Winding number and circular 4-coloring of (signed) graphs

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Abstract

Concerning the recent notion of circular chromatic number of signed graphs, we introduce a bipartite analogue of the generalized Mycielski graphs, denoted $BM_{k,2k-1}$, having the following properties. It has $k(2k-1)+1$ vertices, its shortest negative cycle is of length $2k$ and its circular chromatic number is 4.

In the course of proving our result, we also obtain a simpler proof of the fact that the generalized Mycielski graph $M_\ell(C_{2k+1})$ has circular chromatic number 4. The proof has the advantage that it illuminates, in an elementary manner, the strong relation between algebraic topology and graph coloring problems.

1 Introduction

The problem of building graphs of high girth and high chromatic number is one of the basic questions of graph coloring and its study has lead to many developments. In particular, the original proof of Erdős for the existence of such graphs has led to the development of probabilistic methods in graph theory. Since then several constructive methods are presented, but none easy to grasp. With a weaker condition of high odd girth instead of high girth there are several natural classes of graphs. In particular, in the family of the Kneser graphs one can find examples of high odd girth and high chromatic number. The proof of the lower bound for the chromatic number of the Kneser graphs, by L. Lovász, was the birth place of the connection between algebraic topology and graph coloring. Further developing this method, Stiebitz introduced a generalization of the Mycielski construction to build small graphs of high odd girth and high chromatic number. Generalized Mycielski on odd cycles have been studied independently by several authors and number of results on their chromatic number is proved.

In this work, building up on the ideas from several work in the literature, we first present a relatively short proof that the generalized Mycielski graphs on odd cycles have circular chromatic number 4. The proof has the advantage that captures the connection between algebraic topology and graph coloring with elementary techniques. We then present a class of signed bipartite graphs of high negative girth and circular chromatic number 4. Special subclass of these graphs have been proven to be on nearly optimal number of vertices among 4-chromatic graphs of a given odd girth and are conjectured to achieve the optimal value.

In the next section, we first settle notation and terminology. In the following section, we provide a historical account of what is known. Following that we present our proof. We end with concluding remarks the final section.

2 Notation

We consider simple graphs unless clearly stated otherwise. A *signed (simple) graph* (G, σ) is a graph G together with the assignment σ of signs to the edges. If G is bipartite, then (G, σ) is called a *signed bipartite graph* (in some literature this term is used to refer to a balanced signed graph, that is a signed graph with no negative cycle). The sign of a structure in (G, σ) (such as a cycle, a closed walk, a path) is the product of the signs of edges in the said structure counting multiplicity.

Given an integer n, $n \geq 3$, we denote by C_n the cycle (graph) on n vertices. That is a 2-regular connected graph on n vertices. Furthermore, we view C_n as plane graph, that is the graph together with a planar embedding. For topological use of C_n, one may identify it with the regular polygon on n vertices. Vertices of C_n are normally labeled as $v_1, v_2, \ldots, v_n$. The *exact square* of C_n, denoted $C_n^{\#2}$, is the graph on the same set of vertices where two vertices are adjacent if they are at distance (exactly) 2 in C_n. Observe that for odd values of n, $C_n^{\#2}$ is also a cycle of length n. For even values of n, $C_n^{\#2}$ consists of two connected components, each isomorphic to a cycle of length $\frac{n}{2}$. They are induced on sets of vertices with odd and even indices and will be denoted, respectively, by $C_n^{\#2o}$ and $C_n^{\#2e}$.

Given a positive real number, we denote by O_r the (geometric) circle of circumference r. That would be a circle of radius $\frac{r}{2\pi}$. The *antipodal* of a point x on O_r is the unique point $\overline{x}$ on O_r which is collinear with x and the center of the circle.

Given a real number r, $r \geq 2$, a *circular r-coloring* of a signed graph (G, σ) is a mapping ψ of the vertices of G to the points of O_r such when xy is a negative edge, then distance of $\psi(x)$ from $\psi(y)$ on O_r is at least 1 and if xy is a positive edge, then the distance of $\psi(x)$ from $\overline{\psi(y)}$ is at least 1, equivalently, the distance between $\psi(x)$ and $\psi(y)$ is at most $\frac{r}{2} - 1$. The *circular chromatic number* of (G, σ), denoted $\chi_c(G, \sigma)$, is the infimum of r such that (G, σ) admits a circular r-coloring. When restricted on signed graphs where all edges are negative, we have the classic notion of circular coloring of graphs. This extension to signed graphs is first presented in [17] noting that a different but similar parameter under a similar name has appeared in the literature first [10]. However, the roll of positive and negative edges are exchanged for a better suitability with literature on structural theory on signed graphs, specially in regard with minor theory of signed graphs.

