UNIPOTENT SUBGROUPS OF STABILIZERS

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For James Humphreys

Abstract. We consider semi-continuity of certain dimensions on group schemes.

1. Introduction

Let G be an algebraic group over a field k. Let $d_u(G)$ the maximal dimension of a (smooth) connected unipotent subgroup of $G_{\overline{k}}$. Using techniques à la Demazure-Grothendieck, we show the following result.

Theorem 1.1. *Let S be a scheme and let G be a separated S-group scheme of finite presentation (for example an affine S-group of finite presentation). Then the function d_u on S is upper semi-continuous.*

Upper semi-continuous means informally that the function jumps along closed sets. In particular, the function is locally constant at the points of the minimal value locus. This gives the following useful corollary.

Corollary 1.2. *Let S be an irreducible scheme and let G be a separated S-group scheme of finite presentation. If for the generic point $\xi \in S$, $G_{\overline{\kappa(\xi)}}$ contains a d-dimensional smooth unipotent subgroup, then the same is true for $G_{\overline{\kappa(s)}}$ for all $s \in S$.*

A case of special interest is the group scheme of stabilizers called also the stabilizer of the diagonal [SGA3, V.10.2]. More precisely if G is an S–group scheme acting on a separated S–scheme X of finite presentation (with S noetherian), we consider the fiber product

$$\begin{array}{ccc} F & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ G \times_S X & \longrightarrow & X \times_S X \end{array} \qquad (g,x) \longmapsto (x, g.x).$$

It defines an X-group scheme F which is a closed X-subgroup scheme of $G \times_S X$ of finite presentation such that for each $x \in X$ of image $s \in S$, $F_{\kappa(x)} \subset G \times_{\kappa(s)} \kappa(x)$ is the stabilizer of the point x for the action of $G \times_{\kappa(s)} \kappa(x)$ on $X \times_{\kappa(s)} \kappa(x)$. In case $\kappa(s) = \kappa(x)$, the usual notation for $F_{\kappa(x)}$ is G_x.

One motivation for this question was related to the base size of finite groups acting primitively on a set and the existence of regular orbits for nontransitive actions. An important

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case is that of a finite simple group of Lie type over a finite field where the action comes from the algebraic group. See [BGS]. Another interesting case is when G is a reductive algebraic group acting linearly on X (or acting on the Grassmanian of a rational module). See [GG, GL, PV] for more on this. We give more details on this in Section 5.

Note that by the Lang-Steinberg theorem, an algebraic group defined over $\mathbb{F}_q$ has a Borel subgroup defined over $\mathbb{F}_q$. We also know that if U is a d-dimensional unipotent subgroup defined over $\mathbb{F}_q$, then $|U(\mathbb{F}_q)| = q^d$ [B, GG]. We can apply Corollary 1.2 to the stabilizer scheme to obtain the following result.

Corollary 1.3. *Let G be a algebraic group acting faithfully on an irreducible variety X and assume that the G, X and the action are defined over a finite field $\mathbb{F}_q$. Assume that there is a nonempty open subset X_0 of X such that the stabilizer G_x of $x \in X_0$ has a d-dimensional unipotent subgroup. Then for all $x \in X(\mathbb{F}_q)$, $G_x(\mathbb{F}_q)$ contains a subgroup of size q^d.*

One can ask more generally what other functions are upper semi-continuous. Of course, dimension is [SGA3, VI_B.4.3]. In fact, we will show that other such functions are also upper semi-continuous. On the other hand, if we define $d^0(G)$ to be the dimension of the derived subgroup of the connected component of G, it is not true that d^0 need be upper semi-continuous. We study this in Section 6 and particularly for the smooth case. Smoothness rarely holds in the case of the stabilizer scheme and we give an example to show the failure of upper semi-continuity for $d^0(G)$ in stabilizer schemes.

We use mostly the terminology of Borel's book [B] and time to time the more general setting Demazure-Gabriel's book [D-G] and the SGA3 seminar [SGA3] where in particular an algebraic group is not supposed smooth. All definitions are coherent.

In the next sections, we prove some preliminary results. We prove Theorem 1.1 and other upper semi-continuity results in Section 4. In Section 6, we consider the dimension of the derived subgroup. In the final section, we present an alternate proof of Theorem 1.1 due to Brian Conrad.

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2. Definition of the rank functions

Let k be a field and let G/k be an algebraic group. We remind the reader that a linear algebraic group G is unipotent if for each (or any) faithful $\overline{k}$–representation $\rho : G_{\overline{k}} \to \mathrm{GL}_{n,\overline{k}}$, $\rho(G(\overline{k}))$ consists in unipotent elements. This agrees with the general definition given in [D-G, 4.2.2.1], see 4.3.26.b of this reference. In practice we use the equivalent definition that G admits a closed k-embedding in a k–group of strictly upper triangular matrices (*ibid*, 4.2.2.5).

