the 2-planes  $H = \langle h, h' \rangle$  in  $H_4$  such that  $\omega_2(h, h') = 0$  and  $f_1^\vee(h) = f_1^\vee(h') = 0$ . Such a 2-plane must be contained in the kernel  $K_3$  of  $f_1^\vee$ , and it has to contain the kernel  $K_1$  of the restriction of  $\omega_2$  to  $K_3$ . We finally get for fiber a pencil of planes.  $\square$

To summarize, the exceptional locus is the union of five planes  $\pi_1, \dots, \pi_5$  in  $G(2, V_4)$ , where  $\pi_i$  parametrizes the planes in  $V_4$  containing  $\omega_i$ . Any two of these five planes meet at a single point, over which the fiber of  $p_1$  is a quadratic surface.

If  $U_2$  does not belong to any of the five exceptional planes, we have seen that  $U_3$  is the span of the kernels of the two-forms  $\theta(v)$ , for  $v \in U_2$ . Since this kernel can be computed as  $\theta(v) \wedge \theta(v)$ , there is a natural associated conic bundle over  $G(2, V_4)$  minus the five exceptional planes. This also stresses the analogy with the construction of the Segre primal  $C_3$  as the image of a rational map  $\mathbb{P}(V_4) \dashrightarrow \mathbb{P}(V_5)$  defined by  $\theta$ . Here we get  $C_4$  as the image of a rational map  $G(2, V_4) \dashrightarrow G(2, V_5)$  also defined by  $\theta$ . We will put its equations in simple form in the next section.

**4.4. Blow-up and contract.** Proposition 4.6 suggests to construct  $X_4$  by first blowing-up  $G(2, V_4)$  along the 10 points of  $L_0$ , then the strict transforms of the 5 planes, which are Del Pezzo surfaces of degree five. The first blow-up  $Bl_0 : G_0 \rightarrow G(2, V_4)$  gives 10 exceptional divisors  $E_{ij} \simeq \mathbb{P}^3$  for  $1 \leq i < j \leq 5$ , each with a pair of skew lines  $\ell_i, \ell_j$  coming from the two planes  $\pi_i$  and  $\pi_j$  intersecting at  $p_{ij}$ . The second blow-up  $Bl_P : G_1 \rightarrow G_0$  produces five other exceptional divisors

$F_k$  for  $1 \leq k \leq 5$ , while the strict transform of  $\tilde{E}_{ij}$  of  $E_{ij}$  is the blowup of  $E_{ij}$  along  $\ell_i \cup \ell_j$ . Since the blowup of  $\mathbb{P}^3$  along two skew lines is the total space of  $\mathbb{P}(\mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1))$  over  $\mathbb{P}^1 \times \mathbb{P}^1$ , we deduce that the rational map to  $X_4$  is a morphism. More precisely, it has to coincide with the blowup  $Bl_Q : G_1 \rightarrow X_4$  of the ten quadratic surfaces  $S_{ij} = p_1^{-1}(p_{ij})$  in  $X_4$ . This explains in particular why the Picard number is equal to 6.



Let  $F = F_1 + \dots + F_5$ , and let  $E$  be the sum of the ten divisors  $\tilde{E}_{ij}$  in  $G_1$ . from the identity

$$K_{G_1} = -4H_1 + 3E + F = K_{X_4} + E = -H_1 - H_2 + E$$

we deduce the relation  $3H_1 = H_2 + 2E + F$ .

The exceptional locus of  $p_2$  defines a collection of 10 planes in  $X_4$ , contracted to the ten singular points of  $C_4$ , and that we can identify with their isomorphic images in  $G(2, V_4)$ . Recall that in this Grassmannian we have the five planes  $\pi_1, \dots, \pi_5$ .

**Proposition 4.7.** *The resulting collection of 10+5 planes in  $G(2, V_4)$  is in natural correspondance with the Cremona-Richmond configuration.*

**4.5. Incidences with the Segre cubic.** Now we relate the two varieties  $X_3$  and  $X_4$  by considering the incidence correspondence

$$I = \{(A_1, B_1), (U_2, U_3) \in X_3 \times X_4, A_1 \subset U_2, B_1 \subset U_3\}.$$

Recall that by definition,  $B_1$  is (contained in) the kernel of  $\theta(v)$  for  $v \in A_1$ , while  $U_3$  is the linear span of the kernels of the two forms  $\theta(u)$  for  $u \in U_2$ ; this kernel depends quadratically on  $u$  since it is given by  $\theta(u) \wedge \theta(u)$ . This implies that  $I$  is (generically) a  $\mathbb{P}^1$ -bundle over  $X_4$ , and (generically) a  $\mathbb{P}^2$ -bundle over  $X_3$ . We have

a commutative diagram

$$\begin{array}{ccccc}
 & & I & & \\
 & \swarrow & \downarrow & \searrow & \\
 X_3 & & Fl(1, 2, V_4) & & X_4 \\
 \downarrow & \swarrow & & \searrow & \downarrow \\
 \mathbb{P}(V_4) & & & & G(2, V_4)
 \end{array}$$

One easily checks that:

**Proposition 4.8.** *The base locus of the birational projection  $I \rightarrow Fl(1, 2, V_4)$  is the union of five disjoint planes.*

*Proof.* Denote by  $\psi_i$  the plane  $G(2, V_4)$  parametrizing the lines in  $\mathbb{P}(V_4)$  that pass through  $p_i$ , and by  $\Psi_i$  its lift in  $Fl(1, 2, V_4)$ . The preimage in  $I$  of a point in  $\psi_i$  is given by a flag  $B_1 \subset U_3 \subset V_5$  such that  $U_1$  is contained in the kernel  $K_i$  of  $\omega_i$  and  $U_3$  contains

**4.6. Projective duality.** We have seen that  $X_4$  is birationally equivalent to the projective bundle  $\mathbb{P}_{X_2}(Q)$  over the del Pezzo surface  $X_2$ . Since  $Q^\vee$  has no section, we would rather write it as  $\mathbb{P} = \mathbb{P}_{X_2}(\wedge^2 Q^\vee)$ , in which case the relative tautological bundle  $\mathcal{O}_{\mathbb{P}}(-1)$  sends  $\mathbb{P}$  to  $\mathbb{P}(\wedge^2 V_5^\vee) \simeq \mathbb{P}(\wedge^3 V_5)$ , the image being  $C_4 \subset G(3, V_5)$ . We are then in the context of Homological Projective Duality for projective bundles, according to which  $\mathbb{P} \rightarrow \mathbb{P}(\wedge^2 V_5^\vee)$  is dual to  $\mathbb{P}^* \rightarrow \mathbb{P}(\wedge^2 V_5)$ , with  $\mathbb{P}^*$  the projective bundle  $\mathbb{P}_{X_2}(W \wedge V_5)$ , where  $W$  denotes the rank two tautological bundle.