Among basic results the following should be noted for the purpose of this work. The infimum in the definition is always attained for finite graphs, even allowing multi-edges and positive loops, but a negative loop cannot be colored with a finite r. For the class of signed bipartite (multi)graphs we have the trivial upper bound of $\chi_c(G, \sigma) \leq 4$, to see this, map the vertices of one part of G to the north pole of O_4 and the vertices of the other part to the east point. Even with such a strong upper bound the problem of determining the exact value of the circular chromatic number of a given signed bipartite graph is of high importance and, in general, quite a difficult problem. In particular, as it is pointed out in [17], using some basic graph operations, namely indicators, one can transform a graph G into a signed bipartite graph $F(G)$ such that the circular chromatic number of $F(G)$ determines the circular chromatic number of G. A basic example of this sort is to replace each edge uv of G with a negative 4-cycle $ux_{uv}vy_{uv}$ where x_{uv} and y_{uv} are new and distinct vertices. It is then shown in [17] that $\chi_c(S(G)) = 4 - \frac{4}{\chi_c(G)+1}$. Further connections with some well known study and theorems, such as the four-color theorem, is discussed in [11] and [16].

Motivated by these observations and in connection with some other studies, some of which are mentioned in the last section, the question of constructing signed bipartite graphs of high negative girth but circular chromatic number 4 is of high interest. In this work, we present a bipartite analogue of the generalized Mycielski graph on odd cycles as such examples of signed bipartite graphs.

Our proof also leads to an elementary understanding of the relation between coloring problems of graphs and basic notions of algebraic topology, namely the *winding number*. Recall that given a closed curve γ on the plane, the winding number of γ, defined rather intuitively, is the number of times γ is winded around the origin in the clockwise direction, noting that: if the origin is not in the part bounded by γ, then the winding number is 0 and that winding in counterclockwise direction is presented by a negative number. Here the closed curves we work with are mappings to O_r with the center of O_r being the center of the plane. They can be thought of as continuous mappings of $[0,1]$ to O_r with the condition that the two end points, i.e., 0 and 1 are mapped to the same point.

3 A historical note

Mycielski introduced in 1955 [14] the construction that is now known as the Mycielski construction. His goal of the construction was to build triangle-free graphs of high chromatic number. In this construction, given a graph G one adds a vertex v' for each vertex v of G which is joined to all neighbors of v in G and then adds a vertex u which is joined to all vertices v'. It is not difficult to prove that the resulting graph has chromatic number $\chi(G)+1$.

Generalization of the construction, where one adds several layers of copy vertices before adding a universal vertex to the last layer, was first considered, independently, in Ph.D. thesis of M. Stiebitz [23] and Ph.D. thesis of N. Van Ngoc [18]. The former being written in German and the latter being in Hungarian, they have not available to the author. Stiebitz applied methods of algebraic topology to prove that if one starts with K_2 and iteratively builds a generalized Mycielski, at each step the chromatic number would increase by 1. This does not hold for every graph though. For example, the chromatic number of the complement of C_7 is 4, and any generalized Mycielski of it, except the original one is also of chromatic number 4. It is recently been shown in [13] that the result of Stiebitz is equivalent to the Borsuk-Ulam theorem.

First English publications of the fact that the generalized Mycielski based on an odd cycle has chromatic number 4 appeared independently in [20], [25] and [26]. The proof of Payan is about the special case of $M_k(C_{2k+1})$ as they appear as subgraphs of nonbipartite Cayley graphs on binary groups but it works the same for any $M_\ell(C_{2k+1})$. This proof has strongly motivated the work presented here. The proof of [25] is presented quite differently, but the hidden idea behind the proof is the same. The result of [26] is more general. It is shown that if G is not bipartite but admits an embedding on the projective plane where all facial cycles are 4-cycles, then $\chi(G)=4$. That such structures are necessary in a 4-chromatic triangle-free projective planar graphs was conjectured in [26] and proved in [5]. The well known fact that $M_\ell(C_{2k+1})$ quaderangulate the projective plane is evident from our presentation of these graphs in the next section.

The circular chromatic number of Mycielski constructions was first studied in [2]. That of the generalized Mycielski is studied in [7], [12], [22], and [21] among others. In particular, that $\chi_c(M_\ell(C_{2k+1}))=4$ follows from the general result of [22] where it is shown that if the lower bound for the chromatic number is proved using topological connectivity, then the same lower bound works for the circular chromatic number as well. The fractional chromatic number of generalized Mycielski graphs is studied in [24].