Similarly a closed smooth k–subgroup G of GL_n is trigonalizable if there exists $h \in \mathrm{GL}_n(k)$ such that $hGh^{-1} \subset B_n$ where B_n is the k–Borel subgroup of GL_n consisting in upper triangular matrices; the same holds for any linear representation $G \to \mathrm{GL}_r$ [B, 15.5]. That definitions holds more for an arbitrary affine algebraic k–group since it is equivalent to the Demazure-Gabriel's definition [D-G, 4.2.3.4].

2.1. **Relative version.** We define the following invariants:

(i) $D_u(k,G)$ = Maximal dimension of an unipotent k-subgroup of G;
(ii) $D_{cu}(k,G)$ = Maximal dimension of a commutative unipotent k-subgroup of G;
(iii) $D_t(k,G)$ = Maximal dimension of a trigonalizable k-subgroup of G;
(iv) $D_n(k,G)$ = Maximal dimension of an affine nilpotent k-subgroup of G;
(v) $D_{nt}(k,G)$ = Maximal dimension of a nilpotent trigonalizable k-subgroup of G;
(vi) $D_s(k,G)$ = Maximal dimension of an affine solvable k-subgroup of G;
(vii) $D_c(k,G)$ = Maximal dimension of an affine commutative k-subgroup of G;
(viii) $D'_c(k,G)$ = Maximal dimension of a commutative k-subgroup of G;
(ix) $D'_n(k,G)$ = Maximal dimension of a nilpotent k-subgroup of G;
(x) $D'_s(k,G)$ = Maximal dimension of a solvable k-subgroup of G.

We have $D_c(k,G) \leq D'_c(k,G)$, $D_n(k,G) \leq D'_n(k,G)$ with equalities if G is affine. We have the obvious inequalities $D'_c(k,G) \leq D'_n(k,G) \leq D'_s(k,G)$ and $D_{cu}(k,G) \leq \mathrm{Min}(D_c(k,G), D_u(k,G))$ and $D_u(k,G) \leq D_n(k,G)$, $D_{nt}(k,G) \leq \mathrm{Max}\big(D_n(G), D_t(k,G)\big) \leq D_s(k,G)$.

All these functions are increasing by change of fields.

Lemma 2.1. *If k is algebraically closed, then $D(k,G) = D(F,G)$ for any field extension F/k and for each function D as above.*

Proof. We do it for d_u, the other cases being similar. We can assume that F is algebraically closed so that G_F admits a smooth unipotent F–subgroup U of dimension d. There exists a finitely generated k–subextension E of F such that U is defined over E. The field E is function field of a smooth k-variety X. Up to shrinking X, U extends to a closed subgroup scheme $\mathfrak{U}$ of $G \times_k X$ which is smooth in view of [SGA3, VI_B.10]. Up to shrinking one more time, $\mathfrak{U}$ is a closed subgroup scheme of the X-group scheme of strictly upper triangular matrices. Since k is algebraically closed, we have $X(k) \neq \emptyset$. The fiber at $x \in X(k)$ provides a smooth unipotent k–subgroup $\mathfrak{U}_x \subset G$ of dimension d. Thus $D(k,G) \geq d$. □

2.2. **Absolute version.** We define now

(i) $d_u(G) = D_u(\overline{k}, G)$, i.e. the maximal dimension of an unipotent $\overline{k}$-subgroup of $G_{\overline{k}}$;
(ii) $d_{cu}(G) = D_{cu}(\overline{k}, G)$;
(iii) $d_t(G) = D_t(\overline{k}, G)$;
(iv) $d_n(G) = D_n(\overline{k}, G)$;
(v) $d_{nt}(G) = D_{nt}(\overline{k}, G)$;
(vi) $d_s(G) = D_s(\overline{k}, G)$;
(vii) $d_c(G) = D_c(\overline{k}, G)$;
(viii) $d'_c(G) = D'_c(\overline{k}, G)$;
(ix) $d'_n(G) = D'_n(\overline{k}, G)$;
(x) $d'_s(G) = D'_s(\overline{k}, G)$.

Clearly it does not depend of the choice of $\overline{k}$. All these functions are insensitive to change of fields.

Lemma 2.2. *We have $d(k,G) = d(F,G)$ for any field extension F/k and for each function d as above.*

Proof. The function d is the absolute version of a relative rank function D. Let $\overline{F}$ be an algebraic closure of F containing $\overline{k}$. Lemma 2.1 shows that $D(\overline{k},G) = D(\overline{F},G)$ whence $d(k,G) = d(F,G)$. □

Up to considering the connected reduced fiber, we have also

$$d_u(G) = \text{Maximal dimension of a smooth connected unipotent } \overline{k}\text{-subgroup of } G_{\overline{k}};$$

and similarly for the other absolute rank functions. Since affine smooth connected solvable $\overline{k}$–subgroups trigonalizable by the Lie-Kolchin's theorem [B, 10.5] it follows that $d_s(k,G) = d_t(k,G)$ and similarly we have $d_n(k,G) = d_{nt}(k,G)$.

We have $d_c(G) \leq d'_c(k,G)$, $d_n(G) \leq d'_n(G)$ and $d_c(G) \leq d'_c(G)$ with equalities if G is affine. We have the obvious inequalities $d'_c(G) \leq d'_n(G) \leq d'_s(G)$ and $d_{cu}(G) \leq \mathrm{Min}(d_c(G), d_u(G))$ and $d_u(G) \leq d_n(G) \leq d_s(G)$.

