On the infimum of the absolute value of successive derivatives of a real function defined on a bounded interval

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Abstract

A study of the greatest possible ratio of the smallest absolute value of a higher derivative of some function, defined on a bounded interval, to the L^p-norm of the function.

Keywords

Chebyshev polynomials, Legendre polynomials, extremal problems, inequalities for derivatives
MSC classification: 26D10, 41A10

To the memory of Eduard Wirsing,
master of analysis,
and of its applications to number theory.

1 Introduction

Let n be a positive integer, $I = [a, b]$ a bounded segment of the real line, of length $L = b - a$. Define $\mathcal{D}^n(I)$ as the set of real functions f defined on I, with successive derivatives $f^{(k)}$ defined and continuous on I for $0 \leqslant k \leqslant n-1$, and $f^{(n)}$ defined on $\mathring{I} =]a, b[$. We will use the notation

$$m_n(f) = \inf_{a<t<b} |f^{(n)}(t)|.$$

Let p be a positive real number, or ∞.

The problem addressed in this article is that of determining the best constant $C^* = C^*(n, p, I)$ in the inequality

$$m_n(f) \leqslant C^* \, \|f\|_p \quad (f \in \mathcal{D}^n(I)),$$

where

$$\|f\|_p = \Big(\int_a^b |f(t)|^p \, dt \Big)^{1/p},$$

with the usual convention when $p = \infty$, here: $\|f\|_\infty = \max |f|$.

This problem has been posed by Kwong and Zettl in their 1992 Lecture Notes [11] (see Lemma 1.1, p. 6). They give upper bounds for $C^*(n,p,I)$, but their reasoning and results are erroneous. In her 1993 PhD Thesis [5], Huang has pointed out that this problem is equivalent to a classical problem in the theory of polynomial approximation: that of determining the minimal L^p-norm of a monic polynomial of given degree on a given bounded interval. Our purpose in this text is to give a new proof of the equivalence, and to list the consequences of the known results about this extremal problem for the evaluation of $C^*(n,p,I)$.

2 First observations

2.1 Homogeneity

Defining $g(u) = f(a+uL)$ for $f \in \mathcal{D}^n(I)$ and $0 \leqslant u \leqslant 1$, one has

$$g \in \mathcal{D}^n([0,1]) \quad ; \quad g^{(n)}(u) = L^n f^{(n)}(a+uL) \quad (0 < u < 1) \quad ; \quad \|g\|_p = L^{-1/p} \|f\|_p \, .$$

Hence,

$$C^*(n,p,I) = C^*(n,p,[0,1]) \, L^{-n-1/p}, \tag{1}$$

and one is left with determining $C^*(n,p,[0,1]) = C(n,p)$, or in fact $C^*(n,p,I)$ for any fixed, chosen segment I. We will see that $I = [-1,1]$ is particularly convenient.

2.2 An extremal problem

One has

$$\begin{aligned} C^*(n,p,I) &= \sup\{m_n(f)/\|f\|_p \, , \ f \in \mathcal{D}^n(I), \ m_n(f) \neq 0\} \\ &= \sup\{m_n(f)/\|f\|_p \, , \ f \in \mathcal{D}^n(I), \ m_n(f) = \lambda\} \quad \text{(for every } \lambda > 0) \\ &= \lambda / D^*(n,p,\lambda,I), \end{aligned}$$

where

$$\begin{aligned} D^*(n,p,\lambda,I) &= \inf\{\|f\|_p \, , \ f \in \mathcal{D}^n(I), \ m_n(f) = \lambda\} \\ &= \inf\{\|f\|_p \, , \ f \in \mathcal{D}^n(I), \ m_n(f) \geqslant \lambda\}, \end{aligned}$$

the last equality being true since $D^*(n,p,\mu,I) = \frac{\mu}{\lambda} D^*(n,p,\lambda,I) \geqslant D^*(n,p,\lambda,I)$ if $\mu \geqslant \lambda$.

Also, since a derivative has the intermediate value property (cf. [3], pp. 109-110), the inequality $m_n(f) \geqslant \lambda > 0$ implies that $f^{(n)}$ has constant sign on I, so that

$$D^*(n,p,\lambda,I) = \inf\{\|f\|_p\,,\ f \in \mathcal{D}^n(I),\ f^{(n)}(t) \geqslant \lambda \text{ for } a < t < b\}.$$

Thus, determining $C^*(n,p,I)$ is equivalent to minimizing $\|f\|_p$ for $f \in \mathcal{D}^n(I)$ with the constraint $f^{(n)}(t) \geqslant \lambda > 0$ for $a < t < b$. We will denote this extremal problem by $\mathcal{E}^*(n,p,\lambda,I)$.