4 The construction

The main body of the construction we will work with is an almost quadrangulation of the cylinder which we define here. Given positive integers ℓ and k, $C_{\ell\times(2k+1)}$ is the graph whose vertex set is $V=\{v_{i,j} \mid 1\leq i\leq \ell, 1\leq j\leq 2k+1\}$ with the edge set $E=\{v_{i,j}v_{i+1,j-1}, v_{i,j}v_{i+1,j+1} \mid 1\leq i\leq \ell-1, 1\leq j\leq 2k+1\}$. Here, and in the rest of this work, the addition on the indices is taken modular the maximum value of the said index, which is $\pmod{2k+1}$ in this case. We note that, as a graph $C_{\ell\times(2k+1)}$ is isomorphic to the categorical product $P_\ell\times C_{2k+1}$, but the standard labeling of this product does not fit well with our purpose. A general picture of this graph is depicted in Figure 1 where the dashed circles are only presenting the layers, but they will play a key role.

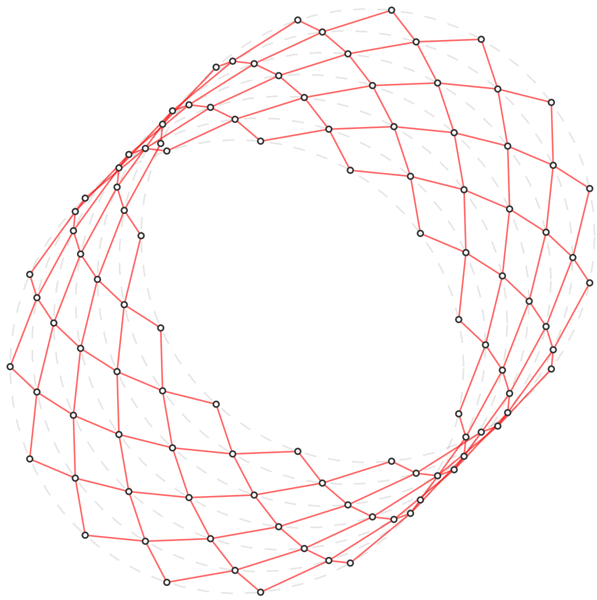

Figure 1: $C_{\ell\times(2k+1)}$ with layers highlighted

Given positive integers ℓ and k, the generalized Mycielski graph $M_\ell(C_{2k+1})$ is built from $C_{\ell\times(2k+1)}$ by the following two steps:

- Connect $v_{1,j}$ to $v_{1,j+k}$ (Figure 2 right).
- Add a new vertex u and connect it to all vertices $v_{\ell,j}$ (Figure 2 left).

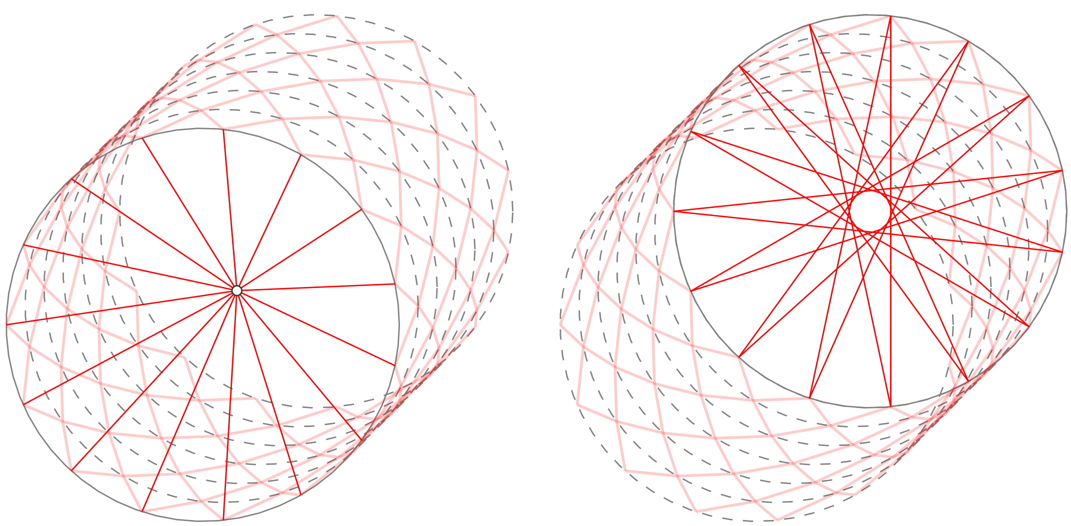

Figure 2: Constructions on bottom and top layers

Observe that the added edges in the first item form an isomorphic copy of C_{2k+1}. One can easily observe that, starting with this cycle, the classic definition of generalized Mycielski graph results in the same graph. The graph $M_1(C_3)$ is K_4. The graph $M_2(C_5)$ is the well known Grözsch graph. It serves as an example that the assumption of planarity in the Grözsch theorem is necessary [6]. That $M_2(C_5)$ is the smallest

4-chromatic triangle-free graph is proposed as an exercise in [8]. Chvátal showed [3] that, furthermore, $M_2(C_5)$ is the only 4-chromatic triangle-free graph on 11 vertices.

