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Abstract

In this paper, we deal with a Markov chain on a measurable state space $(\mathbb{X}, \mathcal{X})$ which has a transition kernel P admitting an aperiodic small-set S (i.e. $P \geq \nu(\cdot)1_S$ for some positive measure ν on $\mathbb{X}$ such that $\nu(1_S) > 0$), and satisfying the standard geometric-drift condition. Under these assumptions, it can be easily checked that there exists $\alpha_0 \in (0, 1]$ such that the following property holds: $PV^{\alpha_0} \leq \delta^{\alpha_0} V^{\alpha_0} + \nu(V^{\alpha_0})1_S$. Hence P is $V^{\alpha_0}-$geometrically ergodic and its "second eigenvalue" ϱ_{α_0} provides the best rate of convergence. Setting $R := P - \nu(\cdot)1_S$ and $\Gamma = \{\lambda \in \mathbb{C},\ \delta^{\alpha_0} < |\lambda| < 1\}$, this "second eigenvalue" is shown to satisfy, either $\varrho_{\alpha_0} = \max\big\{|\lambda| \,:\, \lambda \in \Gamma,\ \sum_{k=1}^{+\infty} \lambda^{-k}\, \nu(R^{k-1}1_S) = 1\big\}$ if this set is not empty, or $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$. Actually the set is finite in the first case and is composed by the spectral values of P in Γ. The second case occurs when P has no spectral value in Γ. Moreover, a bound of the operator-norm of $(zI - P)^{-1}$ allows us to derive an explicit formula for the multiplicative constant in the rate of convergence, which can be evaluated provided that any information of the "second eigenvalue" is available. To get such an information, we obtain a simple and explicit bound of the operator-norm of $(I - P + \pi(\cdot)1_{\mathbb{X}})^{-1}$ involved in the definition of the so-called fundamental solution to Poisson's equation. This allows us to specify the location of the eigenvalues of P and, then, to obtain a new explicit bound on ϱ_{α_0}. The case of reversible Markov kernel is also discussed and an application to MCMC algorithms is proposed. In fact the bound for the operator-norm of $(I - P + \pi(\cdot)1_{\mathbb{X}})^{-1}$ is based on an estimate, only depending on δ^{α_0}, of the operator-norm of $(I - R)^{-1}$ which provides another way to get a solution to Poisson's equation. This estimate is also shown to be of greatest interest to generalize the error bounds obtained for perturbed discrete and atomic Markov chains in [LL18] to the case of general geometrically ergodic Markov chains. These error estimates are the simplest that can be expected in this context. All the estimates in this work are expressed in the standard $V^{\alpha_0}-$weighted operator norm.

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1 Introduction

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space, and let $\mathcal{M}^+$ denote the set of finite non-negative measures on $(\mathbb{X}, \mathcal{X})$. For any $\mu \in \mathcal{M}^+$ and any μ-integrable function $f : \mathbb{X} \to \mathbb{C}$, $\mu(f)$ denotes the integral $\int f d\mu$. For any measurable function $W \geq 1$ we denote by $(\mathcal{B}_W, \|\cdot\|_W)$ the Banach space of measurable functions $f : \mathbb{X} \to \mathbb{C}$ such that $\|f\|_W := \sup_{x\in\mathbb{X}} |f(x)|/W(x) < \infty$. The identity map on $\mathcal{B}_W$ is denoted by I, and $(\mathcal{B}'_W, \|\cdot\|'_W)$ stands for the topological dual space of $\mathcal{B}_W$ (i.e. the Banach space of $\mathbb{C}$-valued bounded linear maps on $\mathcal{B}_W$). For any $\mu \in \mathcal{M}^+$ satisfying $\mu(W) < \infty$, the map $f \mapsto \mu(f)$ belongs to $\mathcal{B}'_W$, and for any such $(\mu_1, \mu_2) \in (\mathcal{M}^+)^2$, the norm $\|\mu_1 - \mu_2\|'_W$ coincides with the standard W-weighted total variation norm, that is:

$$\|\mu_1 - \mu_2\|'_W := \sup_{|f|\leq W} \big|\mu_1(f) - \mu_2(f)\big|. \tag{1}$$

Throughout the paper P is a Markov kernel on $(\mathbb{X}, \mathcal{X})$, and the existence of a small-set S for P is assumed, that is: there exist $S \in \mathcal{X}$ and $\nu \in \mathcal{M}^+$ such that

$$\nu(1_S) > 0 \quad \text{and} \quad \forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad P(x, A) \geq \nu(1_A)\, 1_S(x). \tag{$\boldsymbol{S}$}$$