2.3. **Connections with the literature.** (a) In [SGA3, XII.1], Grothendieck defines related rank functions but which are different. For example the Grothendieck unipotent rank $\rho_u(G)$ of a smooth connected group G over an algebraically closed field is $d_u(C)$ where C is a Cartan subgroup of G^0. This function ρ_u is upper semi-continuous if G is smooth affine over a base [SGA3, XII.2.7.(i)]. Our result does not require smoothness (nor flatness).

(b) If G is a smooth connected affine algebraic group over an algebraically closed field k, the Borel subgroups are the maximal smooth connected solvable subgroups, they are all conjugate and so $d_s(G)$ is nothing but the dimension of a Borel subgroup. The unipotent radicals are then the maximal smooth connected unipotent subgroups, they are all conjugate and so $d_u(G)$ is the dimension of a the unipotent radical of a Borel subgroup of G. For $d_n(G)$ the situation is more complicated since maximal smooth connected nilpotent subgroups of G do not consist of a single conjugacy class. However there are finitely many conjugacy classes (Platonov, [P, thm 2.13]).

3. Specialization over a regular local ring

3.1. **Group schemes over a DVR.** Let A be a discrete valuation ring of fraction field K and of residue field k. If G is an A–group scheme of finite presentation, we would like to list properties on the generic fiber G_K which are inherited by the closed fiber G_k. For example, if G is flat, then G_K and G_k share the same dimension [SGA3, VI_B.4.3]. Also if G is separated and flat and G_K is affine, then G is affine (Raynaud, [PY, prop 3.1]) so that G_k is affine. Flatness is then an important property, we recall that an A–scheme $\mathfrak{X}$ is flat if and only if it is torsion free, this second condition being equivalent to the density of the generic fiber $\mathfrak{X}_K$ in $\mathfrak{X}$ [GW, §14.3]. For the study of the function d_u, the next statement is the key step.

Lemma 3.1. *We assume that G is flat and affine. If G_K is trigonalizable (resp. unipotent) so is G_k.*

Proof. We assume that G_K is trigonalizable, that is, G_K is an extension of a diagonalizable K-group by an unipotent K-group. According to [BT, 1.4.5], there exists a closed monomorphism $\rho : G \to \mathrm{GL}_N$. Since G_K is trigonalizable, its stabilizes a flag of K^N [D-G, prop. 4.2.3.4, $(i) \Longrightarrow (iv)$]. In other words ρ factorizes through a Borel subgroup B_K of GL_N. Since the A–scheme of Borel subgroups of GL_N is projective [SGA3, XXII.5.8.3], B_K extends uniquely to a Borel A–subgroup scheme B of GL_N [a concrete way is to use a filtration $V_0 = 0 \subsetneq V_1 \subsetneq V_2 \cdots \subsetneq V_N = K^N$ and put $\widetilde{V}_i = A^n \cap V_i$ for each i. It defines an A-flag of lattices of A^n which is stabilized by G]. Its reduction to k provides an embedding of G_k to a Borel k–subgroup of $\mathrm{GL}_{N,k}$. The last quoted result, $(v) \Longrightarrow (i)$, enables us to conclude that G_k is trigonalizable.

We assume furthermore that G_K is unipotent. We have an exact sequence of A–group schemes $1 \to U \to B \xrightarrow{q} \mathbb{G}_m^N \to 1$. Since G_K is unipotent $q_K \circ \rho_K = G_K \to (\mathbb{G}_m^N)_K$ is trivial according to [D-G, IV.2.2.4]. Since G_K is dense in G it follows that $q \circ \rho = G \to \mathbb{G}_m^N$ is trivial so that ρ factorizes through the unipotent radical U of B. A fortiori G_k admits a representation in a strictly upper triangular k-group so is unipotent according to [D-G, IV.2.2.5, $(vi) \Longrightarrow (i)$]. □

Remark 3.1. (a) If G_K is split K–unipotent, a result of Veĭsfeĭler-Dolgachev on unipotent group schemes [VD, thm. 1.1] shows also that G_k is unipotent. This is a very different proof.

(b) If G has smooth fibers and G_K is unipotent, the fact that G_k is unipotent follows from [SGA3, VI_B.8.4.(ii)]. This is a quite different proof.

(c) One simpler proof of (b) occurs in the alternate proof below of Theorem 1.1 using the smoothness of the scheme of maximal tori of G, see §7.

For dealing the other rank functions, we need more facts.

Lemma 3.2. *Let H be a K–subgroup of G_K and let $\mathfrak{H}$ be the schematic closure of H in G.*

(1) $\mathfrak{H}$ is a closed A–subgroup scheme of G which is flat of finite presentation. If H is central, then $\mathfrak{H}$ is central.

(2) The fppf quotient $G/\mathfrak{H}$ is representable by a separated A-scheme of finite presentation. Furthermore if G is flat so is $G/\mathfrak{H}$.