The following is a key property of $M_\ell(C_{2k+1})$.

Proposition 1. *The shortest odd cycle of $M_\ell(C_{2k+1})$ is the minimum of $2k+1$ and $2l+1$.*

Since this is folklore fact, we do not provide a proof but we note that the main idea to verify it is also presented in the next proposition.

Next, given integers ℓ and k satisfying $\ell, k \geq 2$, we define a signed bipartite graph $BM_{\ell,2k-1}$ also from $C_{\ell\times(2k-1)}$ as follows.

- Edges of $C_{\ell\times(2k-1)}$ are all negative.
- Connect $v_{1,j}$ to $v_{2,j+k}$ by a positive edge (Figure 3, right).
- Add a new vertex u and connect it to each of the vertices $v_{\ell,j}$, $j = 1, \dots, 2k-1$, with a negative edge (Figure 2, left).

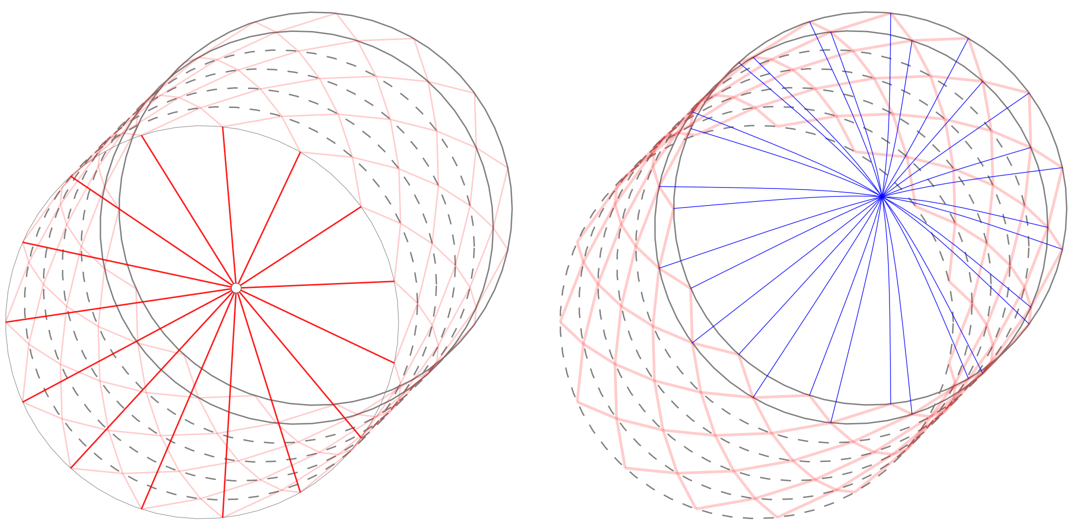

Figure 3: Construction of $BM_{\ell,2k-1}$

We view this construction as a bipartite analogue of the generalized Mycielski. The second item of the construction, which is presented in Figure 3, right, is the main difference with the previously know constructions: While in construction of $M_\ell(C_{2k+1})$ we add some edges between vertices of the first layer, in this new construction we add some connection between vertices of the first layer and the second layer. This operation preserves the bipartition. The underlying graph of the induced subgraph on the first two layers here then is isomorphic to what is known as the Möbuis ladder with $2k-1$ steps. We will refer to it as such.

The case of $BM_{1,3}$ is $(K_{3,4}, M)$ depicted in Figure 4. That is the signed bipartite graph where all the edges of a maximum matching of $K_{3,4}$ are assigned each a positive sign and all the other edges are assigned each a negative sign.

That the underlying graph of $BM_{\ell,2k-1}$ is bipartite is easily observed. Parity of the levels is a natural bipartition of the graph. We show that based on the choice of k and l this signed bipartite graph does not have a short negative cycle.