**Proposition 4.9.** *The image of  $\mathbb{P}^* \rightarrow \mathbb{P}(\wedge^2 V_5)$  is an octic hypersurface in  $\mathbb{P}(\wedge^2 V_5)$ , containing the Grassmannian  $G(2, V_5)$  in its singular locus.*

*Proof.* First consider the full projective bundle  $\mathbb{P}_{G(2, V_5)}(W \wedge V_5)$  and its projection to  $\mathbb{P}(\wedge^2 V_5)$ . The generic fiber is a copy of  $\mathbb{Q}^3$  (while the special fibers, that occur over  $G(2, V_5)$ , are codimension two Schubert cycles). When we restrict to  $X_2$ , we cut the fibers by linear spaces of codimension four. Generically, they meet the span of the fiber at one point; in codimension one, this point will be on the fiber itself. This implies that  $\mathbb{P}^* \rightarrow \mathbb{P}(\wedge^2 V_5)$  is birational onto its image, which must be a hypersurface. As usual, we compute the degree of this hypersurface as

$$\int_{\mathbb{P}^*} \mathcal{O}_{\mathbb{P}^*}(1)^8 = \int_{X_2} s_2(W \wedge V_5) = \int_{G(2, V_5)} (2\sigma_2 + \sigma_{11})\sigma_1^4 = 8.$$

Here we used exact sequences to compute the Segre class  $s(W \wedge V_5) = c(Q)^\vee c(S^2 U)$ , with  $c(Q) = 1 + \sigma_1 + \sigma_2 + \sigma_3$  and  $c(U) = 1 - \sigma_1 + \sigma_{11}$ .

Over a point  $W^0$  of the Grassmannian, the fiber of  $\mathbb{P}_{G(2, V_5)}(W \wedge V_5) \rightarrow \mathbb{P}(\wedge^2 V_5)$  is the Schubert cycle of planes  $W$  meeting  $W^0$  along at least a line. It is desingularized by a  $\mathbb{P}^3$ -bundle over  $\mathbb{P}(W^0)$ . If we fix a line  $L \subset W^0$ , there exists a plane  $W \supset L$  in  $X_2$  if and only if the four linear forms  $\theta(L, \bullet)$  on  $V_5/L$  are linearly dependent. This defines a section of  $\wedge^4(Q^\vee(1)) = \mathcal{O}(3)$  over  $\mathbb{P}^1$ , and we conclude that the general fiber of  $\mathbb{P}_{X_2}(W \wedge V_5) \rightarrow \mathbb{P}(\wedge^2 V_5)$  over  $G(2, V_5)$  consists in three points. Since this morphism is birational onto its image, Zariski's main theorem implies that  $G(2, V_5)$  is contained in the singular locus.  $\square$

## 5. SYMMETRIES

The symmetries of the Segre cubic primal must be reflected in  $X_4$ . In this section we describe the symmetries of  $X_4$  in some detail. In particular we will prove:

**Proposition 5.1.** *The generic stabilizer of the action of  $PGL(V_4) \times PGL(V_5)$  on  $\mathbb{P}(V_4^\vee \otimes \wedge^2 V_5^\vee)$  is the symmetric group  $\mathcal{S}_5$ .*

What is classically known, as we mentioned in the introduction, is that the action of  $PGL(V_4) \times PGL(V_5)$  on  $V_4^\vee \otimes \wedge^2 V_5^\vee$  is prehomogeneous. The representative of the open orbit given in [23] is

$$\theta = e_1^\vee \otimes (f_{25} - f_{34}) + e_2^\vee \otimes (f_{15} - f_{24}) + e_3^\vee \otimes (f_{23} - f_{14}) + e_4^\vee \otimes (f_{45} - f_{12}),$$

with the notation  $f_{ij} = f_i^\vee \wedge f_j^\vee$ . The corresponding points in  $\mathbb{P}(V_4)$  and rank two forms are easy to identify; we get

$$\begin{aligned} p_1 &= e_2 + ie_4, & \omega_1 &= (f_1 + if_4) \wedge (f_2 + if_5), \\ p_2 &= e_2 - ie_4, & \omega_2 &= (f_1 - if_4) \wedge (f_2 - if_5), \\ p_3 &= e_1 + e_3 + e_4, & \omega_3 &= (f_2 + f_4) \wedge (f_1 + f_3 + f_5), \\ p_4 &= e_1 + je_3 + j^2e_4, & \omega_4 &= (f_2 + j^2f_4) \wedge (f_1 + j^2f_3 + jf_5), \\ p_5 &= e_1 + j^2e_3 + je_4, & \omega_5 &= (f_2 + jf_4) \wedge (f_1 + jf_3 + j^2f_5). \end{aligned}$$

Here  $j$  and  $i$  are primitive fourth and third roots of unity. Each pair  $\omega_p, \omega_q$  defines two planes in  $V_5^\vee$  whose common orthogonal is a line  $[e_{pq}]$ . Then the planes of the Cremona-Richmond configuration are obtained as follows:  $P_{pq}$  is generated by the three points  $e_{ij}, e_{jk}, e_{ik}$  for  $ijk$  distinct from  $pq$ ; and  $P_p$  is generated by the four points  $e_{ip}$  for  $i \neq p$ . Explicitly, the ten vectors  $e_{pq}$  can be chosen as follows:

$$\begin{aligned} e_{12} &= (0, 0, 1, 0, 0) & e_{24} &= (i, -j^2, -2ij, 1, ij^2) \\ e_{13} &= (1, -i, -2, i, 1) & e_{25} &= (i, -j, -2ij^2, 1, ij) \\ e_{14} &= (-i, -j^2, 2ij, 1, -ij^2) & e_{34} &= (1, 0, j^2, 0, j) \\ e_{15} &= (-i, -j, 2ij^2, 1, -ij) & e_{35} &= (1, 0, j, 0, j^2) \\ e_{23} &= (1, i, -2, -i, 1) & e_{45} &= (1, 0, 1, 0, 1). \end{aligned}$$

Each  $\omega_i$  defines a plane  $\pi_i$  in  $V_5^\vee$ , from which we can deduce a collection of hyperplanes  $\pi_{ij} = \pi_i + \pi_j$  and points  $p_{ijk} = \pi_i \cap (\pi_j + \pi_k)$ .