We also assume that there exists a measurable function $V : \mathbb{X} \to [1, +\infty)$ (called a Lyapunov function) satisfying the following geometric drift condition with $S^c := \mathbb{X} \setminus S$:

$$\exists \delta \equiv \delta(P) \in (0, 1),\ \forall x \in S^c, \quad (PV)(x) \leq \delta\, V(x) \tag{$\boldsymbol{D_{S^c}}$}$$

$$\text{and} \quad K := \sup_{x\in S} (PV)(x) < \infty. \tag{$\boldsymbol{K}$}$$

Throughout the paper, Assumptions $(\boldsymbol{A})$ will stand for the set of the three assumptions $(\boldsymbol{S})$-$(\boldsymbol{D_{S^c}})$-$(\boldsymbol{K})$. Under Assumptions $(\boldsymbol{A})$ we know that there exists a unique $P-$invariant probability measure denoted by π on $(\mathbb{X}, \mathcal{X})$ and that $\pi(V) < \infty$, e.g. see [MT93, RR04, Bax05, DMPS18]. In this paper, replacing the Lyapunov function V with V^{α_0} for some suitable constant $\alpha_0 \in (0, 1]$ derived from the data in $(\boldsymbol{A})$, we present new results concerning the spectral properties of P on the space $\mathcal{B}_{V^{\alpha_0}}$ in relation with the so-called $V^{\alpha_0}-$geometrical ergodicity of P. These spectral results are applied to the study of the sensitivity with respect to the parameter θ of the invariant probability measure of transition kernels P_θ satisfying Assumptions $(\boldsymbol{A})$ in a uniform way in θ.

Let us recall some facts before specifying the main results of the paper. Under Assumptions $(\boldsymbol{A})$, we know from [HL22, Cor. 4.2] that there exists $\alpha_0 \equiv \alpha_0(P) \in (0, 1]$ such that

$$PV^{\alpha_0} \leq \delta^{\alpha_0}\, V^{\alpha_0} + \nu(V^{\alpha_0}) 1_S. \tag{$\boldsymbol{D^{\alpha_0}}$}$$

It follows from $(\boldsymbol{D^{\alpha_0}})$ and $(\boldsymbol{S})$ that P is $V^{\alpha_0}-$geometrically ergodic, e.g. see [MT93, RR04, Bax05, DMPS18]: There exist $\rho \in (0, 1)$ and $C_\rho \in (0, +\infty)$ such that

$$\forall f \in \mathcal{B}_{V^{\alpha_0}},\ \forall n \geq 1, \quad \|P^n f - \pi(f) 1_{\mathbb{X}}\|_{V^{\alpha_0}} \leq C_\rho\, \rho^n\, \|f\|_{V^{\alpha_0}}. \tag{2}$$

We denote by ϱ_{α_0} the infimum bound of the positive real numbers ρ satisfying (2). The real number ϱ_{α_0} is sometimes called the "second eigenvalue" of P on $\mathcal{B}_{V^{\alpha_0}}$ (even though ϱ_{α_0} is not necessarily an eigenvalue of P), while $1 - \varrho_{\alpha_0}$ is called the spectral gap of P on $\mathcal{B}_{V^{\alpha_0}}$. When P satisfies (2) and is reversible with respect to π, it follows from [Bax05, Th. 6.1] that

$$\forall f \in \mathbb{L}^2(\pi),\ \forall n \geq 1, \quad \|P^n f - \pi(f) 1_{\mathbb{X}}\|_{\mathbb{L}^2(\pi)} \leq 2\, \varrho_{\alpha_0}^{\,n}\, \|f\|_{\mathbb{L}^2(\pi)} \tag{3}$$

where $\mathbb{L}^2(\pi)$ is the standard Lebesgue space equipped with the norm $\|f\|_{\mathbb{L}^2(\pi)} = \pi(|f|^2)^{1/2}$. Thus, in this case, ϱ_{α_0} is an upper bound of the second eigenvalue of P on $\mathbb{L}^2(\pi)$. Finally recall that $\lambda \in \mathbb{C}$ is a spectral value of P on $\mathcal{B}_{V^{\alpha_0}}$ if $\lambda I - P$ is not invertible on $\mathcal{B}_{V^{\alpha_0}}$. The spectral value $\lambda \in \mathbb{C}$ is an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ if $\lambda I - P$ is not injective on $\mathcal{B}_{V^{\alpha_0}}$.

Under Assumptions ($\boldsymbol{A}$) we obtain the following statements with $\alpha_0 \in (0,1]$ given in ($\boldsymbol{D^{\alpha_0}}$).

- (Section 2) Let $a \in (\delta^{\alpha_0}, 1)$. The set $\mathcal{S}_a$ of spectral values λ of P on $\mathcal{B}_{V^{\alpha_0}}$ such that $a \leq |\lambda| \leq 1$ is finite and composed of eigenvalues of P. Note that $\lambda = 1 \in \mathcal{S}_a$. If $\mathcal{S}_a = \{1\}$, then $\varrho_{\alpha_0} \leq a$; Otherwise $\varrho_{\alpha_0} = \max\big\{|\lambda|,\, \lambda \in \mathcal{S}_a,\, \lambda \neq 1\big\}$. Moreover
$$\lambda \in \mathcal{S}_a \iff \mu_\lambda(1_S) = 1 \tag{4}$$
where $\mu_\lambda(1_S) := \sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k(1_S)$, with $\beta_k = \nu \circ (P - \nu(\cdot)1_S)^{k-1} \in \mathcal{B}'_{V^{\alpha_0}}$.