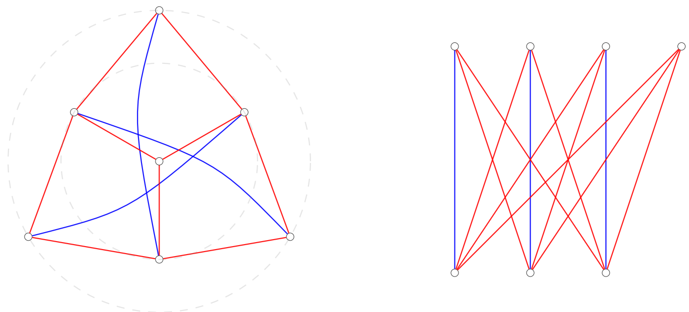

Figure 4: $BM_{1,3}$, presented two different ways

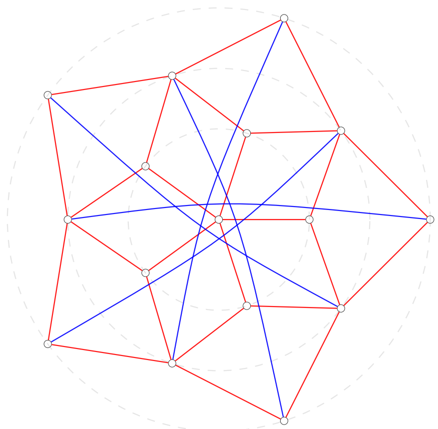

Figure 5: $BM_{2,5}$

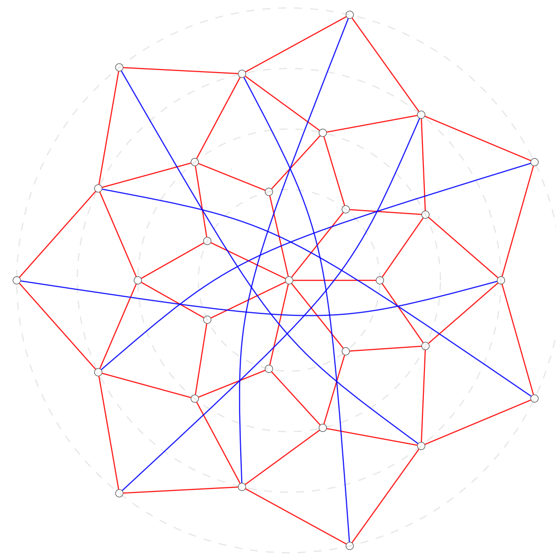

Figure 6: $BM_{3,7}$

Proposition 2. *Given integers l and k where $l, k \geq 2$, the shortest negative cycle of $BM_{\ell,2k-1}$ is of length $\min\{2l+2, 2k\}$.*

Proof. We first present two natural choices for a negative cycle, one of length $2k$ and another of length $2l+2$. The first is a negative cycle on the first two layers. Take a positive edge and connect its two ends with one of the two paths using only the negative edges that are connecting the two layers. This would result in negative cycle of length $2k$. The second negative cycle we consider is by taking a positive edge and connecting each of its ends to the vertex u by a shortest path (all edges negative). One of these paths will be of length l and the other would be of length $l+1$. Together with the first chosen edge itself then they form a negative cycle of length $2l+2$.

It remains to show that the shortest of these two types of cycles gives us the negative girth. To that end we will first show that a shortest negative cycle can only use one positive edge of $BM_{\ell,2k-1}$. Toward a contradiction, let C be a negative cycle with more than two positive edges. Our goal is to present a negative cycle C' whose length is at most $|C|-2$. To that end we take two positive edges of C that come consecutively on the cyclic order of C induced on the positive edges. Assume xy and $x'y'$ are these two edges and that x' is followed by y in the cyclic order of C (that is to say there is no positive edge in the $x'-y$ path in C). We remove the two positive edges xy and $x'y'$ and the $x'y$ path connecting them in C, but then we add yy' copy of this path (which also has no positive edge). The result is a closed walks whose sign is the same as that of C, and whose length is $|C|-2$. But then this closed walk must contain a negative cycle, whose length then is also at most $|C|-2$, a contradiction.

Finally if C is a cycle that uses exactly one negative edge, say xy, then the $x-y$ path $P_{xy} = C - xy$ either passes through u in which case we have at least $2l+2$ edges in C, or the natural image of P_{xy} to the cycle in between the first and second layers also connects x to y. But the shortest such path is of length $2k-1$, thus P_{xy}^{+} is of length at least $2k-1$, and the negative cycle is of length at least $2k$. □

5 Winding number and coloring

Given a simple closed curve γ on the plane, and a continuous mapping φ of γ to O_r, we define the *winding number* of the pair (γ, φ) to be the winding number of the curve $\varphi(\gamma)$ with center of O_r considered as the center of the plane. Intuitively speaking, (γ, φ) tells us how many times the curve γ is wrapped around O_r in the clockwise direction noting that a negative number reflects an anticlockwise mapping. This value then will be denoted by $\omega(\gamma, \varphi)$.

A mapping c of the vertices of the cycle C_n to the points of O_r can be extended to a continuous mapping of C_n to O_r with the former being viewed as the closed curve or the polygon. There are 2^n natural ways to do this. For each pair v_i, v_{i+1} of the vertices of C_n, the pair $c(v_i), c(v_{i+1})$ partitions the circle O_r into two parts. The segment of the polygon that represents the edge v_iv_{i+1} can be projected into one of these two parts. We note that c is allowed to map several vertices of C_n to the same point and that even if v_i and v_{i+1} are mapped to the same point, in our view, they partition the circle O_r into two parts: a part of length 0 and a part of length r.