**Proposition 5.2.** *For any permutation  $i, j, k, l, m$  of  $1, \dots, 5$ ,  $p_{ijk} = p_{ilm}$ .*

*Proof.* Explicit check. □

We have no convincing explanation of this coincidence, but as a consequence, we don't get thirty but only fifteen points in  $\mathbb{P}(V_5^\vee)$ . Obviously,  $p_{ijk}$  belongs to  $\pi_i$ , hence to any of the four hyperplanes  $\pi_{il}$ ,  $l \neq i$ . Conversely,  $\pi_{ij}$  contains the three points  $p_{iab}$  plus the three points  $p_{jcd}$ .

**Proposition 5.3.** *The fifteen points  $\pi_{ijk}$  and the ten hyperplanes  $\pi_{ij}$  in  $\mathbb{P}(V_4)$  form a configuration  $(15_4, 10_6)$ .*

We thus recover the abstract configuration classically defined by the Segre primal. In particular the fifteen points  $\pi_{ijk}$  should be in natural correspondence with planes in the Segre primal.

Automorphisms in  $PGL(V_4) \times PGL(V_5)$  that fix  $\langle \theta \rangle$  are in bijective correspondence with elements of  $PGL(V_5)$  fixing the four-plane generated by the  $\omega_i$ 's. Automatically such an automorphism will preserve the set of five planes  $\pi_1, \dots, \pi_5$ , hence the collection of the thirty points  $p_{ijk}$ .

In order to show that any permutation of the five planes can be lifted to  $PGL(V_5)$ , it is enough to lift two generators of  $\mathcal{S}_5$ , say a transposition and a complete cycle. By sending  $f_i$  to  $\epsilon_i f_i$  with  $\epsilon_i = 1$  for  $i$  odd and  $\epsilon_i = -1$  for  $i$  even, we exchange  $\pi_1$  and  $\pi_2$  and let the three other planes fixed. So let us turn to a maximal cycle. We claim that the cycle  $(12345) \in \mathcal{S}_5$  can be lifted to the transformation of  $GL(V_5)$  given by

$$\begin{aligned} f_1 &\mapsto \frac{j}{3}f_1 - 2ijf_2 + \frac{j}{3}f_3 - ijf_4 + \frac{4j}{3}f_5, \\ f_2 &\mapsto -\frac{2i}{3}f_1 - f_2 + \frac{j}{3}f_3 + \frac{i}{3}f_5, \\ f_3 &\mapsto \frac{4j}{3}f_1 + 4ij^2f_2 - \frac{2j^2}{3}f_3 - 4ij^2f_4 + \frac{4j^2}{3}f_5, \\ f_4 &\mapsto -\frac{2i}{3}f_1 - \frac{2i}{3}f_3 - jf_4 + \frac{2ij}{3}f_5, \\ f_5 &\mapsto \frac{4}{3}f_1 + if_2 + \frac{1}{3}f_3 + 2if_4 + \frac{1}{3}f_5. \end{aligned}$$

**Corollary 5.4.** *The automorphism group of the Fano fourfold  $X_4$  is  $Aut(X_4) = \mathcal{S}_5$ .*

*Proof.* An automorphism of  $X_4$  is induced by a linear transformation in  $PGL(V_4) \times PGL(V_5)$  preserving  $\theta$ . Considered as a homomorphism from  $V_4$  to  $\wedge^2 V_5^\vee$ ,  $\theta$  defines a codimension four linear section of  $G(2, V_5)$ , that is a degree five del Pezzo surface  $S_5$ . This implies that  $Stab(\theta)$  embeds into  $Aut(S_5)$ , which is well-known to be  $\mathcal{S}_5$ . Since we know by the previous computations that  $Stab(\theta)$  contains  $\mathcal{S}_5$ , we are done.  $\square$

Once we identify  $\mathcal{S}_5$  with the stabilizer of  $\theta$  in  $SL(V_4) \times SL(V_5)$ , we get actions of  $\mathcal{S}_5$  on  $V_4$  and  $V_5$ , clearly irreducible. Up to the sign representation there is a unique irreducible representation of  $\mathcal{S}_5$  of dimension 4, and a unique one of dimension 5. The complex (1) shows that  $C_4$  is cut out by a family of quadrics on  $G(2, V_5)$  parametrized by  $V_4$ , hence a  $\mathcal{S}_5$ -invariant copy of  $V_4$  inside  $S_{22}V_5$ . We will show later on that this copy is unique. (This point of view from finite group representation theory is typically used in [12]. Something with the same flavour has been done in [2] for the quintic del Pezzo surface.)

We use the character table of  $\mathcal{S}_5$  (see for example [13]) to compute some plethysm and tensor product representations. Recall that  $\mathcal{S}_5$  has irreducible representations of dimension 1, 1, 4, 4, 5, 5, 6 that we denote by  $U_1, U_1^-, U_4, U_4^-, U_5, U_5^-, U_6$ . All these representations are self-dual, being defined over the real numbers. Concretely,  $U_1$  is the trivial representation,  $U_1^-$  is the sign representation,  $U_4$  is the natural representation,  $U_4^- = U_4 \otimes U_1^-$  and  $U_6 = \wedge^2 U_4$ . One computes that

$$S^2 U_4 = U_5 \oplus U_4 \oplus U_1, \quad \wedge^2 U_5 = U_4^- \oplus U_6.$$

The last decomposition implies in particular that  $(U_4^-)^\vee \otimes \wedge^2 U_5^\vee$  contains a unique  $\mathcal{S}_5$ -invariant tensor  $\theta_{\mathcal{S}_5}$ , up to scalars.