- (Section 3) For every $z \in \mathbb{C}\setminus\{1\}$ such that $|z| \in \big(\max(\delta^{\alpha_0}, \varrho_{\alpha_0}), 1\big]$, the operator $zI - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$ and we have
$$\forall f \in \mathcal{B}_{V^{\alpha_0}}, \quad \|(zI-P)^{-1}f\|_{V^{\alpha_0}} \leq \frac{1}{|z|-\delta^{\alpha_0}}\left(1+\frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{\big|1-\mu_z(1_S)\big|\big(|z|-\delta^{\alpha_0}\big)}\right)\|f\|_{V^{\alpha_0}}. \tag{5}$$
Moreover, for any $\rho \in \big(\max(\delta^{\alpha_0}, \varrho_{\alpha_0}), 1\big)$, Inequality (2) holds with
$$C_\rho = \frac{\rho}{2\pi(\rho-\delta^{\alpha_0})}\left(1+\frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{m_\rho\big(\rho-\delta^{\alpha_0}\big)}\right) \quad \text{with } m_\rho := \min_{z\in\mathbb{C}:|z|=\rho} |1-\mu_z(1_S)| > 0. \tag{6}$$

- (Section 4) For any $f \in \mathcal{B}_{V^{\alpha_0}}$ such that $\pi(f) = 0$, the two functions $\widetilde{f} := \sum_{n=0}^{+\infty} R^n f$ with $R := P - \nu(\cdot)1_S$ and $\widehat{f} := \sum_{n=0}^{+\infty} P^n f$ are solutions in $\mathcal{B}_{V^{\alpha_0}}$ to Poisson's equation $(I-P)g = f$. Moreover
$$\|\widetilde{f}\|_{V^{\alpha_0}} \leq \frac{1}{1-\delta^{\alpha_0}}\,\|f\|_{V^{\alpha_0}} \quad \text{and} \quad \|\widehat{f}\|_{V^{\alpha_0}} \leq \frac{1+\pi(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}{1-\delta^{\alpha_0}}\,\|f\|_{V^{\alpha_0}}. \tag{7}$$

- (Section 5) Using (5) and the second bound in (7), we present results concerning the location of the eigenvalues of P on $B_{V^{\alpha_0}}$, from which we deduce a upper bound of the second eigenvalue ϱ_{α_0} (Corollary 5.1). In particular, when P is reversible with respect to π and satisfies a slight additional condition (see (44)), we obtain that (Corollary 5.2)
$$\varrho_{\alpha_0} \leq \psi(\eta_\infty) \leq \psi(\eta_n) \quad \text{with} \quad \psi(t) := \frac{\delta^{\alpha_0}(1-\delta^{\alpha_0})t + \nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{(1-\delta^{\alpha_0})t + \nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}} \tag{8}$$
where $\forall n \geq 1$, $\eta_n := 2\sum_{k=1}^{n} \beta_{2k-1}(1_S)$ and $\eta_\infty := 2\sum_{k=1}^{+\infty} \beta_{2k-1}(1_S)$. An application to Markov chain Monte Carlo (MCMC) algorithms is addressed in Corollary 5.5. Finally, if P is also positive (i.e. $\forall f \in \mathbb{L}^2(\pi)$, $\langle f, Pf\rangle_{\mathbb{L}^2(\pi)} \geq 0$), then $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$ (Corollary 5.3). Recall that, in reversible case, any bound of ϱ_{α_0} is of interest in (3) too.

- (Section 6) Let Θ be an open subset of some metric space, and let $\{P_\theta\}_{\theta\in\Theta}$ be a family of Markov kernels on $(\mathbb{X}, \mathcal{X})$ satisfying Assumptions ($\boldsymbol{A}$) in a uniform way in $\theta \in \Theta$, as well as the following condition:
$$\forall x \in \mathbb{X}, \quad \lim_{\theta\to\theta_0} \Delta_{\theta,\alpha_0}(x) = 0 \quad \text{with} \quad \Delta_{\theta,\alpha_0}(x) := \|P_\theta(x,\cdot) - P_{\theta_0}(x,\cdot)\|'_{V^{\alpha_0}} \tag{9}$$

where θ_0 is fixed in Θ. Let π_θ denote the $P_\theta-$invariant probability measure. Then $\lim_{\theta\to\theta_0}\|\pi_\theta-\pi_{\theta_0}\|'_{V^{\alpha_0}}=0$ and we have for every $\theta\in\Theta$

$$\big\|\pi_\theta-\pi_{\theta_0}\big\|'_{V^{\alpha_0}} \le \frac{1+\pi_{\theta_0}(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}{1-\delta^{\alpha_0}}\times\pi_\theta\big(\Delta_{\theta,\alpha_0}\big) \quad\text{with}\quad \lim_{\theta\to\theta_0}\pi_\theta\big(\Delta_{\theta,\alpha_0}\big)=0. \tag{10}$$