These 2^n extensions are in a one-to-one correspondence with the 2^n possible orientations of C_n: orient the edge v_iv_{i+1} in such a way that the mapping follows the clockwise direction of O_r.

Given a coloring c of the vertices of the cycle C_n, two extensions of c to a mapping of the polygon to O_r are of special importance. The first is the extension corresponding to the directed cycle C_n. Here v_iv_{i+1} is mapped to the part of the circle where $c(v_{i+1})$ follows $c(v_i)$ in the clockwise direction. Let us denote this extension by c^D. A trivial observation here is that the winding number of (C_n, c^D) is never 0.

The other natural extension is to choose the shortest of the two parts of the circle determined by $c(v_i)$ and $c(v_{i+1})$ to extend the mapping on the segment corresponding to the edge v_iv_{i+1}. The orientation corresponding to this extension then depends on whether $c(v_i)$ is the start or the end of this shorter part of the circle with respect to the clockwise orientation. We denote this extension by c^{sh} and observe that this extension may result in winding number 0 for some choices of c (and r).

Given the cycle C_n, a mapping c of its vertices to O_r and an extension φ of c to the polygon, a combinatorial way to compute $\omega(C_n, \varphi)$ is as follows: take an (open) interval I on O_r which does not contain any image of the vertices of C_n. Then in an extension φ of c to a mapping of the polygon to O_r, each edge of C_n either traverses I completely, or does not touch any point of it. Now the winding number $\omega(C_n, \varphi)$ is the number of edges that traverse I in the clockwise direction minus the number of edges that traverse it in the anticlockwise direction.

Let c be a mapping of the vertices of a cycle C_n to the circle O_r. Consider the continuous mapping (C_n, c^D) and an (open) interval I of O_r which does not contain any point $c(v_i)$. Color the edges of C_n with two colors, say green and orange, as follows: if the image of an edge e under c^D contains I, then color it green, otherwise color it orange. We are interested in the pairs of consecutive edges $v_{i-1}v_i$ and v_iv_{i+1} which are colored differently. If in such a pair the first edge is colored green, then in the next pair of this sort, (next in the cyclic order of indices), the first edge must be orange and vice versa. Thus, the total number of such pairs is even, regardless of the choices of n and c.

To use this observation, we will work with certain types of mappings c. We say a mapping c of the vertices of C_n to the points on O_r is *semi-proper* if the followings hold: for each i the pair $c(v_{i-1})$ and $c(v_{i+1})$ of the points on O_r partitions it into two unequal parts and that $c(v_i)$ is on the larger of the two parts.

In the following, we present how the condition of semi-proper provides a connection between c^D extension of c on C_n and c^{sh} extension of the mapping c on $C_n^{\#2}$.

Lemma 3. *Let c be a semi-proper mapping of C_n to O_r and let I be an interval of O_r which does not contain any $c(v_i)$. Then in the extension c^{sh} as a mapping of one or two cycles in $C_n^{\#2}$ to O_r, the number of edges $v_{i-1}v_{i+1}$ that does not cross over I is an even number.*

Proof. That is because this number is the number of consecutive pairs of the edges of C_n which are not of the same color in the {orange, green}-coloring corresponding to the extension c^D of c. In counting such edges, following the cyclic order, each time we see a green-orange, we have to see an orange-green next. As we must return to the starting point we have an even number of them. □

We may now observe that, given a real number r, $r < 4$, any circular r-coloring of C_n must be a semi-proper coloring. Thus we have the following two consequences depending on the parity of n.

Lemma 4. *Let c be a circular r-coloring of an even cycle C_n. Let c_o (resp. c_e) be its restriction on the vertices with odd (resp. even) indices. Then the winding numbers of $(C^{\#2o}, c_o^{sh})$ and $(C^{\#2e}, c_e^{sh})$ are of the same parity.*

Proof. That is because after choosing a suitable interval I, by Lemma 3, the total number of edges of $C^{\#2}$ that does not cross over I in the extension c^{sh} is even. As the total number of edges is also even (that is n), the number of edges of $C^{\#2}$ that cross over I is also even. However, the winding number of each of $(C^{\#2o}, c_o^{sh})$ and $(C^{\#2e}, c_e^{sh})$, which is the difference of the number of edges crossing I in the clockwise direction and the number edge crossing it in the anticlockwise direction, has the same parity as the total number of the edges of the cycle in consideration that cross over I (in the c^{sh} extension). This proves our claim as the sum of the two winding numbers is an even number. □

Using this lemma we can build cylinder of many layers, as shown in the example of Figure 1, with the property that in any circular r-coloring c of the red graph ($r < 4$) all of the dashed gray cycles must have winding numbers of the same parity. Observe that in this construction the zigzag red cycle between two layers is an even cycle and its exact square consists of the two gray cycles presenting the two levels. If we then add structures to the two ends in such a way that one forces an odd winding number of one of the gray cycles and the other forces an even winding number to one of them, then the result would be a graph which admits no circular r-coloring for $r < 4$.