At this point it could therefore be reasonable to reverse the whole process and start from the representation theory of  $\mathcal{S}_5$ . One should be able to check directly that  $\theta_{\mathcal{S}_5}$  is generic, and then we should get  $\theta_{\mathcal{S}_5}$ -invariant descriptions of all the objects we have been studying.

Note that  $S^2 U_4 = U_5 \oplus U_4 \oplus U_1$  allows to construct  $U_5$  from  $U_4$ , as the space of quadrics which are apolar to the obvious invariant cubic. In coordinates  $x_1, \dots, x_5$  permuted by  $\mathcal{S}_5$ , the representation  $U_4$  is the hyperplane  $x_1 + \dots + x_5 = 0$ , the invariant cubic is  $x_1^3 + \dots + x_5^3$  and the apolar quadrics are of the form  $\sum_{i \neq j} a_{ij} x_i x_j$  with

$$a_{ij} = a_{ji} \quad \forall i \neq j, \quad \sum_{i \neq k} a_{ik} = 0 \quad \forall k.$$

We get ten indeterminates and five independent relations, consistently with the fact that these quadrics should span a copy of  $V_5$ .

Inside the space  $V_5$  of apolar quadrics to the invariant cubic, note that we have  $q_{ij,kl} = (x_i - x_j)(x_k - x_l)$  for  $i, j, k, l$  distinct integers. These quadrics are subject to the Plücker type relations  $q_{ij,kl} - q_{ik,jl} + q_{il,jk} = 0$ . This suggests to define the following elements of  $\wedge^2 V_5$ :

$$\begin{aligned} Q_1 &= q_{23,45} \wedge q_{24,35}, \\ Q_2 &= q_{13,45} \wedge q_{14,53}, \\ Q_3 &= q_{12,45} \wedge q_{14,25}, \\ Q_4 &= q_{12,35} \wedge q_{13,52}, \\ Q_5 &= q_{12,34} \wedge q_{13,24}. \end{aligned}$$

Obviously, for any permutation  $\sigma \in \mathcal{S}_5$  one must have  $\sigma(Q_i) = \pm Q_{\sigma(i)}$ . We also let, for a pair  $i \neq j$  with complement  $p, q, r$  in  $1 \dots 5$ ,

$$Q_{i,j} = q_{ip,qr} \wedge q_{jp,qr} + q_{iq,rp} \wedge q_{jq,rp} + q_{ir,pq} \wedge q_{jr,pq}.$$

**Proposition 5.5.** *The action of  $\mathcal{S}_5$  on  $\langle Q_1, \dots, Q_5 \rangle$  gives a copy of the representation  $U_4^-$  in  $\wedge^2 U_5$ . Similarly, the action of  $\mathcal{S}_5$  on  $\langle Q_{i,j}, 1 \leq i < j \leq 5 \rangle$  gives a copy of the representation  $U_6$ .*

What have we gained in doing all that? First, we get a better, more symmetric normal form for the generic  $\theta$  than that of Ozeki, as

$$\theta_{\mathcal{S}_5} = e_1 \otimes Q_1 + e_2 \otimes Q_2 + e_3 \otimes Q_3 + e_4 \otimes Q_4 + e_5 \otimes Q_1,$$

with  $e_1 + \dots + e_5 = 0$ .

Also, we can make explicit the quadratic equations of  $\mathcal{C}_4$ . A character computation yields:

**Lemma 5.6.** *The multiplicity of  $U_4^-$  inside  $S^2(\wedge^2 U_5)$  is equal to one.*

So the space of quadratic equations we are looking for is uniquely defined in terms of the  $\mathcal{S}_5$ -action. Moreover, recall that  $\wedge^2 U_5 = U_4^- \oplus U_6$ . Another character computation shows that the copy of  $U_4^-$  that we are looking for inside  $S^2(\wedge^2 U_5)$  is in fact contained inside  $U_4^- \otimes U_6 = U_4^- \otimes \wedge^2(U_4^-) \subset U_4^- \otimes \text{End}(U_4^-)$  (recall that  $U_4^-$  is self-dual). So there is an obvious map to  $U_4^-$ , and dually, this says that the space of quadrics we are looking for is generated by the five quadrics

$$CQ_i = \sum_{j \neq i} Q_{i,j} Q_j, \quad 1 \leq i \leq 5.$$

*Remark.* Since  $\text{Aut}(C_3) = \mathcal{S}_6$ , certain automorphisms of the Segre primal do not lift to  $X_4$ . would it be possible that  $\mathcal{S}_6$  act on  $X_4$  by birational transformations?

## 6. THE CHOW RING OF $X_4$

In this section we completely determine the Chow ring of  $X_4$ , with its structure of  $\mathcal{S}_5$ -module. Let us start with the Picard group.

From the relation  $3H_1 = H_2 + 2E + F$  that we found on  $G_1$ , we compute that

$$H_1^4 = 2, \quad H_1^3 H_2 = 6, \quad H_1^2 H_2^2 = 13, \quad H_1 H_2^3 = 14, \quad H_2^4 = 12.$$

The Picard group is generated by  $H_1, H_2$  and the five components of  $F$ , which are permuted by  $\mathcal{S}_5$ . We deduce:



**Proposition 6.1.** *The Chow ring of  $X_4$  is generated by  $A^*(G)$  and the five divisors  $F_1, \dots, F_5$ . As a representation of  $\mathcal{S}_5$ , the Picard group decomposes as*

$$\text{Pic}(X_4) \otimes_{\mathbb{Z}} \mathbb{C} = 2U_0 \oplus U_4.$$

We know by Proposition 4.1 that the middle dimensional Chow group  $A^2(X_4)$  has dimension 17, and we expect that the invariant part has dimension four, with two classes coming from  $G(2, V_4)$  and two other classes from  $G(2, V_5)$ . We will show that all come (at least over  $\mathbb{Q}$ ) from products of divisor classes.

We compute the multiplicative structure of the Chow ring by embedding it in the Chow ring of  $G_1$ , that we shall now describe. First, the Chow ring of  $G_0$  is generated by the Chow ring of  $G = G(2, V_4)$  and the ten exceptional divisors  $E_{pq}^0$  of the blow-up  $b_0 = Bl_0$ , such that

$$(E_{pq}^0)^4 = -1, \quad E_{pq}^0 E_{p'q'}^0 = 0 \text{ for } \{p, q\} \neq \{p', q'\}, \quad E_{pq}^0 \cdot b_0^* C = 0$$

for any class  $C \in A^*(G)$  of positive degree. After this first blow-up, the five planes  $\pi_1, \dots, \pi_5$  give five disjoint surfaces  $\Sigma_1, \dots, \Sigma_5$ , each one being a plane blow-up in five points, that is a del Pezzo surface of degree 5. We denote the four exceptional lines in  $\Sigma_p$  by  $\ell_p^q$ , whose image in  $G$  is the point  $\pi_{pq}$ , for  $q \neq p$ .