Recall that finding effective and computable rate of convergence in geometric ergodicity is a difficult issue, see [MT94, Bax05, and the references therein]. The results of Section 2 based on the quasi-compactness of P ensure that the following alternative holds for the second eigenvalue: Either ϱ_{α_0} equals to the largest solution (in modulus) to the equation $\mu_z(1_S)=1$ in $\{z\in\mathbb{C}:\delta^{\alpha_0}<|z|<1\}$ if such a solution exists; Or $\varrho_{\alpha_0}\le\delta^{\alpha_0}$ (see Corollary 2.1 for details). This algebraic issue is difficult to apply in practice since it involves the power series $\sum_{k=1}^{+\infty}z^{-k}\,\beta_k(1_S)$. When the second eigenvalue of P is known (or at least bounded), Property (6) which gives explicit constants C_ρ in (2) is relevant, provided that the numerical computation of m_ρ in (6) is tractable. Note that the equivalence (4) is crucial to prove the positivity of m_ρ. Property (7) gives true computable bounds for the $V^{\alpha_0}-$norm of two particular solutions to Poisson's equation. Note that the second bound in (7) concerning the classical solution $\widehat{f}=\sum_{n=0}^{+\infty}P^nf$ to Poisson's equation when $\pi(f)=0$ is deduced from the first bound in (7) for the solution $\widetilde{f}=\sum_{n=0}^{+\infty}R^nf$. These two solutions to Poisson's equation are of interest in this work: $\widehat{f}$ is relevant in Section 5, while $\widetilde{f}$ is used in Section 6. Note that the inequality $\pi(V^{\alpha_0})\le\nu(V^{\alpha_0})/(1-\delta^{\alpha_0})$ easily derived from $(\boldsymbol{D^{\alpha_0}})$, may be used in (7) when π is unknown. The general bound of ϱ_{α_0} obtained in Corollary 5.1 requires the numerical computation of the positive real number $m_0:=\min_{\vartheta\in[\vartheta_0,2\pi-\vartheta_0]}|1-\mu_{e^{i\vartheta}}(1_S)|$ for some $\vartheta_0\in(0,\pi/2)$. When P is reversible, this issue is greatly simplified since m_0 is replaced with η_∞ (see (8)). The first bound in (7) is applied in the perturbation issue of Section 6, so that the constant in (10) is truly computable too (use again $\pi_{\theta_0}(V^{\alpha_0})\le\nu(V^{\alpha_0})/(1-\delta^{\alpha_0})$ when π is unknown). The real number $\pi_\theta\big(\Delta_{\theta,\alpha_0}\big)$ in (10) is available when the function Δ_{θ,α_0} in (9) is known (or can be bounded) and π_θ is computable for $\theta\ne\theta_0$. This holds for instance for discrete set $\mathbb{X}$ when the perturbed Markov kernels are truncated stochastic matrices on a finite state space (see Remark 6.1). Finally note that $\|\pi_\theta-\pi_{\theta_0}\|'_{TV}\le\|\pi_\theta-\pi_{\theta_0}\|'_{V^{\alpha_0}}$ where $\|\pi_\theta-\pi_{\theta_0}\|'_{TV}$ denotes the total variation distance between the two probability measures π_θ and π_{θ_0} (use (1) with $W=1_{\mathbb{X}}$).

Under Assumptions $(\boldsymbol{A})$, it is proved in [Bax05, Th. 1.1] that P is $V-$geometrically ergodic. However it is worth noticing that our results only focus on the $V^{\alpha_0}-$weighted operator norm in Sections 2–5 and on $V^{\alpha_0}-$weighted total variation norm in Section 6, where $\alpha_0\in(0,1]$ is given in $(\boldsymbol{D^{\alpha_0}})$. Hence, when $\alpha_0<1$, our results involve the smaller space $\mathcal{B}_{V^{\alpha_0}}$ in place of the expected one $\mathcal{B}_V$. This is the price to pay when working with the drift condition $(\boldsymbol{D^{\alpha_0}})$. The benefit is that the results obtained on $\mathcal{B}_{V^{\alpha_0}}$ from $(\boldsymbol{D^{\alpha_0}})$ have a fairly simple form. In the reversible case, recall that any rate of convergence in (2) provides a rate of convergence in the $\mathbb{L}^2(\pi)-$geometrical ergodicity (3), whatever the value of α_0,

The constant $\alpha_0\in(0,1]$ in $(\boldsymbol{D^{\alpha_0}})$ can be easily computed from the data in Assumptions $(\boldsymbol{A})$ (see [HL22, (28)]). For convenience the proof that $(\boldsymbol{D^{\alpha_0}})$ holds true and the explicit computation of α_0 are recalled in Appendix A. The real number K in Condition $(\boldsymbol{K})$ plays an important role in the computation of α_0: roughly speaking, the larger K is compared to $\nu(V)$, the smaller α_0 is. If the small-set S in $(\boldsymbol{S})$ is an atom with ν given by $\nu=P(s,\cdot)$ for some $s\in S$, then $(\boldsymbol{D^{\alpha_0}})$ holds with $\alpha_0=1$ (see Appendix A). Note that the case $\alpha_0=1$ is not equivalent to the atomic case, in other words Property $(\boldsymbol{D^{\alpha_0}})$ may hold with $\alpha_0=1$

for non-atomic small set S, see [HL22, Sec. 6]. Of course there are probably instances of Markov chains satisfying Assumptions ($\boldsymbol{A}$), for which the use of Property ($\boldsymbol{D^{\alpha_0}}$) is not relevant because α_0 is too close to zero, so that δ^{α_0} is too close to one for the bounds (6), (7), (8) or (10) to be of interest. We believe that these unfavourable cases correspond to instances for which the minorization/drift conditions are not well suited for finding interesting rates of convergence in geometrical ergodicity context, whatever the method used (see [QH21]).