3 The relevance of monic polynomials

Let $\mathcal{P}_n$ be the set of monic polynomials of degree n, with real coefficients, identified with the set of the corresponding polynomial functions on I, which is a subset of $\mathcal{D}^n(I)$. Since $m_n(f) = n!$ for $f \in \mathcal{P}_n$, one has

$$D^*(n,p,n!,I) \leqslant D^{**}(n,p,I), \tag{2}$$

where

$$D^{**}(n,p,I) = \inf\{\|Q\|_p\,,\ Q \in \mathcal{P}_n\}.$$

A basic fact in the study of the extremal problem $\mathcal{E}^*(n,p,\lambda,I)$ is that (2) is in fact an equality.

Proposition 1 *For all n,p,I, one has $D^*(n,p,n!,I) = D^{**}(n,p,I)$.*

It follows from this proposition that $C^*(n,p,I) = n!/D^{**}(n,p,I)$ and, by (1),

$$C(n,p) = L^{n+1/p}n!/D^{**}(n,p,I). \tag{3}$$

Let us review the history of Proposition 1.

For $p = \infty$, it is a corollary to a theorem of S. N. Bernstein from 1937. Denoting by $E_k(f)$ the distance (for the uniform norm on I) between f and the set of polynomials of degree at most k, he proved in particular that

$$E_{n-1}(f_0) > E_{n-1}(f_1) \quad (f_0, f_1 \in \mathcal{D}^n(I)),$$

provided that the inequality $f_0^{(n)}(\xi) > |f_1^{(n)}(\xi)|$ is valid for every $\xi \in \mathring{I}$ (cf. [2], p. 48, inequalities (47bis)-(48bis)). Proposition 1 follows by taking $f_1(x) = x^n$ and $f_0(x) = \lambda f(x)$, where f is a generic element of $\mathcal{D}^n(I)$ such that $f^{(n)}(t) \geqslant n!$ for $a < t < b$, and $\lambda > 1$, then letting $\lambda \to 1$.

This theorem of Bernstein was generalized by Tsenov in 1951 to the case of the L^p-norm on I, where $p \geqslant 1$ (cf. [15], Theorem 4, p. 477), thus providing a proof of Proposition 1 for $p \geqslant 1$. The case $0 < p < 1$ was left open by Tsenov.

The study of the extremal problem $\mathcal{E}^*(n,p,\lambda,I)$ was one of the themes of the 1993 PhD thesis of Xiaoming Huang [5]. In Lemma 2.0.7, pp. 9-10, she gave another proof (due to Saff) of Proposition 1 in the case $p = \infty$. For $1 \leqslant p < \infty$, she gave a proof of Proposition 1 which is unfortunately incomplete (cf. [5], pp. 28-30). Again, the case $0 < p < 1$ was left open.

We present now a self-contained proof of Proposition 1, valid for $0 < p \leqslant \infty$. As it proceeds by induction on n, we will need the following classical-looking division lemma, for which we could not locate a reference (compare with [16] or [13]).

Proposition 2 *Let $n \geqslant 2$ and $f \in \mathcal{D}^n(I)$. Let $c \in [a, b]$. Put*

$$g(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c} & (x \in I,\, x \neq c) \\ f'(c) & (x = c). \end{cases} \tag{4}$$

Then $g \in \mathcal{D}^{n-1}(I)$. For every $x \in \,]a, b[\,$, one has

$$g^{(n-1)}(x) = \frac{f^{(n)}(\xi)}{n},$$

where $\xi \in \,]a, b[$.

Proof

Since f' is continuous, one has

$$g(x) = \int_0^1 f'\big(c + t(x - c)\big)\, dt \quad (x \in I).$$

Using the rule of differentiation under the integration sign, one sees that g is $n - 2$ times differentiable on I, with

$$g^{(n-2)}(x) = \int_0^1 t^{n-2} f^{(n-1)}\big(c + t(x - c)\big)\, dt \quad (x \in I).$$

As $f^{(n-1)}$ is continuous on I, this formula yields the continuity of $g^{(n-2)}$ on I.