(3) If H is normal in G_K, then $\mathfrak{H}$ is a normal A–subgroup scheme of G and $G/\mathfrak{H}$ carries a natural structure of A-group scheme.

Proof. (1) The first part is in [EGA4, (2.8.1)]. Assume that H is central, that is the commutator map $G_K \times_K H \to G_K$ is trivial. Since $G_K \times_K H$ is dense in $G \times_A \mathfrak{H}$, it follows that the commutator map $G \times_A \mathfrak{H} \to G$ is trivial, so that $\mathfrak{H}$ is central in G.

(2) The representability is result by Anantharaman [A, IV, th. 4.C] so that $G/\mathfrak{H}$ is separated and of finite presentation [SGA3, VI_B.9.2.(x) and (xiii)]. If G is flat, $G/\mathfrak{H}$ is flat according to [SGA3, VI_B.9.2.(xi)].

(3) Assume that H is normal in G_K, that is, the commutator map $G_K \times_K H \to G_K/H$ is trivial. Since $G_K \times_K H$ is dense in $G \times_A \mathfrak{H}$, it follows that the commutator map $G \times_A \mathfrak{H} \to G/\mathfrak{H}$ is trivial so that $\mathfrak{H}$ is a normal A–subgroup scheme of

$\mathfrak{G}$. According to [SGA3, VI_B.9.2.(iv)], it follows that $G/\mathfrak{H}$ carries a natural structure of A-group scheme.

□

Lemma 3.3. *We assume that G is flat. If G_K is commutative (resp. nilpotent, solvable). Then G_k is commutative (resp. nilpotent, solvable).*

Proof. If G_K is commutative, so is G and G_k according to Lemma 3.2.(1).

We assume now that G_K is nilpotent, that is, admits a central composition serie $H_0 = 1 \subset H_1 \subset H_2 \subset \cdots \subset H_{n-1} \subset H_n = G_K$ where the H_i's are normal K–subgroups of G_K and such that each H_{i+1}/H_i is central in G_K/H_i. Let G_i be the schematic closure of H_i in G, this is a flat A-group scheme and all H_i's are normal A–subgroups of G according to Lemma 3.2.(3). Furthermore each quotient G_{i+1}/G_i is central in G/H_i. By extending the scalars to k we get then a central composition series for G_k.

The argument is similar for the solvable case. □

3.2. **The regular local ring case.** Let A be a regular local ring with fraction field K and residue field k.

Proposition 3.4. *Let G be an A-group scheme of finite presentation.*

(1) Assume that k is infinite and that G_K contains an algebraic subgroup (resp. normal subgroup) of dimension d. Then G_k contains an algebraic subgroup (resp. normal subgroup) of dimension d.

(2) Assume that G_K contains an algebraic subgroup which is commutative (resp. nilpotent, solvable) of dimension d. Then G_k contains an algebraic subgroup which is commutative (resp. nilpotent, solvable) of dimension d.

(3) Assume that that G is separated and that G_K contains an algebraic subgroup which is affine commutative (resp. unipotent, commutative unipotent, affine nilpotent, nilpotent trigonalizable, trigonalizable, affine solvable) of dimension d. Then G_k contains an algebraic subgroup which is affine commutative (resp. unipotent, commutative unipotent, affine nilpotent, nilpotent trigonalizable, trigonalizable, affine solvable) of dimension d.

Proof. According to [EGA4, lemma 15.1.1.6], there exists a discrete valuation ring B which dominates A and such that its residue field is a purely (finitely generated) transcendental extension of k. We denote by L the fraction field of B and by l its residue field. We have $l = k$ or $k(t_1, \dots, t_n)$.

(1) Our assumption is that G_K contains an algebraic subgroup (resp. normal subgroup) H which is of dimension d. We consider the schematic closure of H_L in G_B, this defines a flat B–group scheme (resp. normal B–subgroup scheme according to Lemma 3.2.(3)) $\mathfrak{H}$ of closed fiber $\mathfrak{H}_l$ which is a subgroup of $(G_k)_l$. Since k is infinite, one may "specialize" at a rational k–point to obtain a k–subgroup of G_k of dimension d.

(2) Our assumption is that G_K contains an algebraic subgroup H which is commutative (resp. nilpotent, solvable) of dimension d. We consider the schematic closure of H_L in G_B, this defines a flat B–group scheme $\mathfrak{H}$ of closed fiber $\mathfrak{H}_l$ which is a subgroup of $(G_k)_l$. We apply Lemma 3.3 to $\mathfrak{H}$ and obtain that $\mathfrak{H}_l$ is commutative (resp. nilpotent, solvable) of dimension d.

By induction on n, we may assume that $l = k(t)$. We have that $G_{k((t))}$ admits the subgroup $\mathfrak{H}_{k((t))}$ which is commutative (resp. nilpotent, solvable) of dimension d. By performing the same method as above in the case of the DVR $k[[t]]$, it follows that G_k contains a commutative (resp. nilpotent, solvable) subgroup of dimension d.