The spectral properties for geometrically ergodic Markov chains have been investigated in many papers, e.g. see [KM03, KM05, Hen06, HL14a, HL14b, Del17]. The novelty of this work is that we obtain more simple and explicit results due to Condition ($\boldsymbol{D^{\alpha_0}}$). To the best of our knowledge, the results in this work are new. In particular, when P is positive reversible and satisfies Assumptions ($\boldsymbol{A}$) with an atom S (thus $\alpha_0 = 1$), the bound $\varrho_1 \leq \delta$ was obtained in [Bax05, Sec. 2.3]. Thus the bound $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$ in Corollary 5.3 extends this result to the non-atomic case. Using the numerical values of α_0 given in [HL22, Sec. 6.3-6.4] for Metropolis-Hastings algorithm of the Gaussian distribution and for Gaussian autoregressive Markov chain, we can check that the bound $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$ is relevant in comparison with those provided in [Bax05, Sec. 2.3 (non-atomic case)], see Remark 5.3. That $\widetilde{f} := \sum_{n=0}^{+\infty} R^n f$ is solution to Poisson's equation when $\pi(f) = 0$ seems to be a new result which extends to our framework the statement [Kem81, Th. 2] involving generalized fundamental finite matrix. The bounds (7) and (10) have been proved for discrete state space Markov chains with a finite atom in [LL18, Prop. 1, Th. 2] thanks to renewal theory. Theorems 4.1 and 6.1 extend these results to the non-atomic case and to general state spaces. The bound (10) improves all the error bounds obtained under Condition (9) in the literature for the stationary distribution of perturbed geometrically ergodic Markov chains, provided we use the Lyapunov function V^{α_0} in place of V. Indeed, the bound (10) involves neither the iterates of the unperturbed Markov kernels, nor those of the perturbed Markov kernels, exactly as in the case of discrete Markov chains with an atom investigated in [LL18]. It is worth noticing that Condition (9) is much weaker than the standard operator norm continuity assumption introduced in [Kar86] (see Remark 6.2). The comparison with the weak operator norm continuity assumptions as in [SS00, FHL13, HL14a, RS18, MARS20] is also addressed in Remark 6.2. Finally mention that the operator $R = P - \nu(\cdot)1_S$ and its iterates have been considered in [KM03] to investigate the eigenvectors belonging to the dominated eigenvalue of the Laplace kernels associated with the Markov kernel P. This issue called "multiplicative Poisson equation" in [KM03] is used to prove limit theorems for the underlying Markov chain. This question is not addressed in our work.

2 Quasi-compactness of P

For any measurable function $W \geq 1$, if L is a bounded linear operator on $(\mathcal{B}_W, \|\cdot\|_W)$, we also denote by $\|L\|_W := \sup\{\|Lf\|_W, f \in \mathcal{B}_W, \|f\|_W \leq 1\}$ the operator norm of L on $\mathcal{B}_W$. Assume that P satisfies Assumptions ($\boldsymbol{A}$). Note that $\nu(V) < \infty$ due to ($\boldsymbol{S}$). Let $\alpha_0 \in (0, 1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Then P and $T := \nu(\cdot)1_S$ are bounded linear operators on $\mathcal{B}_{V^{\alpha_0}}$. Define

$$R := P - T = P - \nu(\cdot)1_S.$$

We deduce from ($\boldsymbol{S}$) that R is a non-negative operator on $\mathcal{B}_{V^{\alpha_0}}$ (i.e. $\forall f \in \mathcal{B}_{V^{\alpha_0}} : f \geq 0 \Rightarrow Rf \geq 0$). Moreover ($\boldsymbol{D^{\alpha_0}}$) reads as $RV^{\alpha_0} \leq \delta^{\alpha_0} V^{\alpha_0}$ due to the definition of T. Iterating this inequality gives: $\forall k\geq 1,\ R^kV^{\alpha_0}\leq \delta^{\alpha_0 k}\,V^{\alpha_0}$. Then we deduce from the non-negativity of R^k that

$$\forall k\geq 1,\quad \|R^k\|_{V^{\alpha_0}}\leq \delta^{\alpha_0 k} \tag{11}$$

since for every $f\in\mathcal{B}_{V^{\alpha_0}}$ we have $|R^kf|\leq R^k|f|\leq \|f\|_{V^{\alpha_0}}R^kV^{\alpha_0}$. Moreover let us define

$$\forall k\geq 1,\quad \beta_k:=\nu\circ R^{k-1}\in\mathcal{B}'_{V^{\alpha_0}} \tag{12}$$

with the convention $R^0=I$ so that $\beta_1=\nu$. Recall that for every $n\geq 1$ the operator T_n on $\mathcal{B}_{V^{\alpha_0}}$ defined by $T_n:=P^n-R^n$ satisfies (see [HL20, Prop. 2.1])

$$T_n=\sum_{k=1}^n \beta_k(\cdot)P^{n-k}1_S. \tag{13}$$

Hence T_n is finite-rank. This fact and (11) are the key points to prove the next Theorem 2.1 using the notion of essential spectral radius and quasi-compactness. Various equivalent definitions of the essential spectral radius of a bounded linear operator on a Banach space can be found in the literature in link with, either the essential spectrum, or the quasi-compactness property, e.g. see [Hen93, Hen07] and [HH01, Chapter XIV] for a general context and [Wu04, Hen06, AP07, HL14b, HL14a, Del17] in the framework of V-geometrically ergodic Markov kernels. For the link between geometrical ergodicity and spectral theory, see also [MT09, Chap. 20]. The adjoint operator of P acting on $\mathcal{B}'_{V^{\alpha_0}}$ is denoted by P^*.