The function g is n times differentiable on $\mathring{I} \setminus \{c\}$ (this set is just $\mathring{I}$ if $c = a$ or $c = b$), being a quotient of n times differentiable functions, with non-vanishing denominator. In the case $a < c < b$, we have now to check that g is $n - 1$ times differentiable at the point c.

The function $f^{(n-1)}$ being continuous on I and differentiable at the point c, there exists a function $\varepsilon(h)$, defined and continuous on the segment $[a - c, b - c]$ (the interior of which contains 0), vanishing for $h = 0$, such that

$$f^{(n-1)}(c + h) = f^{(n-1)}(c) + h f^{(n)}(c) + h\varepsilon(h) \quad (a \leqslant c + h \leqslant b).$$

Hence,

$$\begin{aligned}
g^{(n-2)}(x) &= \int_0^1 t^{n-2} f^{(n-1)}\big(c+t(x-c)\big)\,dt \\
&= \int_0^1 t^{n-2}\Big(f^{(n-1)}(c)+t(x-c)f^{(n)}(c)+t(x-c)\varepsilon\big(t(x-c)\big)\Big)\,dt \\
&= \frac{f^{(n-1)}(c)}{n-1}+\frac{f^{(n)}(c)}{n}(x-c)+(x-c)\int_0^1 t^{n-1}\varepsilon\big(t(x-c)\big)\Big)\,dt
\end{aligned}$$

When x tends to c, the last integral tends to 0, so that the function $g^{(n-2)}$ is differentiable at the point c, with

$$g^{(n-1)}(c)=\frac{f^{(n)}(c)}{n}\cdot$$

If $x \in \mathring{I} \setminus \{c\}$, one may use the general Leibniz rule and Taylor's theorem with the Lagrange form of the remainder in order to compute $g^{(n-1)}(x)$:

$$\begin{aligned}
g^{(n-1)}(x) &= \frac{d^{n-1}}{dx^{n-1}}\Big(\big(f(x)-f(c)\big)\cdot\frac{1}{x-c}\Big) \\
&= \big(f(x)-f(c)\big)\cdot\frac{(-1)^{n-1}(n-1)!}{(x-c)^n}+\sum_{k=1}^{n-1}\binom{n-1}{k}f^{(k)}(x)\cdot\frac{(-1)^{n-1-k}(n-1-k)!}{(x-c)^{n-k}} \\
&= \frac{(n-1)!}{(c-x)^n}\Big(f(c)-f(x)-\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}(c-x)^k\Big) \\
&= \frac{(n-1)!}{(c-x)^n}\cdot\frac{f^{(n)}(\xi)}{n!}(c-x)^n \quad \text{(where } \xi \text{ belongs to the open interval bounded by } c \text{ and } x\text{)} \\
&= \frac{f^{(n)}(\xi)}{n}\cdot
\end{aligned}$$

□

In the next proposition, we stress the main element of our proof of Proposition 1, namely the fact that the condition $f^{(n)} \geqslant n!$, for some $f \in \mathcal{D}^n(I)$, implies that the absolute value of f dominates the absolute value of some monic polynomial of degree n.

Proposition 3 *Let $n \geqslant 1$ and $f \in \mathcal{D}^n(I)$ such that $f^{(n)}(x) \geqslant n!$ for every $x \in]a,b[$.*

Then there exists a monic polynomial P of degree n, with all its zeros in I, such that the inequality $|f(x)| \geqslant |P(x)|$ is valid for every $x \in I$.

Moreover, if $|f(x)| = |Q(x)|$ for every $x \in I$, where Q is a monic polynomial of degree n with real coefficients, then $f(x) = Q(x)$ for every $x \in I$.

Proof

The assertion about the zeros may be obtained *a posteriori*, by replacing the zeros of P by their projections on I. The following proof leads directly to a polynomial P with all zeros in I.

We use induction on n.

For $n = 1$, the function f is continuous on $[a, b]$, differentiable on $]a, b[$, with $f'(x) \geqslant 1$ for $a < x < b$.

If $f(a) \geqslant 0$, one has, for $a < x \leqslant b$, $f(x) = f(a) + (x - a)f'(\xi)$ (where $a < \xi < x$), thus $f(x) \geqslant x - a$. Hence, one has $|f(x)| \geqslant |x - a|$ for every $x \in I$.

If $f(b) \leqslant 0$, one proves similarly that $|f(x)| \geqslant |x - b|$ for every $x \in I$.