(3) Though the argument is very similar, we provide all details. Our assumption is that G_K contains an algebraic subgroup H which is affine commutative (resp. unipotent, commutative unipotent, affine nilpotent, nilpotent trigonalizable, trigonalizable, affine solvable). We consider the schematic closure of H_L in G_B, this defines a flat B–group scheme $\mathfrak{H}$ of closed fiber $\mathfrak{H}_l$ which is a subgroup of $(G_k)_l$. Since G_B is separated so is $\mathfrak{H}$. Raynaud's affiness criterion [PY, prop 3.1] ensures that $\mathfrak{H}$ is affine over B.

We combine Lemma 3.1 and 3.3 for $\mathfrak{H}$ and obtain that $\mathfrak{H}_l$ is affine commutative (resp. unipotent, commutative unipotent, affine nilpotent, nilpotent trigonalizable, trigonalizable, affine solvable) of dimension d.

Once again, by induction on n, we may assume that $l = k(t)$. We have that $G_{k((t))}$ admits the subgroup M which is affine commutative (resp. unipotent, affine nilpotent, nilpotent trigonalizable, trigonalizable, affine solvable). By performing the same method as above in the case of the DVR $k[[t]]$, it follows that G_k contains an affine (resp. unipotent, affine nilpotent, nilpotent trigonalizable, trigonalizable, affine solvable) subgroup of dimension d. □

3.3. **More permanence properties.**

Lemma 3.5. *Let G be an algebraic group defined over a field k.*

(1) For each function D as above, we have $D(k, G) = D(k(t), G) = D(k((t)), G)$.

(2) If X is a connected smooth k–variety such that $X(k) \neq \emptyset$, we have $D(k, G) = D(k(X), G)$.

Proof. (1) We have the obvious inequalities $D(k, G) \leq D(k(t), G) \leq D(k((t)), G)$. The fact that $D(k((t)), G) \leq D(k, G)$ follows from Proposition 3.4 applied to $A = k[[t]]$ and the group scheme $G \times_k k[[t]]$.

(2) We have $D(k, G) \leq D(k(X), G)$. For proving the converse inequality, we pick a point $x \in X(k)$. Let $(t_1, \dots, t_d)$ be a system of parameters of the regular local ring $R = \mathcal{O}_{X,x}$. Then its completion is k–isomorphic to $k[[t_1, \dots, t_d]]$ which embeds in the field of iterated Laurent series $k((t_1)) \dots ((t_d))$. It follows that $k(X)$ embed in $k((t_1)) \dots ((t_d))$ so that $D(k(X), G) \leq D(k((t_1)) \dots ((t_d)), G)$. By induction on d, (1) provides $D(k, G) = D(k((t_1)) \dots ((t_d)), G)$ so that $D(k(X), G) \leq D(k, G)$.

□

4. Upper semi-continuity

Theorem 4.1. *define $d_\bullet(s) = d_\bullet(G_{\kappa(s)})$ for each $s \in S$.*

(1) The functions d'_c, d'_n, d'_s on S arer upper semi-continuous.

(2) Assume that G is separated. The functions $d_c, d_u, d_{cu}, d_n, d_s$ on S arer upper semi-continuous.

Proof. We prove both statements simultaneously. Let $d_\bullet$ one of the function. The problem is local so we can assume that $S = \mathrm{Spec}(A)$ for A an integral local ring of fraction field K and residue field k. We have to show that $d_\bullet(G_k) \geq d_\bullet(G_K) = d$. By using the standard yoga of noetherian reduction [SGA3, VI_B.10.2], we can assume that A is furthermore noetherian. If A is a field, we have $K = k$ and this is obvious. We assume that A is not a field. According to [EGA2, prop. 7.1.7], there exists an extension L of K (of finite type) and equipped with a discrete valuation such that its valuation ring B dominates A, that is, $A \subset B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$. We denote by l the residue field of B. Since $d_\bullet(G_k) = d_\bullet(G_l)$ and $d_\bullet(G_K) = d_\bullet(G_L)$, it is enough to show that $d_\bullet(G_l) \geq d_\bullet(G_L) = d$.

In other words the problem reduces to the case of a discrete valuation ring A with fraction field K. Our assumption is that $G_{\overline{K}}$ contains a closed subgroup of dimension d which is commutative (resp. nilpotent, solvable, affine commutative, unipotent, commutative unipotent, nilpotent, nilpotent trigonalizable, nilpotent, nilpotent trigonalizable, nilpotent, trigonalizable, solvable). Then there exists a finite K–subextension K' of $\overline{K}$ such that the same holds for $G_{K'}$. Let v' be an extension of the valuation v_K to K' and denote by B' its valuation ring and by k' its residue field. Once again we have $d_\bullet(G_k) = d_\bullet(G_{k'})$ and $d = d_\bullet(G_K) = d_\bullet(G_{K'})$, so that it is enough to show that $d_\bullet(G_{k'}) \leq d$. Proposition 3.4.(2) and (3) shows that $G_{k'}$ contains a closed subgroup of dimension d which is commutative (resp. nilpotent, solvable, affine commutative, unipotent, commutative unipotent, nilpotent, nilpotent trigonalizable, nilpotent, nilpotent trigonalizable, nilpotent, trigonalizable, solvable). We get then $D_\bullet(k', G_k) \geq d$ so a fortiori $d_\bullet(G_k) \geq d$. □

5. Finite Groups

One motivation for considering this question comes from a problem about finite groups.