Theorem 2.1 *Suppose that P satisfies Assumptions ($\boldsymbol{A}$), and let $\alpha_0\in(0,1]$ be given in ($\boldsymbol{D}^{\boldsymbol{\alpha_0}}$). Then, for any $a\in(\delta^{\alpha_0},1)$, the set $\mathcal{S}_a$ of spectral values λ of P on $\mathcal{B}_{V^{\alpha_0}}$ (or of P^* on $\mathcal{B}'_{V^{\alpha_0}}$) such that $a\leq|\lambda|\leq 1$ is finite and composed of eigenvalues of both P and P^*. Moreover the second eigenvalue ϱ_{α_0} of P on $\mathcal{B}_{V^{\alpha_0}}$ (see (2)) is such that:*

(a) Either $\mathcal{S}_a=\{1\}$ and $\varrho_{\alpha_0}\leq a$.

(b) Or $\mathcal{S}_a\neq\{1\}$ and $\varrho_{\alpha_0}=\max\big\{|\lambda|,\,\lambda\in\mathcal{S}_a,\,\lambda\neq 1\big\}$.

The quasi-compactness of P on $\mathcal{B}_V$ is proved in [HL14a, Th. 5.2] under Assumptions ($\boldsymbol{A}$). However the bound obtained in [HL14a, Th. 5.2] for the essential spectral radius of P on $\mathcal{B}_V$ is not accurate. Here Property ($\boldsymbol{D}^{\boldsymbol{\alpha_0}}$) allows us to prove the explicit bound (15) below for the essential spectral radius of P on $\mathcal{B}_{V^{\alpha_0}}$. First we prove the following simple lemma.

Lemma 2.1 *$\lambda=1$ is the only eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ such that $\varrho_{\alpha_0}<|\lambda|\leq 1$.*

Proof. Let $\lambda\in\mathbb{C}\setminus\{1\}$ be any eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$. Prove that $|\lambda|\leq\varrho_{\alpha_0}$. Let $f\in\mathcal{B}_V$, $f\neq 0$, be such that $Pf=\lambda f$. Then $\pi(f)=0$, so that (2) gives $|\lambda|^n=\mathrm{O}(\rho^n)$, thus $|\lambda|\leq\rho$. Hence $|\lambda|\leq\varrho_{\alpha_0}$ since ρ in (2) can be chosen arbitrarily close to ϱ_{α_0}.

□

Proof of Theorem 2.1. Recall that the essential spectral radius $r_{ess}(P)$ of P on $\mathcal{B}_{V^{\alpha_0}}$ is defined by

$$r_{ess}(P):=\lim_n\big(\inf_{H\in\mathcal{K}}\|P^n-H\|_{V^{\alpha_0}}\big)^{1/n} \tag{14}$$

where $\mathcal{K}$ denotes the space of all compact operators on $\mathcal{B}_{V^{\alpha_0}}$. Then we have

$$r_{ess}(P)\leq\delta^{\alpha_0}<1 \tag{15}$$

from (14) and (11) since $P^n - T_n = R^n$ where T_n in (13) is a finite-rank operator so is compact on $\mathcal{B}_{V^{\alpha_0}}$. Hence P is quasi-compact on $\mathcal{B}_{V^{\alpha_0}}$ since the spectral radius of P on $\mathcal{B}_{V^{\alpha_0}}$ is one. It follows from quasi-compactness that the set $\mathcal{S}_a$ is composed of finitely many spectral values which are in fact eigenvalues, e.g. see [Hen93]. The alternative (a)-(b) then follows from the definition of ϱ_{α_0} (see (2)) and from classical arguments of spectral theory. For the sake of completeness, let us present the main arguments. First assume that $\mathcal{S}_a \neq \{1\}$ and define $\gamma_a = \max\{|\lambda|,\, \lambda \in \mathcal{S}_a,\, \lambda \neq 1\}$. From Lemma 2.1 we have $\gamma_a \leq \varrho_{\alpha_0}$. Moreover, it follows from the standard spectral theory that, for any $\gamma \in (\gamma_a, 1)$, we have the following equality

$$\forall n \geq 1, \quad P^n = \pi(\cdot)1_{\mathbb{X}} + \frac{1}{2i\pi}\oint_{|z|=\gamma} z^n (zI - P)^{-1}\, dz, \tag{16}$$

from which we deduce that the value $\rho = \gamma$ is allowed in (2). Thus $\varrho_{\alpha_0} \leq \gamma$, so that $\varrho_{\alpha_0} \leq \gamma_a$ since γ is arbitrarily close to γ_a. We have proved that $\varrho_{\alpha_0} = \gamma_a$ in Case (b). Finally assume that $\mathcal{S}_a = \{1\}$. Then property (16) applies to $\gamma = a$, so that the value $\rho = a$ is allowed in (2). Thus $\varrho_{\alpha_0} \leq a$. □