The second blow-up  $b_1 = Bl_P$  is the blow-up of these five surfaces. We denote by  $F_p^1$  the five exceptional divisors, and by  $E_{pq}^1$  the strict transforms of the divisors  $E_{pq}^0$ . Since  $F_p^1 = \mathbb{P}(N_p)$ , for  $N_p$  the normal bundle of  $\Sigma_p$  inside  $G_0$ , we need to describe this normal bundle. Recall that when one blows up one point in a smooth variety  $X$ , creating an exceptional divisor  $E$  inside the blow-up  $Y \xrightarrow{\pi} X$ , the tangent exact sequence is  $0 \rightarrow TY \rightarrow \pi^*TX \rightarrow i_*TE \rightarrow 0$ , where  $i : E \rightarrow Y$  denotes the inclusion. Since the normal bundle of  $\pi_p$  inside the Grassmannian  $G$  is the quotient bundle  $Q$ , we get the following diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T\Sigma_p & \longrightarrow & TG_{0|\Sigma_p} & \longrightarrow & N_p \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & b_0^*T\pi_p & \longrightarrow & b_0^*TG|_{\pi_p} & \longrightarrow & b_0^*Q \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \oplus_{q \neq p} T\ell_p^q & \longrightarrow & \oplus_{q \neq p} TE_{pq}^0 & \longrightarrow & \oplus_{q \neq p} N_p^q \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Here we denoted by  $N_p^q$  the normal bundle to  $\ell_p^q \simeq \mathbb{P}^1$  inside  $E_{pq}^0 \simeq \mathbb{P}^3$ , which is just  $\mathcal{O}_{\ell_p^q}(1) \oplus \mathcal{O}_{\ell_p^q}(1)$ . We deduce the Segre class

$$s(N_p) = s(b_0^*Q) \prod_{q \neq p} c(\mathcal{O}_{\ell_p^q}(1))^2 \in A^*(\Sigma_p).$$

On the one hand, the Segre class  $s(Q)$  equals the Chern class of the tautological bundle on  $G$ , that is  $s(Q) = 1 - H_1 + \sigma_{11}$ , and the Schubert class  $\sigma_{11}$  restricts to zero

on  $\pi_p$ . On the other hand, on the del Pezzo surface  $\Sigma_p$  we have  $\mathcal{O}_{\ell_p^q}(1) = \mathcal{O}(-\ell_p^q)|_{\ell_p^q}$ , from which we get the Segre class  $s(\mathcal{O}_{\ell_p^q}(1)) = 1 + \ell_p^q + 2(\ell_p^q)^2$ . Finally,

$$s(N_p) = 1 - H_1 + 2 \sum_{q \neq p} \ell_p^q + 2 \sum_{q \neq p} (\ell_p^q)^2.$$

We can deduce several intersection numbers, since for any class  $C_{3-k}$  of degree  $3-k$  on  $G_0$ , we have the classical formulas

$$F_p^{k+1} b_1^* C_{3-k} = \int_{F_p} F_p^k b_1^* C_{3-k} = (-1)^k \int_{\Sigma_p} s_{k-1}(N_p) C_{3-k}.$$

**Lemma 6.2.**

$$(F_p^1)^4 = 8, \quad (F_p^1)^3 H_1 = -1, \quad (F_p^1)^2 H_1^2 = -1, \quad F_p^1 H_1^3 = 0.$$

Note also that  $F_p^1$  does not meet  $E_{r,q}^1$  for  $r, q \neq p$ , but it meets  $E_{p,q}^1$  transversally along the surface  $S_p^q = b_1^{-1}(\ell_p^q)$ . Therefore

$$\mathcal{O}_{G_1}(E_{p,q}^1|_{F_p^1}) = \mathcal{O}_{F_p^1}(S_p^q) = b_1^* \mathcal{O}_{\Sigma_p}(\ell_p^q).$$

Applying the previous formula to  $C_{3-k} = (E_{p,q}^0)^{3-k}$  we get:

**Lemma 6.3.**  $F_p^1 E_{r,q}^1 = 0$  if  $r, q \neq p$ , but

$$(F_p^1)^3 E_{p,q}^1 = -2, \quad (F_p^1)^2 (E_{p,q}^1)^2 = 1, \quad F_p^1 (E_{p,q}^1)^3 = 0.$$

On the other hand,  $E_{p,q}^0$  gets blown-up along the two-skew lines  $\ell_p^q$  and  $\ell_q^p$ , and its strict transform  $E_{p,q}^1$  is contracted to the quadratic surface  $\ell_p^q \times \ell_q^p$  in  $X_4$ . This surface is also the intersection of  $F_p$  and  $F_q$  in  $X_4$ , in particular it is contained in  $F_p$ . We deduce, denoting  $Bl_Q$  by  $c$ , that

$$c^* F_p = F_p^1 + \sum_{q \neq p} E_{p,q}^1.$$

Summing up over  $p$ , we get the relation  $c^* F = F^1 + 2E^1$ .

**Corollary 6.4.**  $C_4 \subset G(3, V_5)$  is the image of  $G = G(2, V_4)$  by the linear system  $|I_\pi(3H_1)|$  of cubics vanishing along the union  $\pi$  of the five planes  $\pi_1, \dots, \pi_5$ .

We have enough information to describe the full intersection product on  $X_4$ .

**Proposition 6.5.** The nonzero intersection numbers among the divisor classes  $H_1, F_1, \dots, F_5$  are the following, for  $1 \leq p \neq q \leq 5$ :

$$F_p^4 = 12, \quad F_p^3 F_q = -2, \quad F_p^2 F_q^2 = 1, \quad F_p^3 H_1 = -1, \quad F_p^2 H_1^2 = -1, \quad H_1^4 = 2.$$

Moreover we always have  $H_1 F_p F_q = 0$  for  $p \neq q$  and  $F_p F_q F_r = 0$  for  $p \neq q \neq r \neq p$ .