Let G be a group acting on a set X. A base of G acting on X is a subset Y of X such any element of $g \in G$ which fixed Y pointwise acts trivially on X. The base size $b(G, X)$ is the minimal cardinality of a base. In the case of finite groups, this has been classical object of study for more than 150 years. This has had many applications (e.g. in computational group theory). One is also interested in this from a probabilistic point of view; what is the proportion of the subsets of size b which are a base.

In [BGS], this was considered for G a simple algebraic group acting on the homogeneous space G/M with M a maximal closed subgroup and in almost cases $b(G, M)$ was determined exactly (in a few cases, there was a small range of possible values). In this case one can consider two other quantities. We define $b^0(G, X) = c$ to be the smallest positive integer c so that there is subset Y of X of size c so that the pointwise stabilizer of Y is finite and $b^1(G, X) = e$ where e is the smallest positive integer such that the pointwise stabilizer of e points is trivial. It is easy to show that $b^0 \leq b \leq b^1 \leq b^0 + 1$.

Note that this can be rephrased in terms of G acting on $(G/M)^e$ and asking if there is a regular orbit or an orbit of $\dim G$ or if the generic orbit is regular.

More generally let G be an algebraic group acting on a variety X and assume that the action is defined over a finite field. We are interested the stabilizers in $G(\mathbb{F}_q)$ of a point $x \in X(\mathbb{F}_q)$ (and more generally we consider Steinberg-Lang endomorphisms with finite group of fixed points). Note that $X(\mathbb{F}_q)$ may not be a single orbit for $G(\mathbb{F}_q)$ if the stabilizer G_x is not connected.

As noted the stabilizer scheme $\{(g,x)|g \in G, x \in X, gx = x\}$ satisfies our hypotheses and so our results apply in this case. In particular, if for generic x, G_x contains a d-dimensional connected unipotent subgroup, then G_y does for all $y \in X$. Then Corollary 1.3 implies that if $y \in X(\mathbb{F}_q)$ and G_y has a smooth d-dimensional connected unipotent subgroup, then $G_y(\mathbb{F}_q)$ contains a subgroup of order q^d. In particular, this gives lower bounds for the base size for $G(\mathbb{F}_q)$ in terms of the base size of G (there are examples where the base size of the finite group can be smaller or larger although not by much – see [BGS]).

6. Derived Subgroups

6.1. For an algebraic group G defined over a field k, we define $d(G)$ (resp. $d^0(G)$) the dimension of the derived group of the smooth $\overline{k}$–group $G_{\overline{k},red}$ (resp. the smooth connected $\overline{k}$–group $G^0_{\overline{k},red}$). If G is smooth, we have $d(G) = \dim_k(DG)$ and $d^0(G) = \dim_k(D(G^0))$.

It is convenient to introduce a third dimension function $d^+(G)$ which is the supremum of the dimensions of the C_n's where C_n stands for the schematic image of the commutator map $c_n : G^{2n} \to G$, $c_n(x_1, y_1, \dots, x_n, y_n) = [x_1, y_1] \dots [x_n, y_n]$.

Since the formation of the schematic image commutes with flat base change, $d^+(G)$ is insensitive to an arbitrary field extension. We have $d^0(G) \le d(G) \le d^+(G)$ and $d(G) = d^+(G)$ for G smooth.

Proposition 6.1. *Let S be a scheme and let G be a flat S–group scheme of finite presentation. Then the function $s \mapsto d^+(G_{\kappa(s)})$ is lower semi-continuous.*

Proof. Using the same kind of argument as in the proof of Theorem 4.1 it is enough to deal with the case $S = \mathrm{Spec}(A)$ where A is a DVR with fraction field K and of residue field k. Let $n \ge 1$ be an integer and consider the commutator map $c_n : G^{2n} \to G$. Let $\mathfrak{C}_n \subset G$ be the schematic closure of $C_{n,K}$; it is flat over A so equidimensional of dimension d according to [EGA4, 12.1.1.5]. Since G_K^{2n} is dense in G^{2n}, it follows that c_n factorizes through $\mathfrak{C}_n$ so that $c_{n,k}$ factorizes through $(\mathfrak{C}_n)_k$. Therefore $C_{n,k} \subset (\mathfrak{C}_n)_k$ whence $d^+(G_k) \le d^+(G_K)$. □

Corollary 6.1. *Let S be a scheme and let G be an S–group scheme of finite presentation.*

(1) Assume that G is smooth. Then the functions $s \mapsto d(G_{\kappa(s)})$ and $s \mapsto d^0(G_{\kappa(s)})$ are lower semi-continuous.

(2) Assume that S is irreducible with generic point ξ such that $G_{\kappa(\xi)}$ is smooth. Then $d(G_{\kappa(\xi)}) \ge d(G_{\kappa(s)})$ for each $s \in S$.