If $f(a) < 0 < f(b)$, there exists $c \in]a, b[$ such that $f(c) = 0$. One has then, for every $x \in I$,

$$f(x) = f(x) - f(c) = (x - c)f'(\xi) \quad (\text{where } a < \xi < b).$$

Hence $|f(x)| \geqslant |x - c|$ for every $x \in I$, and the result is proven for $n = 1$.

Let now $n \geqslant 2$, and suppose that the result is valid with $n - 1$ instead of n. Let $f \in \mathcal{D}^n(I)$ such that $f^{(n)}(x) \geqslant n!$ for every $x \in]a, b[$.

If f vanishes at some point $c \in I$, it follows from Proposition 2 that the function g defined on I by

$$g(x) = \begin{cases} \dfrac{f(x)}{x - c} & (x \in I,\, x \neq c) \\ f'(c) & (x = c), \end{cases}$$

belongs to $\mathcal{D}^{n-1}(I)$ and that, for every $x \in]a, b[$, one has

$$g^{(n-1)}(x) = \frac{f^{(n)}(\xi)}{n},$$

where $\xi \in]a, b[$, thus $g^{(n-1)}(x) \geqslant (n - 1)!$. By the induction hypothesis, there exists a monic polynomial Q of degree $n - 1$, with all its roots in I, such that $|g(x)| \geqslant |Q(x)|$ for every $x \in I$. Hence, one has the inequality $|f(x)| \geqslant |P(x)|$ for every $x \in I$, where $P(x) = (x - c)Q(x)$ is a monic polynomial of degree n, with all its roots in I.

If $f > 0$, it reaches a minimum at some point $c \in I$. Again, it follows from Proposition 2 that the function g defined on I by (4) satisfies the required hypothesis for degree $n - 1$. Thus there exists a monic polynomial Q of degree $n - 1$, with all its roots in I, such that $|g(x)| \geqslant |Q(x)|$ for every $x \in I$. Hence, one has the inequality

$$f(x) - f(c) = |f(x) - f(c)| \geqslant |P(x)| \quad (x \in I),$$

where $P(x) = (x - c)Q(x)$. It follows that

$$|f(x)| = f(x) \geqslant f(c) + |P(x)| > |P(x)| \quad (x \in I)$$

If $f < 0$, the reasoning is similar by considering a point $c \in I$ where f reaches a maximum.

Let us prove the last assertion. The hypothesis $|f| = |P|$ is equivalent to the equality $f^2 = P^2$, that is $(f - P)(f + P) = 0$. The set $E = \{x \in I,\ f(x) + P(x) = 0\}$ has empty interior, since $f^{(n)}(x) + P^{(n)}(x) = 0$ on every open subinterval of E, whereas $f^{(n)}(x) + P^{(n)}(x) \geqslant 2n!$ on $\mathring{I}$. The set $I \setminus E$ is therefore dense in I ; its elements x all verify $f(x) = P(x)$, hence $f = P$ on I by continuity. □

Proposition 1 is an immediate corollary of Proposition 3: by taking f and P as stated there, one has $|f(x)| \geqslant |P(x)|$ for every $x \in I$, so that

$$\int_a^b |f(x)|^p \, dx \geqslant \int_a^b |P(x)|^p \, dx, \tag{5}$$

for every $p > 0$ (for $p = \infty$: $\max |f| \geqslant \max |P|$).

Moreover, if $p < \infty$, equality in (5) implies that $|f| = |P|$ on I, hence $f = P$.

In other words, if $0 < p < \infty$, the extremal problem $\mathcal{E}^*(n, p, n!, I)$ has exactly the same solutions (value of the infimum and extremal functions) as the problem $\mathcal{E}^{**}(n, p, I)$ obtained by considering only monic polynomials of degree n, which one may even take with all their roots in I.

For $p = \infty$, our reasoning does not prove that an extremal function for $\mathcal{E}^*(n, p, n!, I)$ (if it exists) must be a polynomial. This is true anyway, as proved by Huang in [5], pp. 10-13.

4 Extremal polynomials

One may now use the results of the well developed theory of the extremal problem $\mathcal{E}^{**}(n, p, I)$ for polynomials. Thus, since the integral

$$\int_a^b |(x - x_1) \cdots (x - x_n)|^p \, dx \quad (x_1, \ldots, x_n \in I)$$

(or the value $\max_{x \in I} |(x - x_1) \cdots (x - x_n)|$) is a continuous function of $(x_1, \ldots, x_n)$, the compactness of I^n yields the existence of an extremal (polynomial) function for $\mathcal{E}^{**}(n, p, I)$, hence for $\mathcal{E}^*(n, p, n!, I)$.