*Proof.* The values of  $F_p^3 H_1$  and  $F_p^2 H_1^2$  can be computed directly by restricting to a general hyperplane or a general codimension two section of  $G$ ; then we avoid the points  $\pi_{qr}$  and we are reduced to compute the self-intersection of the exceptional divisor for the blow-up of a line in a three-dimensional quadric, or a point in a surface. Then we can deduce the value of  $F_p^4$  by computing the self-intersection of  $H_2 = 3H_1 - F$ , which we know is equal to

$$12 = 81H_1^2 - 108H_1^3 F + 54H_1^2 F^2 - 12H_1 F^3 + F^4 = 162 - 270 + 60 + F^4.$$

This gives  $F^4 = F_1^4 + \dots + F_5^4 = 60$ , hence  $F_p^4 = 12$ . (But note that this is not equal to  $(c^*F_p)^4 = -4$ , as a consequence of the fact that  $F_p$  contains four of the quadratic surfaces blown-up by  $c$ .)

The other intersection numbers can be computed by pulling-back by  $c$  and using Lemma 6.2.  $\square$

**Proposition 6.6.** *The square map  $S^2A_1(X_4) \rightarrow A^2(X_4)$  is surjective. As a consequence, the  $\mathcal{S}_5$ -module structure of  $A^2(X_4)$  is*

$$A^2(X_4) = 4U_0 \oplus 2U_4 \oplus U_5.$$

*Proof.* The decomposition of the  $\mathcal{S}_5$ -module  $S^2A^1(X_4)$  is  $4U_0 \oplus 3U_4 \oplus U_5$ , the sum of three isotypic components, and the kernel of the square map must decompose accordingly.

First consider the four invariant classes  $H_1^2, H_1F, F^{(2)}, F^{(11)}$ , where

$$F^{(2)} = \sum_p F_p^2, \quad F^{(11)} = \sum_{p < q} F_p F_q.$$

Suppose that there is a relation  $aH_1^2 + bH_1F + cF^{(2)} + dF^{(11)} = 0$ . Multiplying successively by  $H_1^2, H_1F_p, F_p^2, F_pF_q$  and using the results of Proposition 6.5, we deduce that  $2a - 5c = 0$ ,  $b + c = 0$ ,  $a + b + 16c - 8d = 0$ ,  $4c - d = 0$ , hence  $a = b = c = d = 0$ .

Now consider the possibility that  $U_5$  be contained in the kernel of the square map. We claim that  $U_5$  is embedded inside  $S^2A^1(X_4)$  as the space of linear combinations  $\sum_{p \neq q} a_{pq} F_p F_q$  with  $a_{pq} = a_{qp}$  and  $\sum_r a_{pr} = 0$  for all  $p, q$ . Indeed, this defines an invariant five-dimensional subspace of  $S^2A^1(X_4)$ , not containing any invariant class, so it must be  $U_5$ . A typical element is

$$3F_p F_q - (F_p + F_q) \sum_{r \neq p, q} F_r + \sum_{s, t \neq p, q} F_s F_t.$$

If this was zero in  $A^2(X_4)$ , multiplying by  $F_p F_q$  would imply that the intersection number  $F_p^2 F_q^2 = 0$ , which is not the case.

We can conclude that the kernel of the square map must be contained in the isotypic component  $3U_4$  of  $S^2A^1(X_4)$ , which is generated by the three copies of  $U_4$  respectively obtained as the linear combinations  $\sum_p a_p H_1 F_p$ ,  $\sum_p a_p F F_p$  and  $\sum_p a_p F_p^2$  for  $\sum_p a_p = 0$ . A copy of  $U_4$  in the kernel corresponds to a relation of the form

$$uH_1 F_p + vF F_p + wF_p^2 = I \quad \forall p,$$

for  $I$  an invariant class. Since  $I$  is invariant, multiplying by  $H_1 F_p$  and  $H_1 F_q$  must then give the same intersection number, which gives the relation  $-u - v - w = 0$ . Similarly, multiplying by  $F_p^2$  or  $F_q^2$  must give the same result, that is  $-u + 4v + 12w = -v + w$ . Finally, multiplying by  $F_p F_q$  or  $F_q F_r$  with  $q, r$  distinct from  $p$  must also give the same result, that is  $-v - 2w = 0$ . These three equations are linearly dependent and reduce to  $u = w$  and  $v = -2w$ , which proves that there is a unique copy of  $U_4$  in the kernel of the square map. This concludes the proof.  $\square$

**Threefolds.** Consider the two families of divisors in  $X_4$  given by sections of  $H_1$  and  $H_2$ , respectively. Since a general hyperplane section in  $G(2, V_4) \simeq \mathbb{Q}^4$  will avoid the ten points  $\pi_{pq}$ , the first ones are just blowups of five disjoint lines in  $\mathbb{Q}^3$ . For the same reason, the second ones, say  $Z_3$ , are isomorphic with their images in  $G(3, V_5) \cap$

$H_2$ , which are codimension two degeneracy loci defined by the condition that the morphism  $\wedge^2 V \rightarrow V_4^\vee$  has rank exactly two. Its image is then the pullback from  $G(2, V_4)$  of the dual quotient bundle  $Q^\vee$ . In particular we get an exact sequence

$$0 \rightarrow \mathcal{O}(1, -2) \rightarrow p_2^*(\wedge^2 V) \rightarrow p_1^* Q^\vee \rightarrow 0$$

on  $Z_3$ . This shows in particular that  $\mathcal{O}(-1, 2)$ , the restriction of  $2H_2 - H_1 = 5H_1 - 2F$ , is generated by sections on  $Z_3$ . The image in  $Z_3$  is the closure of the planes  $U \subset V_4$  such that the image of  $S^2 U \rightarrow \wedge^2 V_5 \simeq V_5$  is isotropic with respect to some three-form on  $V_5$ . This defines a section of  $\wedge^3(S^2 U)^\vee = \det(U^\vee)^3$ , so that the image of  $Z_3$  in  $G(2, V_4)$  is a singular cubic hypersurface.