A basic method to achieve these conditions, which results in the generalized Mycielski graphs, is presented next.

Lemma 5. *Given an odd integer n, a positive real number r, and a semi-proper mapping c of C_n to O_r, the winding number $\omega(C_n^{\#2}, c^{sh})$ is an odd number.*

Proof. By Lemma 3, the total number of edges of $C_n^{\#2}$ that does not cross over I is even. As n is an odd number, $C_n^{\#2}$ is isomorphic to C_n, and, hence, the number of edges crossing over I is odd. This is the sum of the number of edges crossing over I in the clockwise direction and in the counterclockwise direction. Thus the winding number, which is the difference of these two numbers, is also an odd number. □

Applying this lemma on circular r-coloring for $r < 4$ we have the following.

Lemma 6. *Given an odd integer n, $n = 2k+1$, a real number r satisfying $2 + \frac{1}{k} \leq r < 4$, and a circular r-coloring c of C_n, the winding number $\omega(C_n^{\#2}, c^{sh})$ is an odd number.*

Proof. We observe that if $r < 4$ and c is a circular r-coloring of C_n, then it is, in particular, a semi-proper mapping of C_n. That is because for three consecutive vertices v_{i-1}, v_i, and v_{i+1}, having partitioned O_r to two parts based on $c(v_{i-1})$ and $c(v_{i+1})$, the part that contains $c(v_i)$ must be of length at least 2. As $r < 4$, this must be the larger part. □

Observation 7. *Let G be the star $K_{1,n}$ with u being the central vertex and A being the independent set of order n. Let c be a circular r-coloring of G with $r < 4$. Then for any cycle C built on A the winding number of (C, c^{sh}) is 0.*

This is observed by taking a sufficiently small interval containing $c(u)$ and noting that since $r < 4$, for any pair x and y of vertices in A in the partition of O_r to two parts by $c(x)$ and $c(y)$, the part containing u is of length at least 2 and thus it is the larger of the two.

We may now give a new proof of the following theorem.

Theorem 8. *For any positive integers ℓ and k, we have $\chi_c(M_\ell(C_{2k+1}) = 4$.*

Proof. It is enough to observe that $M_\ell(C_{2k+1})$ is obtained from the $l \times (2k+1)$ cylindrical grid of Figure 1 by adding diagonal edges to bottom layer (that is connecting pairs at distance k of the gray cycle) and adding a universal vertex to the top layer. As any circular r-coloring with $r < 4$ is also semi-proper, any such a coloring would imply an odd winding number for the the layer in C^{sh} extension from one end and an even winding number for the layers from the other end. □

Next we show that $BM_{\ell,2k-1}$ shares the same property. We will note later that Theorem 8 follows from the next theorem.

Theorem 9. *For given positive integers ℓ and k, satisfying $l, k \geq 2$, we have $\chi_c(BM_{\ell,2k-1}) = 4$.*

Proof. Toward a contradiction, let c be a circular r-coloring of $BM_{\ell,2k-1}$ with $r < 4$. We will have a contradiction if we show that the cycle C' formed on $v_{1,1}v_{1,2}\cdots v_{1,2k-1}$ in this cyclic order has an odd winding number under the mapping c^{sh} (restricted on the vertices of this cycle). We emphasize that edges of C' are not in $BM_{\ell,2k-1}$.

To this end we first consider another cycle, $C^\star$, (also not part of our graph) by considering the following sequence of vertices of the first layer of $BM_{\ell,2k-1}$: $v_{1,1}v_{1,k+1}v_{1,2}v_{1,k+2}\cdots v_{1,k}$. Note that in this cycle $v_{1,j}$ is followed by $v_{1,j+k}$ where the addition is taken $\pmod{2k-1}$. We may also note that this is the diagonally drawn cycle on the first layer of Figure 2 (right).