It is a known fact that the polynomial extremal problem $\mathcal{E}^{**}(n, p, I)$ has a unique solution for all $p \in]0, \infty]$, but there is no proof valid uniformly for all values of p.

• For $p = \infty$, uniqueness was proved by Young in 1907 (cf. [18], Theorem 5, p. 340)) and follows from the general theory of uniform approximation (cf. [12], Theorem 1.8, p. 28).

• For $1 < p < \infty$, as proved by Jackson in 1921 (cf. [7], §6, pp. 121-122), this is a consequence of the strict convexity of the space $L^p(I)$.

• For $p = 1$, this is also due to Jackson in 1921 (cf. [6], §4, pp. 323-326).

• For $0 < p < 1$, the uniqueness of the extremal polynomial was proved in 1988 by Kroó and Saff (cf. [10], Theorem 2, p. 184). Their proof uses the uniqueness property for $p = 1$ and the implicit function theorem.

We will denote by $T_{n,p,I}$ the unique solution of the extremal problem $\mathcal{E}^{**}(n,p,I)$. Uniqueness gives immediately the relation

$$T_{n,p,I}(a+b-x) = (-1)^n T_{n,p,I}(x) \quad (x \in \mathbb{R}).$$

Another property of these polynomials is the fact that all their roots are simple. For $p = 1$, this fact was proved by Korkine and Zolotareff in 1873 (cf. [8], pp. 339-340), before their explicit determination of the extremal polynomial (see §5.4 below), and their proof extends, *mutatis mutandis*, to the case $1 < p < \infty$. For $p = \infty$, this is a property of the Chebyshev polynomials of the first kind (see §5.2 below). Lastly, for $0 < p < 1$, this was proved by Kroó and Saff in [10], p. 187.

Define $T_{n,p} = T_{n,p,[-1,1]}$, and write $n = 2k + \varepsilon$, where $k \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$. It follows from the mentioned results that

$$T_{n,p}(x) = x^{\varepsilon}(x^2 - x_{n,1}(p)^2) \cdots (x^2 - x_{n,k}(p)^2) \quad (x \in \mathbb{R}), \tag{6}$$

where

$$0 < x_{n,1}(p) < \cdots < x_{n,k}(p) \leqslant 1.$$

Kroó, Peherstorfer and Saff have conjectured that all the $x_{n,k}$ are increasing functions of p (cf. [9], p. 656, and [10], p. 192).

5 Results on $C(n,p)$

5.1 The case $n = 1$

The value $n = 1$ is the only one for which $C(n,p)$ is explicitly known for all p.

Proposition 4 *One has* $C(1,p) = 2(p+1)^{1/p}$ *for* $0 < p < \infty$, *and* $C(1,\infty) = 2$.

Proof

By (6), one has $T_{1,p}(x) = x$, so that, for $0 < p < \infty$,

$$D^{**}(1,p,[-1,1]) = \Big(\int_{-1}^{1} |t|^p \, dt\Big)^{1/p} = \big(2/(p+1)\big)^{1/p},$$

and, by (3),

$$C(1,p) = 2^{1+1/p}/D^{**}(1,p,[-1,1]) = 2(p+1)^{1/p}. \qquad \square$$

Note that the Lemma 1.1, p. 6 of [11], asserts that $C(1,p) \leqslant 2 \cdot 3^{1/p}$ for $p \geqslant 2$, and that bound is $< 2(p+1)^{1/p}$ for $p > 2$.

5.2 The case $p = \infty$

This is the classical case, solved by Chebyshev in 1853 by introducing the polynomials T_n defined by the relation $T_n(\cos t) = \cos nt$ (now called Chebyshev polynomial of the first kind): the unique solution of the extremal problem $\mathcal{E}^{**}(n, \infty, [-1, 1])$ is $2^{1-n}T_n$. Let us record a short proof of this fact.