Proof. (1) In this case $d^+(G_{\kappa(s)}) = d(G_{\kappa(s)})$ so that Proposition 6.1 implies that d is lower semi-continuous. For the other function we consider the (smooth) S–group scheme G^0 defined in [SGA3, VI_B.3.10]. We have $d^0(G_{\kappa(s)}) = d(G^0_{\kappa(s)})$ so d^0 is upper semi-continuous.

(2) We have $d(G_{\kappa(\xi)}) = d^+(G_{\kappa(\xi)}) \ge d^+(G_{\kappa(s)}) \ge d(G_{\kappa(s)})$.

□

This function d may fail to be upper semi-continuous even in the case of a smooth affine group scheme over a DVR with connected fibers; in that case it would be locally constant according to Corollary 6.1.(1).

Lemma 6.2. *Let k be a an algebraically closed field and G be a split semisimple simply connected $k((t))$-group assumed almost simple of rank r. Let $\mathfrak{B}$ be Bruhat-Tits $k[[t]]$–group scheme attached to an Iwahori subgroup of $G(k((t)))$. Then we have*

$$d(\mathfrak{B}_k) \leq d(G) - r < d(G) = \dim_{k((t))}(\mathfrak{B}_{k((t))}).$$

Such a $\mathfrak{B}$ is smooth and has connected fibers according to [BT, prop. 4.6.32].

Proof. In this case $G_{k[[t]]}$ is a Bruhat-Tits group scheme attached to the maximal parahoric subgroup $G(k[[t]])$ of $G(k((t)))$. The Bruhat-Tits correspondence [BT, th. 4.6.35] is a bijection between the k–parabolic subgroups of G_k and the parahoric subgroups of $G(k((t)))$ included in $G(k[[t]])$. By taking a Borel subgroup B_k of G_k, we get then a Iwahori subgroup $\mathcal{B}$ of $G(k((t)))$ and a Bruhat-Tits group scheme $\mathfrak{B}$ such that B_k occurs as quotient of $\mathfrak{B}_k$. In particular $\mathfrak{B}_k$ maps onto a Borel k–subgroup of G_k so admits a commutative quotient of dimension r. It follows that $d(\mathfrak{B}_k) \leq \dim(\mathfrak{B}_k) - r = \dim(G) - r = d(G) - r < d(G)$. We have proven that for one specific Iwahori subgroup but this is enough by conjugacy reasons. □

We conclude by giving an example of a stabilizer scheme where the generic stabilizer is simple of dimension 3 but some stabilizer is abelian (also of dimension 3).

Let $G = \mathrm{Sp}_4(k) = \mathrm{Sp}(V)$ for k any algebraically closed field. Let $X = V \oplus V$. If $x = (v_1, v_2)$ is a generic point, then the stabilizer of x is the subgroup acting trivially on the nondegenerate 2-space spanned by v_1 and v_2 and so $G_x \cong \mathrm{Sp}_2(k)$. If $y = (w_1, w_2)$ with w_1 and w_2 spanning a totally singular 2-space, then G_y is the unipotent radical of the parabolic subgroup stabilizing the space spanned by w_1 and w_2. In particular, for a generic point x, G_x is nonsolvable while G_y is abelian.

6.2. **Abelianization.** A variant is the following. For an algebraic group G defined over a field k, we define $d_{ab}(G)$ (resp. $d^0_{ab}(G)$) the dimension of the abelianization of the smooth $\overline{k}$–group $G_{\overline{k},red}$ (resp. the smooth connected $\overline{k}$–group $G^0_{\overline{k},red}$). We have $d(G) + d_{ab}(G) = \dim_k(G)$ and similarly $d^0_{ab}(G) + d^0_{ab}(G) = \dim_k(G)$.

Corollary 6.2. *Let S be a scheme and let G be an S–group scheme of finite presentation.*

(1) Assume that G is smooth. Then the functions $s \mapsto d_{ab}(G_{\kappa(s)})$ and $s \mapsto d^0_{ab}(G_{\kappa(s)})$ are upper semi-continuous.

(2) Assume that S is irreducible with generic point ξ such that $G_{\kappa(\xi)}$ is smooth. Then $d_{ab}(G_{\kappa(\xi)}) \leq d_{ab}(G_{\kappa(s)})$ for each $s \in S$.

Proof. (1) Since the dimension is a locally constant function, the statement follows from Corollary 6.1.(1).

(2) Corollary 6.1.(2) states that $d(G_{\kappa(\xi)}) \geq d(G_{\kappa(s)})$ so that $-d(G_{\kappa(\xi)}) \leq -d(G_{\kappa(s)})$. On the other hand we have $\dim(G_{\kappa(\xi)}) \leq \dim(G_{\kappa(s)})$ according to [SGA3, VI_B.4.1]. By summing the inequalities we obtain $d_{ab}(G_{\kappa(\xi)}) \leq d_{ab}(G_{\kappa(s)})$ as desired.

□

7. Appendix: Alternate proof of Theorem 1.1

We present an alternate proof of the upper semi-continuity of d_u. It involves the following preliminary fact.