Our claim is that the mapping c, viewed as a mapping of the vertices of $C^\star$ to O_r, is a semi-proper mapping. Toward proving the claim, we consider $c(v_{1,j})$, $c(v_{1,j+k})$, and $c(v_{1,j+1})$. The first observation is that since $v_{2,j}$ is adjacent to both $v_{1,j}$ and $v_{1,j+1}$ with negative edges, the points $c(v_{1,j})$ and $c(v_{1,j+1})$ of O_r partition O_r in such a way that the part containing $c(v_{2,j})$ is at least 2. As $r < 4$, it follows that $c(v_{2,j})$ is on the larger part of O_r when it is partitioned by $c(v_{1,j})$ and $c(v_{1,j+1})$. It remains to show that $c(v_{1,j+k})$ is also on the same part. If not, that is if $c(v_{1,j+k})$ is on the shorter side of O_r, then one of the arcs $c(v_{1,j+k})c(v_{2,j})$ and $c(v_{2,j})c(v_{1,j+k})$ contains the shorter side of $c(v_{1,j})c(v_{2,j})$ and the other contains the shorter side of $c(v_{1,j+1})c(v_{2,j})$. As each of these shorter arcs are of length at least one, we conclude that the distance of $c(v_{1,j+k})$ and $c(v_{2,j})$ is at least one. However, since c is a circular r-coloring where $r < 4$ and $v_{1,j+k}v_{2,j}$ is a positive edge, they should be at distance at most $\frac{r}{2} - 1 < 1$, a contradiction.

Finally observing that C' is the exact square of $C^\star$, and by Lemma 5, we conclude that the winding number of C' is odd. □

6 Concluding remarks

1. In recent development of the theory of homomorphisms and colorings of signed graphs it has normally been the case that restriction on the class of signed bipartite graphs strengthen the results on the class of graphs. This has been the case in this work as well. We show here that Theorem 9 easily implies Theorem 8.

First, observe that adding a positive loop to the vertices does not affect the circular coloring and the circular chromatic number of signed graphs. The claim follows from observation that if we identify the two ends of each positive edge in $BM_{\ell,2k+1}$, then we will get a copy of $M_{\ell-1}(C_{2k+1})$ where the vertices on the first layer have positive loops on them but all other edges are negative.

2. It is not difficult to show that removing any edge from $BM_{\ell,2k+1}$ the resulting signed graph admits a circular 3-coloring.

3. The special subclass of $M_k(C_{2k+1})$, on $2k^2+k+1$ vertices, is conjectured in [25] to have the smallest number of vertices among 4-chromatic graphs of odd-girth $2k+1$. In [4], this is verified to be the case with an added assumption that every pair of odd cycles share a vertex. For the general case, a lower bound of $(k-1)^2$ for the number of vertices of a 4-critical graph of odd girth $2k+1$ is given in [9] modifying the method of [19].

The starting point of this work has been a joint work with Lan Anh Pham and Zhouningxin Wang on the study of C_{-4}-critical signed graphs (for a definition see [15]). In a forthcoming work, it is shown that a C_{-4}-critical signed graph of negative girth at least $2k$ must have at least k^2 vertices. Based on the fact that $\chi_c(C_{-4}) = \frac{8}{3}$, our result in this work implies that $BM_{k,k-1}$ is a signed bipartite graph of negative girth $2k$ which does not map to C_{-4}. Thus $BM_{k,k-1}$ contains a C_{-4}-critical signed graph. As $BM_{k,k-1}$ has $2k^2-k+1$ vertices, this implies the smallest number of the vertices of a C_{-4}-critical signed graph of negative girth $2k$ is somewhere between k^2 and $2k^2-k+1$.

4. As mentioned, the fact that $M_l(C_{2k+1})$ is a 4-chromatic graph is generalized by Youngs showing that any quaderangulation of the projective plane, if not bipartite, is of chromatic number 4. A strengthening and stronger connection to this work is to be addressed in a follow up work. That would be to show that, if a signed graph (G,σ) admits an embedding on the projective plane such that each face of it is a positive 4-cycle, then either it is balanced, and thus admits a circular 2-coloring, or, after adding a positive loop to each vertex of it, it would admit a homomorphism from some $BM_{\ell,2k-1}$. This would imply that its circular chromatic number is at least 4, but a circular 4-color is implied by maximum average degree condition on these graphs.

5. In this work, a connection between the circular chromatic number being 4 and winding number is developed. Following Lovasz's proof of the Kneser conjecture using topological connectivity, applications of topological methods to graph coloring problems have been one of the fascinating part of graph theory. Perhaps some other results of this sort can be reduced to notion of winding number using the following development from [15].

Given a graph G, let $T_\ell(G)$ be the signed graph obtained from G by replacing each edge e of it with a path P_e of length ℓ such that internal vertices are distinct, and then assign a sign such that product of the signs of the edges in P_e is negative. It then follows that G is k-colorable if and only if $T_{k-2}(G)$ admits a homomorphism to C_{-k}, or equivalently, if it admits a circular $\frac{2k}{k-1}$-coloring.

6. Finally we should note that Payan's interest in $M_k(C_{2k+1})$ was based on the fact that they appear as subgraph in binary Cayley graphs that are not bipartite and in particular in the projective cubes (see [1]). An extension in this direction to signed bipartite Cayley graphs is to be addressed in a forthcoming joint work with Meirun Chen.

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