Take $I = [-1, 1]$ and suppose that P is a monic polynomial of degree n satisfying the inequality $\|P\|_\infty \leqslant \left\|2^{1-n}T_n\right\|_\infty = 2^{1-n}$. Then, for $\lambda > 1$ the polynomial

$$Q_\lambda = \lambda 2^{1-n}T_n - P$$

is of degree n, with leading coefficient $\lambda - 1$. Moreover, it satisfies

$$(-1)^k Q_\lambda(\cos k\pi/n) = \lambda 2^{1-n} - (-1)^k P(\cos k\pi/n) > 0 \quad (k = 0, \dots, n)$$

By the intermediate value property, Q_λ has at least n distinct roots, hence exactly n, and these roots, say $x_1, \dots, x_n$, have absolute value not larger than 1. Hence,

$$|Q_\lambda(x)| = (\lambda - 1)\,|(x - x_1)\cdots(x - x_n)| \leqslant (\lambda - 1)(1 + |x|)^n \quad (x \in \mathbb{R}).$$

When $\lambda \to 1$, $Q_\lambda(x)$ tends to 0 for every real x, which means that $P = 2^{1-n}T_n$.

One deduces from this theorem the value of $C(n, \infty)$. One has

$$D^{**}(n, \infty, [-1, 1]) = \max_{|x| \leqslant 1} \left|2^{1-n}T_n(x)\right| = 2^{1-n},$$

hence

$$C(n, \infty) = 2^n \cdot n!/D^{**}(n, \infty, [-1, 1]) = 2^{2n-1}n! \tag{7}$$

(compare with the upper bound $C(n, \infty) \leqslant 2^{n(n+1)/2}n^n$ of [4], 3 (a), p. 185). This result is essentially due to Bernstein (1912, cf. [1], p. 65).

Qualitatively, the result expressed by (7) was nicely described by Soula in [14], p. 86, as follows.

> Bernstein's principle: the minimum of the absolute value of the n-th derivative of an n times differentiable function and the maximum of the absolute value of the n-th derivative of an analytic function have similar orders of magnitude.

5.3 The case $p = 2$

In this case, the extremal problem $\mathcal{E}^{**}(n, 2, [-1, 1])$ is an instance of the general problem of computing the orthogonal projection of an element of a Hilbert space onto a finite dimensional subspace. Here, the Hilbert space is $L^2(-1, 1)$, the element is the monomial function x^n, and the subspace is the set of polynomial functions of degree less than n. The solution follows from the theory of orthogonal polynomials: the extremal polynomial for $\mathcal{E}^{**}(n, 2, [-1, 1])$ is

$$\frac{2^n (n!)^2}{(2n)!} P_n(x) \quad (|x| \leqslant 1),$$

where P_n is the n-th Legendre polynomial, defined by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

Hence,

$$D^{**}(n, 2, [-1, 1]) = \frac{2^n (n!)^2}{(2n)!} \|P_n\|_2 = \frac{2^n (n!)^2}{(2n)!} \sqrt{\frac{2}{2n+1}},$$

(see [17], §15·14, p. 305) and

$$C(n, 2) = 2^{n+\frac{1}{2}} \cdot n!/D^{**}(n, 2, [-1, 1]) = \frac{(2n)!}{n!} \sqrt{2n+1}, \tag{8}$$

a result given by Soula in 1932 (cf. [14], pp. 87-88).

5.4 The case $p = 1$

The problem $\mathcal{E}^{**}(n, 1, [-1, 1])$ was solved by Korkine and Zolotareff in [8]: the extremal polynomial is $2^{-n} U_n(x)$, where U_n is the n-th Chebyshev polynomial of the second kind, defined by the relation $U_n(\cos t) = \sin(n+1)t / \sin t$.

Therefore, one has

$$\begin{aligned} D^{**}(n, 1, [-1, 1]) &= 2^{-n} \int_{-1}^{1} |U_n(x)| \, dx = 2^{-n} \int_0^{\pi} |U_n(\cos t)| \sin t \, dt \\ &= 2^{-n} \int_0^{\pi} |\sin(n+1)t| \, dt = 2^{-n} \int_0^{\pi} \sin u \, du \\ &= 2^{1-n}, \end{aligned}$$

and

$$C(n, 1) = 2^{n+1} \cdot n!/D^{**}(n, 1, [-1, 1]) = 2^{2n} n!. \tag{9}$$

5.5 Bounds for $C(n, p)$

We begin with a simple monotony result.