Lemma 7.1. *Let F be a field.*

(1) Let M be an affine smooth connected F–group. The following assertions are equivalent:

(i) M is unipotent;

(i') $M_{\overline{F}}$ is unipotent;

(ii) All F–tori of M are trivial;

(ii') All $\overline{F}$–tori of M are trivial;

(2) Let $f : G \to H$ be an isogeny between affine smooth connected F–groups. Then G is unipotent if and only if H is unipotent.

Proof. (1) Let $\rho : M \to \mathrm{GL}_n$ be faithful linear representation. The equivalence $(i) \Longleftrightarrow (i')$ is by definition since in both cases it says that $\rho(G(\overline{F}))$ consists in unipotent elements.

$(i') \Longrightarrow (ii')$. Let $\mathbb{G}_{m,\overline{F}}^r \hookrightarrow M_{\overline{F}}$ be a $\overline{F}$–subtorus. If $r \geq 1$, we pick an element $t \neq 1 \in (\overline{F})^r$. We have that $\rho(t) \in GL_n(\overline{k})$ is unipotent and semisimple so is 1 contradicting $t \neq 1$. We conclude that $r = 1$.

$(ii') \Longrightarrow (ii)$. This is obvious.

$(ii) \Longrightarrow (ii')$. According to the conjugacy theorem [B, 11.3],n all maximal $\overline{F}$–subtori of $G_{\overline{F}}$ are conjugated. According to a result of Grothendieck [B, 18.2(i)], there is a maximal $\overline{F}$-torus T in $M_{\overline{F}}$ that is defined over F. Our assumption implies that $T = 1$. According to the conjugacy theorem for maximal $\overline{F}$-subtori of $G_{\overline{F}}$, [B, 11.3] we conclude that all $\overline{F}$–tori of M are trivial.

(2) By (1) we can assume F is algebraically closed and it suffices to show that some maximal torus is trivial. For a maximal torus T in G its image $f(T)$ is a maximal torus in H by [B, Prop. 11.14.(1)]. Since $\ker(f)$ is finite we have $f(T) = 1$ if and only if $T = 1$. The proof is completed. □

We come now to the proof of Theorem 1.1. The beginning is verbatim that of the proof of Theorem 4.1, that is, a reduction to the case of the spectrum of a DVR A with fraction field K and residue field k. We are given a separated A–group scheme G of finite presentation and want so show the inequality $d_u(G_k) \geq d_u(G_K)$. We put $d = d_u(G_K)$. Using the insensitivity of d_u by change of fields (Lemma 2.2) we can assume that A is complete. Also we are allowed to make arbitrary finite extensions of K so that we can assume that G_K admits a smooth connected unipotent K–group U of dimension d. Letting $\mathfrak{H}$ be the schematic closure of U in G, it is separated flat over A and of finite presentation. Raynaud's affiness criterion (see §3.1) shows that $\mathfrak{H}$ is affine. Replacing G by $\mathfrak{H}$ we can then assume that the A–group scheme G is affine flat, and that G_K is smooth unipotent connected.

We consider the case G smooth. This enables to deal with the neutral component G^0 of G [SGA3, VI_B.3.10] which is a smooth open A–subgroup G such that G^0_K (resp. G^0_k) is the neutral component of the algebraic group G_K (resp. G_k). Also the A–group scheme G^0 is affine according to Raynaud's criterion.

Lemma 7.1.(1) states that G^0_k is unipotent if and only if all tori of G^0_k are trivial. Let T_0 be a maximal torus of G^0_k. Since A is henselian, T_0 lifts to a subtorus T of G^0 by using Grothendieck's representability theorem [SGA3, XI.4.1]. Since G_K is unipotent, T_K is trivial so that $T = 1$ and $T_0 = 1$. It follows that G^0_k is unipotent so that $d_u(G) = d$.

We consider now the general case. According to [PY, prop. 3.4] (based on [A, app. II]), there exists a local extension A' of A of DVR's such that the normalization $\widetilde{G}'$ of $G' = G \times_A A'$ is smooth over A' and such that the fraction field K' is finite over K. We denote by k' the residue field of A'. According to [PY, thm. A.6] the normalization morphism $h : \widetilde{G}' \to G'$ is finite. In particular the morphism of smooth affine connected K'–groups $h^0_{K'} : \big((\widetilde{G}')_{K'}\big)^0 \to G_{K'}$ is an isogeny between smooth affine connected K'–groups. Since $G_{K'}$ is unipotent, Lemma 7.1.(2) shows that $(\widetilde{G}')^0_{K'}$ is unipotent.

The smooth case applied to $(\widetilde{G}')^0$ over A' shows that $(\widetilde{G}')^0_{k'}$ is unipotent of dimension d. Since the homomorphism $h^0_k : (\widetilde{G}')^0_{k'} \to (G'_{k'})^0 \cong (G_k)^0 \times_k k'$ is finite, Lemma 7.1.(2) shows that the k'–subgroup $(G'_{k'})^0/\ker(h^0_{k'})$ of $G_{k'}$ is unipotent of dimension d. Thus $d_u(G_k) \geq d$.

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