Remark 2.1 *Note that, under Assumptions ($\boldsymbol{A}$) and with $\alpha_0 \in (0,1]$ given in ($\boldsymbol{D^{\alpha_0}}$), we can directly deduce from (2) that P is quasi-compact on $\mathcal{B}_{V^{\alpha_0}}$ and that its essential spectral radius satisfies $r_{ess}(P) \leq \varrho_{\alpha_0}$. Indeed, for any $\rho \in (0,1)$ such that (2) holds, we have $r_{ess}(P) \leq \rho$ from (14) since $\Pi := \pi(\cdot)1_{\mathbb{X}}$ is rank-one thus compact on $\mathcal{B}_{V^{\alpha_0}}$. Thus $r_{ess}(P) \leq \varrho_{\alpha_0}$ from the definition of ϱ_{α_0}. However, this bound of $r_{ess}(P)$ is not interesting in practice since ϱ_{α_0} is unknown. On the contrary the bound (15) is explicit and, as a by-product, it enables to prove the alternative (a)-(b) of Theorem 2.1 which provides informations on the second eigenvalue ϱ_{α_0}. This result is specified in the next Theorem 2.2*

Recall that $\beta_k \in \mathcal{B}'_{V^{\alpha_0}}$ is defined in (12). It follows from (11) that, for every $z \in \mathbb{C}$ such that $|z| > \delta^{\alpha_0}$, the following series

$$\mu_z := \sum_{k=1}^{+\infty} z^{-k}\, \beta_k \tag{17}$$

is absolutely convergent in $\mathcal{B}'_{V^{\alpha_0}}$, so that $\sum_{k=1}^{+\infty} z^{-k}\, \beta_k(1_S)$ is absolutely convergent in $\mathbb{C}$.

Theorem 2.2 *Assume that P satisfies ($\boldsymbol{A}$), and let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Let $\lambda \in \mathbb{C}$ be such that $\delta^{\alpha_0} < |\lambda| \leq 1$. Then the two following assertions are equivalent:*

(i) λ is an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$.

(ii) $\mu_\lambda(1_S) = \sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k(1_S) = 1$.

Moreover, under Condition (i) or (ii), the subspace $E_\lambda := \{\psi \in \mathcal{B}'_{V^{\alpha_0}} : \psi \circ P = \lambda\,\psi\}$ is spanned by μ_λ.

For the proof of Theorem 2.2 we may assume that $\alpha_0 = 1$ in ($\boldsymbol{D^{\alpha_0}}$), that is

$$PV \leq \delta\, V + \nu(V)1_S. \tag{18}$$

If $\alpha_0 < 1$, then replace V, δ with V^{α_0}, δ^{α_0} respectively in the proof below. First we prove the following lemma.

Lemma 2.2 *For any $z \in \mathbb{C}$ such that $|z| > \delta$ we have*

$$\mu_z \circ P = z\mu_z - \nu + \mu_z \circ T. \tag{19}$$

Proof. Let $z \in \mathbb{C}$ be such that $|z| > \delta$. Then

$$\begin{aligned}
\mu_z \circ P &= \sum_{k=1}^{+\infty} z^{-k}\, \nu \circ R^{k-1} \circ P \qquad \text{(from (17) and (12))} \\
&= \sum_{k=1}^{+\infty} z^{-k}\, \nu \circ R^{k} + \sum_{k=1}^{+\infty} z^{-k}\, \nu \circ \big(R^{k-1} \circ T\big) \qquad \text{(since } P = R + T\text{)} \\
&= \sum_{k=1}^{+\infty} z^{-k}\, \beta_{k+1} + \sum_{k=1}^{+\infty} z^{-k}\, \beta_k \circ T \qquad \text{(from (12))} \\
&= z\mu_z - \nu + \mu_z \circ T \qquad \text{(from (17) and } \beta_1 = \nu\text{)}.
\end{aligned}$$

□

Proof of Theorem 2.2. Let λ be an eigenvalue of P (thus of P^*) such that $\delta < |\lambda| \leq 1$. Using $R^n = P^n - T_n$ and (13) we deduce from (11) (with $\alpha_0 = 1$ here) that

$$\forall n \geq 1,\ \forall f \in \mathcal{B}_V, \qquad \Big\| P^n f - \sum_{k=1}^{n} \beta_k(f) P^{n-k} 1_S \Big\|_V \leq \delta^n \, \|f\|_V. \tag{20}$$

Let $\psi \in E_\lambda$, $\psi \neq 0$. Composing on the left by ψ in (20) gives the following equality in $\mathcal{B}'_V$

$$\lambda^n \psi = \psi(1_S) \sum_{k=1}^{n} \lambda^{n-k}\, \beta_k + O(\delta^n),$$

so that $\psi = \psi(1_S) \sum_{k=1}^{n} \lambda^{-k}\, \beta_k + O((\delta/\lambda)^n)$. Hence $\psi = \psi(1_S)\, \mu_\lambda$. Note that $\psi(1_S) \neq 0$ since $\psi \neq 0$, so that $\mu_\lambda(1_S) = 1$. We have proved the last assertion of Theorem 2.2, as well as the implication $(i) \Rightarrow (ii)$, in Theorem 2.2. Now let us prove that $(ii) \Rightarrow (i)$. Let $\lambda \in \mathbb{C}$ be such that $\delta < |\lambda| < 1$ and assume that $\mu_\lambda(1_S) = 1$, so that $\mu_\lambda \neq 0$. Lemma 2.2 applied to $z = \lambda$ gives