**K3 surfaces.** By taking sections of  $H_1 \oplus H_2$  in  $X_4$ , we get a family of smooth K3 surfaces  $S$  in  $X_4$ . We denote by  $h_1, h_2, f_1, \dots, f_5$  the restriction to  $S$  of the divisors  $H_1, H_2, F_1, \dots, F_5$ .

**Proposition 6.7.** *The intersection numbers of these divisors in  $S$  are*

$$h_1^2 = 6, \quad h_2^2 = 14, \quad h_1 h_2 = 13, \quad h_1 f_i = 1, \quad h_2 f_i = 5, \quad f_i f_j = -2\delta_{ij}.$$

*Proof.* This is an immediate consequence of the computations above, since for two divisors  $A, B$  on  $X$  restricting to  $a, b$  on  $S$ , we have  $ab = ABH_1H_2$ .  $\square$

An obvious consequence is that  $h_1, f_1, \dots, f_5$  are linearly independent. Moreover, the curves  $C_i = F_i \cap S$  are  $(-2)$ -curves on  $S$ , mapping to lines on  $G(2, V_4)$  and to rational quintics in  $G(2, V_5)$ . The divisor  $5h_1 - h_2 = 2h_1 + f$  should contract these five  $(-2)$ -curves to the five singular points of a surface  $\bar{S}$ . Note that this is a divisor of degree 34, so  $\bar{S}$  could be a degeneration of a smooth K3 surface of genus 18. Mukai described the generic such K3 surface as the zero locus in  $OG(3, 9)$  of five sections of the rank two spinor bundle. What is the connection? Note that we have a family of surfaces of dimension  $5 + 9 = 14 = 19 - 5$ , which is coherent with the expectation that imposing 5 nodes on a K3 surface of genus 18 should give five independent conditions.

## 7. THE IGUSA QUARTIC AND THE COBLE FOURFOLD

Given a linear form  $h$  on  $V_5$ , there is an associated quadratic form  $Q_h$  on  $V_4$ :

$$Q_h(v) = h \wedge \theta(v) \wedge \theta(v) \in \wedge^5 V_5^\vee \simeq \mathbb{C}.$$

**Proposition 7.1.** *The quartic  $\det(Q_h) = 0$  is the Igusa quartic in  $\mathbb{P}^4 = \mathbb{P}(V_5^\vee)$ .*

*Proof.* Recall that the generic point of the Segre cubic  $C_3 \subset \mathbb{P}(V_5)$  is the kernel of one of the two-forms  $\theta(v)$ , and that we can get this kernel as the line generated by  $\theta(v) \wedge \theta(v) \in \wedge^4 V_5^\vee \simeq V_5$ . At this generic point, the affine tangent space to the Segre cubic is therefore the hyperplane of  $V_5$  generated by the vectors of the form  $\theta(v) \wedge \theta(w) \in \wedge^4 V_5^\vee \simeq V_5$ . This hyperplane is defined by a linear form  $h_v \in V_5^\vee$  that vanishes on these vectors, which exactly means that  $h_v \wedge \theta(v) \wedge \theta(w) = 0$  for any  $w \in V_5$ . In other words,  $Q_{h_v}(\theta(v), \theta(w)) = 0$  for all  $w \in V_5$ , which means that  $\theta(v)$  belongs to the kernel of the quadratic form  $Q_{h_v}$ . In particular the latter is degenerate.

We have thus proved that the generic point of the projective dual variety of the Segre cubic is contained in the quartic hypersurface  $\det(Q_h) = 0$ . But this projective dual is well-known to be the Igusa quartic in  $\mathbb{P}(V_5^\vee)$ , and these two quartics have to coincide.  $\square$

This yields a simple determinantal representation of the Igusa quartic. Using Ozeki's representative we get

$$Q_h = \begin{pmatrix} 2h_1 & -h_2 & -h_3 & -h_5 \\ -h_2 & 2h_3 & -h_4 & 0 \\ -h_3 & -h_4 & 2h_5 & -h_1 \\ -h_5 & 0 & -h_1 & 2h_3 \end{pmatrix},$$

whose determinant is readily computed to be

$$-\det(Q_h) = 4h_3^4 + 4h_3^2(3h_1h_5 - h_2h_4) - 4h_3(h_1^3 + h_5^3 + h_1h_4^2 + h_2^2h_5) + (h_1h_2 - h_4h_5)^2.$$

One can consider inside  $\mathbb{P}(V_5^\vee) \times G(2, V_4)$  the locus  $J_5$  of pairs  $([h], U)$  such that  $U$  is isotropic with respect to  $Q_h$ . Recall that  $OG_Q(2, 4) = \mathbb{P}^1 \cup \mathbb{P}^1$  is the disjoint union of two smooth conics when  $Q$  is non degenerate. When  $Q$  is a quadratic form of corank one on  $V_4$ , the corresponding orthogonal Grassmannian  $OG_Q(2, V_4)$  is a single conic (while if  $Q$  has corank two,  $OG_Q(2, V_4)$  is the union of two planes meeting at one point, defined by the kernel). This means that the Stein factorization of the projection of  $J_5$  to  $\mathbb{P}(V_5^\vee)$  is  $J_5 \rightarrow Cob_4 \rightarrow \mathbb{P}(V_5^\vee)$  where  $Cob_4$  is the double cover of  $\mathbb{P}(V_5)$  branched over the Igusa quartic: that is, the *Coble fourfold* [8].

On the other hand, denote by  $\mathcal{Q}_{X_4}$  the pull-back to  $X_4$  of the rank two quotient bundle on  $G(3, V_5)$ . The  $\mathbb{P}^1$ -bundle  $\mathbb{P}(\mathcal{Q}_{X_4})$  over  $X_4$  has a natural map to  $G(2, V_4) \times \mathbb{P}(V_5^\vee)$  and we claim that its image is precisely  $J_5$ . Indeed, a generic element  $(U, V \subset H)$  of  $G(2, V_4) \times Fl(3, 4, V_5)$  belongs to  $\mathbb{P}(\mathcal{Q}_{X_4})$  when  $V$ , hence  $H$ , contains the kernels of all the two-forms  $\theta(v)$ ,  $v \in U$ . But if  $h$  is a linear form defining  $H$ , the condition that  $H$  contains the kernel of  $\theta(v)$  exactly means that  $h \wedge \theta(v) \wedge \theta(v) = 0$ , hence our claim. Moreover the projection map  $\mathbb{P}(\mathcal{Q}_{X_4}^\vee) \rightarrow J_5$  is birational, being clearly bijective outside the five special planes in  $G(2, V_4)$ . So we get a diagram

$$\begin{array}{ccc} & \mathbb{P}(\mathcal{Q}_{X_4}^\vee) & \\ \swarrow \text{bir} & & \searrow \mathbb{P}^1 \\ J_5 & & X_4 \\ \searrow \text{conic} & & \\ & Cob_4 & \end{array}$$

where the south-west arrow is a conic bundle, at least generically. But the picture does not seem to recover the small resolutions of  $Cob_4$  described in [8].