Proposition 5 *For every positive integer n, the function $p \mapsto C(n, p)$ is decreasing on the interval $0 < p \leqslant \infty$.*

Proof

Let $I = [0,1]$. Equivalently, we will see that the function $p \mapsto D^{**}(n,p,I)$ is increasing. This is due to the fact that, for a fixed $f \in L^\infty(I)$ such that $|f|$ is not equal almost everywhere to a constant, the function $p \mapsto \|f\|_p$ is increasing (a consequence of Hölder's inequality). Thus, for every $Q \in \mathcal{P}_n$ and $0 < p < p' \leqslant \infty$,

$$\|Q\|_{p'} > \|Q\|_p \geqslant D^{**}(n,p,I),$$

which implies that $D^{**}(n,p',I) > D^{**}(n,p,I)$. □

In particular, (7) and (9) yield the inequalities

$$2^{2n-1} n! < C(n,p) < 2^{2n} n! \quad (1 < p < \infty).$$

The next proposition implies that the limit of $C(n,p)$ when p tends to 0 is $(2e)^n n!$.

Proposition 6 *For every positive integer n and every positive real number p, one has*

$$2^n (1+np)^{1/p} n! \leqslant C(n,p) \leqslant (2e)^n n!$$

Proof

Equivalently, we will prove that

$$(2e)^{-n} \leqslant D^{**}(n,p,I) \leqslant 2^{-n}(1+np)^{-1/p}, \tag{10}$$

where $I = [0,1]$.

Let $Q(t) = (t - x_1)\cdots(t - x_n)$, where $0 \leqslant x_1, \ldots, x_n \leqslant 1$. One has

$$\begin{aligned}
\ln \|Q\|_p &= \frac{1}{p} \ln \int_0^1 |Q(t)|^p \, dt \\
&\geqslant \frac{1}{p} \int_0^1 \ln \left(|Q(t)|^p \right) dt \quad \text{(by Jensen's inequality)} \\
&= \int_0^1 \ln |Q(t)| \, dt \\
&= \sum_{k=1}^n \int_0^1 \ln |t - x_k| \, dt.
\end{aligned}$$

Now,

$$\int_0^1 \ln |t - x| \, dt = (1-x)\ln(1-x) + x \ln x - 1 \quad (0 \leqslant x \leqslant 1),$$

attains its minimal value, namely $-1 - \ln 2$, when $x = 1/2$. This implies the first inequality of (10).

To prove the second inequality of (10), we just compute $\|Q\|_p^p$ when $Q(t) = (t - 1/2)^n$:

$$\int_0^1 |t - 1/2|^{np} \, dt = 2\frac{(1/2)^{np+1}}{np+1}. \qquad \square$$

For $0 < p < 1$, we can also prove the following result.

Proposition 7 *Let n be a positive integer, and p such that $0 < p < 1$. One has*

$$1 \leqslant \frac{C(n,p)}{2^{2n} n!} \leqslant \frac{1}{2}(8/\pi)^{1/p}.$$

Proof

The first inequality is just $C(n, 1) \leqslant C(n, p)$.

To prove the second inequality, let r and s such that $1 < s < 2$ and $r^{-1} + s^{-1} = 1$. Define

$$\begin{aligned} I_1(s) &= \int_{-1}^1 \frac{dt}{(1-t^2)^{s/2}} \\ I_2(s) &= \int_{-1}^1 |t|^{(s-1)/s} \, \frac{dt}{\sqrt{1-t^2}}. \end{aligned}$$

The integrals $I_1(s)$ and $I_2(s)$ may be computed, using the eulerian identity

$$\int_0^1 t^{x-1}(1-t)^{y-1} \, dt = \mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (x > 0, \ y > 0).$$

The results are

$$\begin{aligned} I_1(s) &= 2^{1-s} \frac{\Gamma(1-\frac{s}{2})^2}{\Gamma(2-s)} \\ I_2(s) &= \frac{\Gamma\big(1-\frac{1}{2s}\big)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}-\frac{1}{2s})}. \end{aligned}$$

Now, let $Q \in \mathcal{P}_n$ and put $p' = p/r$. By Hölder's inequality, one has

$$\begin{aligned} \int_{-1}^1 |Q(t)|^{p'} \, \frac{dt}{\sqrt{1-t^2}} &\leqslant \Big(\int_{-1}^1 |Q(t)|^{p'r} \, dt\Big)^{1/r} \Big(\int_{-1}^1 \frac{dt}{(1-t^2)^{s/2}}\Big)^{1/s} \\ &= \|Q\|_p^{p'} \, I_1(s)^{1/s}. \end{aligned}$$