## 8. LOCAL RIGIDITY

Since  $V_4^\vee \otimes \wedge^2 V_5^\vee$  is prehomogeneous, we expect that  $X_4$  has strong rigidity properties. What we can prove is the following statement.

**Proposition 8.1.**  *$X_4$  is locally rigid and has finite automorphism group.*

*Proof.* Local rigidity is equivalent to the vanishing of  $H^1(TX_4)$ . In order to check this, as usual we rely on the normal exact sequence, which yields an exact sequence of cohomology groups

$$H^0(TG|_{X_4}) \longrightarrow H^0(E|_{X_4}) \longrightarrow H^1(TX_4) \longrightarrow H^1(TG|_{X_4}).$$

So local rigidity will follow from the following statements, to be proved separately:

- (1)  $H^1(TG|_{X_4}) = 0$ ;
- (2)  $H^0(E|_{X_4}) = V_4^\vee \otimes \wedge^2 V_5^\vee / \langle \theta \rangle$ ;
- (3)  $H^0(TG|_{X_4}) \longrightarrow H^0(E|_{X_4})$  is surjective.

The third statement follows from the fact that  $\mathbb{P}(V_4^\vee \otimes \wedge^2 V_5^\vee)$  is prehomogeneous under  $PGL(V_4) \times PGL(V_5)$ , more precisely from the fact that the orbit of  $[\theta]$  is open, since this implies that the image of the natural differential  $\mathfrak{sl}_4 \times \mathfrak{sl}_5 = H^0(TG) \longrightarrow V_4^\vee \otimes \wedge^2 V_5^\vee / \langle \theta \rangle$  sending  $X$  to  $X(\theta) \bmod \theta$  is surjective; since this morphism can also be defined by restricting first to  $X_4$  and then composing with the morphism we are interested in, the latter must also be surjective.

In order to prove the second statement we twist by  $E$  the Koszul complex resolving the structure sheaf of  $X_4$ . By standard cohomological arguments, it is enough to check that  $H^0(G, \text{End}_0(E)) = 0$  and  $H^i(G, E \otimes \wedge^{i+1} E^\vee) = 0$  for any  $i > 0$ . For the first claim, observe that

$$\text{End}_0(E) = \text{End}_0(U) \oplus \text{End}_0(V) \oplus \text{End}_0(U) \otimes \text{End}_0(V)$$

is in fact acyclic. For the second claim, check that  $E \otimes \wedge^{i+1} E^\vee$  is also acyclic for any  $i > 0$ . Similarly, in order to prove the first statement we need to check that  $H^{i+1}(G, TG \otimes \wedge^i E^\vee) = 0$  for any  $i \geq 0$ , which is again a straightforward application of Bott's theorem.

By the same type of arguments (or using a computer to check that  $\chi(TX_4) = 0$ ), we deduce that  $H^0(TX_4) = 0$ , which implies that the automorphism group is discrete, hence finite since  $X_4$  is Fano.  $\square$

The question remains open, whether  $X_4$  is also globally rigid, which would be remarkable for a Fano fourfold with such a big Picard number. The first thing to check is whether  $X_4$  remains smooth when we degenerate  $\theta$  to the codimension one orbit. If yes, we would get a similar situation to the case of codimension two linear sections of the spinor tenfold (which has Picard number one, though).

Another question one may ask is whether the quotient bundle restricted to  $X_4$  is rigid. In other words, is the morphism to  $G(3, V_5)$  uniquely defined?

## 9. HIGHER DIMENSIONS

Let us briefly describe the higher dimensional models.

**Proposition 9.1.**  *$X_6$  is a rational Fano sixfold of index one and Picard rank two.*

*The projection of  $X_6$  to  $G(3, V_5)$  is birational, with non trivial fibers isomorphic to  $\mathbb{P}^1$  over the smooth locus of  $C_4$ , and to  $\mathbb{P}^2$  over its ten singular points.*

*The projection to  $\mathbb{P}(V_4)$  is a  $\mathbb{Q}^3$ -bundle outside  $P_0$ , with five four-dimensional fibers over  $P_0$ .*

From this description and that of  $X_4$ , we deduce that in the Grothendieck ring of varieties one has the relation  $[X_6] + L^3[Y_0] = [G(3, V_5)] + L[X_4]$ . This yields the Poincaré polynomial of  $X_6$ ,

$$P_{X_6}(t) = 1 + 2t + 8t^2 + 9t^3 + 8t^4 + 2t^5 + t^6.$$

**Proposition 9.2.**  *$X_8$  is a Fano eightfold of pseudo-index three, while  $X'_8$  is Fano eightfold of index three.*

**Proposition 9.3.** *The projections of  $X_8, X'_8$  to  $G(2, V_5)$  are dual  $\mathbb{P}^2$ -fibrations over the complement of a del Pezzo surface of degree five, the exceptional fibers being isomorphic to  $\mathbb{P}(V_4)$  and  $G(2, V_4)$  respectively.*

We can readily deduce that  $X_8$  and  $X'_8$  have pure cohomology, with Poincaré polynomials

$$\begin{aligned} P_{X_8}(t) &= 1 + 2t + 4t^2 + 6t^3 + 11t^4 + 6t^5 + 4t^6 + 2t^7 + t^8, \\ P_{X'_8}(t) &= 1 + 2t + 5t^2 + 11t^3 + 13t^4 + 11t^5 + 5t^6 + 2t^7 + t^8. \end{aligned}$$

Of course  $X_6, X_8, X'_8$  inherit the same symmetries as  $X_4$ . E. Fatighenti and F. Tantarri checked the necessary vanishing conditions to establish, as for  $X_4$ , that they are also infinitesimally rigid.

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