It was proved by Kroó and Saff (cf. [10], pp. 182-183) that

$$\begin{aligned}
2^{(n-1)p'} \int_{-1}^{1} |Q(t)|^{p'} \, \frac{dt}{\sqrt{1-t^2}} &\geqslant \int_{-1}^{1} |T_n(t)|^{p'} \, \frac{dt}{\sqrt{1-t^2}} \\
&= \int_0^{\pi} |\cos nu|^{p'} \, du = \int_0^{\pi} |\cos u|^{p'} \, du \\
&= \int_{-1}^{1} |t|^{p'} \, \frac{dt}{\sqrt{1-t^2}} \\
&\geqslant \int_{-1}^{1} |t|^{1/r} \, \frac{dt}{\sqrt{1-t^2}} \quad \text{(one has } p' = p/r < 1/r) \\
&= I_2(s).
\end{aligned}$$

Therefore, with $I = [-1, 1]$,

$$\|Q\|_p \geqslant 2^{1-n} I_2(s)^{1/p'} I_1(s)^{-1/p's} = 2^{1-n} A(s)^{1/p} \quad (1 < s < 2), \tag{11}$$

where

$$A(s) = I_2(s)^{s/(s-1)} I_1(s)^{-1/(s-1)}.$$

Hence

$$A(s) = 2\Big(\frac{\Gamma\big(1-\frac{1}{2s}\big)^s \Gamma(\frac{1}{2})^s \Gamma(2-s)}{\Gamma(1-\frac{s}{2})^2 \Gamma\big(\frac{3}{2}-\frac{1}{2s}\big)^s}\Big)^{1/(s-1)} \quad (1 < s < 2).$$

Putting $f(s) = \ln \Gamma(s)$, one has

$$\ln A(s) = \ln 2 + \frac{sf(1-1/2s) + sf(1/2) + f(2-s) - 2f(1-s/2) - sf(3/2-1/2s)}{s-1}.$$

When s tends to 1, the last fraction tends to

$$\ln \pi + \frac{3}{2}\psi(1/2) - \frac{3}{2}\psi(1) = \ln \pi - 3 \ln 2,$$

with the usual notation $f' = \Gamma'/\Gamma = \psi$. It follows that

$$A(s) \to \frac{\pi}{4} \quad (s \to 1).$$

Together with (11), this gives the inequality

$$D^{**}(n, p, [-1, 1]) \geqslant 2^{1-n} (\pi/4)^{1/p}$$

and (3) now implies

$$C(n, p) \leqslant 2^{2n-1} n! (8/\pi)^{1/p}. \qquad \square$$

We now prove an inequality involving three values of the function C.

Proposition 8 *Let p, q, r be positive real numbers such that*

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}\cdot$$

Let m and n be positive integers. Then,

$$\frac{C(m+n,p)}{(m+n)!} \geqslant \frac{C(m,q)}{m!} \cdot \frac{C(n,r)}{n!}\cdot$$

Proof

Equivalently, by (3), one has to prove that

$$D^{**}(m+n,p,I) \leqslant D^{**}(m,q,I) \cdot D^{**}(n,r,I),$$

where I is a segment of the real line.

In fact, if $P \in \mathcal{P}_m$ and $Q \in \mathcal{P}_n$, then$PQ \in \mathcal{P}_{m+n}$ hence

$$D^{**}(m+n,p,I)^p \leqslant \int_I |P(t)Q(t)|^p \; dt \leqslant \Big(\int_I |P(t)|^q \; dt\Big)^{p/q} \cdot \Big(\int_I |Q(t)|^r \; dt\Big)^{p/r}$$

by the definition of $D^{**}(m+n,p,I)$ and Hölder's inequality. The greatest lower bound of the last term, when P runs over $\mathcal{P}_m$ and Q runs over $\mathcal{P}_n$, is

$$D^{**}(m,q,I)^p \cdot D^{**}(n,r,I)^p.$$

The result follows. □

5.6 An open question

Finally, observing that

$$C(n,2) \sim \sqrt{\frac{2}{\pi}} \cdot 2^{2n} n! \quad (n \to \infty),$$

(an exercise on Stirling's formula from (8)), we ask the following question.

> Is it true that, for every $p > 0$, the quantity $2^{-2n}C(n,p)/n!$ tends to a limit when n tends to infinity?

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