$$\mu_\lambda \circ P = \lambda\, \mu_\lambda - \nu + \mu_\lambda \circ T.$$

Moreover, since $T = \nu(\cdot)1_S$ we obtain that

$$\mu_\lambda \circ T = \sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k \circ T = \Big(\sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k(1_S) \Big) \nu = \mu_\lambda(1_S)\, \nu = \nu$$

from which it follows that $\mu_\lambda \circ P = \lambda\, \mu_\lambda$. Hence λ is an eigenvalue of P^*, thus of P. □

We deduce the following statement from Theorems 2.1-2.2.

Corollary 2.1 *Assume that P satisfies ($\boldsymbol{A}$) and let $\alpha_0 \in (0, 1]$ be given in ($\boldsymbol{D}^{\boldsymbol{\alpha_0}}$). Then the second eigenvalue ϱ_{α_0} of P on $\mathcal{B}_{V^{\alpha_0}}$ (see (2)) satisfies the following alternative.*

- *Either for every $z \in \mathbb{C}$ such that $\delta^{\alpha_0} < |z| < 1$ we have $\mu_\lambda(1_S) \neq 1$, so that $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$.*
- *Or $\varrho_{\alpha_0} = \max\big\{ |\lambda| \,:\, \lambda \in \mathbb{C},\ \delta^{\alpha_0} < |\lambda| < 1,\ \mu_\lambda(1_S) = 1 \big\}$.*

As a complement to Theorem 2.2, we prove in Appendix B that any eigenvalue λ of P on $\mathcal{B}_{V^{\alpha_0}}$ such that $\delta^{\alpha_0} < |\lambda| \leq 1$ is of order one (i.e $\mathrm{Ker}(P - \lambda I)^2 = \mathrm{Ker}(P - \lambda I)$) if, and only if, $\mu'_\lambda(1_S) \neq 0$ where $\mu'_\lambda(1_S)$ is the derivative at $z = \lambda$ of $z \mapsto \mu_z(1_S)$.

3 A bound for the constant C_ρ in (2)

Let P satisfying Assumptions $(\boldsymbol{A})$ and let $\alpha_0 \in (0,1]$ be given in $(\boldsymbol{D^{\alpha_0}})$. Recall that ϱ_{α_0} denotes the infimum bound of the positive real numbers ρ such that the $V^{\alpha_0}-$geometrical ergodicity property (2) holds true. Property (22a) below provides an explicit constant C_ρ in (2) when $\rho \in \big(\max(\delta^{\alpha_0}, \varrho_{\alpha_0}), 1\big)$. Recall that for every $z \in \mathbb{C}$ such that $|z| > \delta^{\alpha_0}$, the series $\mu_z(1_S) = \sum_{k=1}^{+\infty} z^{-k}\, \beta_k(1_S)$ is absolutely convergent (see (17)).

Theorem 3.1 *Let P satisfying Assumptions $(\boldsymbol{A})$. Let $\alpha_0 \in (0,1]$ be given in $(\boldsymbol{D^{\alpha_0}})$. Then, for every $z \in \mathbb{C} \setminus \{1\}$ such that $|z| \in \big(\max(\delta^{\alpha_0}, \varrho_{\alpha_0}), 1\big]$, the operator $zI - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$, and*

$$\|(zI-P)^{-1}\|_{V^{\alpha_0}} \le \frac{1}{|z|-\delta^{\alpha_0}}\left(1 + \frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{\big|1-\mu_z(1_S)\big|\big(|z|-\delta^{\alpha_0}\big)}\right). \tag{21}$$

Moreover, for every $\rho \in \big(\max(\delta^{\alpha_0}, \varrho_{\alpha_0}), 1\big)$, we have

$$\forall n \ge 1, \quad \|P^n - \pi(\cdot)1_{\mathbb{X}}\|_{V^{\alpha_0}} \ \le\ \frac{\rho}{2\pi(\rho-\delta^{\alpha_0})}\left(1 + \frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{m_\rho\big(\rho-\delta^{\alpha_0}\big)}\right)\rho^n \tag{22a}$$

$$\textit{with} \qquad m_\rho := \min_{z\in\mathbb{C}:|z|=\rho} \big|1-\mu_z(1_S)\big| > 0. \tag{22b}$$

The explicit $V^{\alpha_0}-$geometrical ergodicity property (22a) is only interesting, on the one hand when ϱ_{α_0} is known or can be at least bounded from above, and on the other hand when m_ρ can be numerically computed or at least bounded from below by a positive real number.

Again, for the following proofs, we may assume that $\alpha_0 = 1$ in $(\boldsymbol{D^{\alpha_0}})$, that is (18) holds. Moreover ϱ stands for ϱ_1 to simplify. If $\alpha_0 < 1$, then replace V, δ and ϱ with V^{α_0}, δ^{α_0} and ϱ_{α_0} respectively in the proof below. The following lemmas are used to prove Theorem 3.1.

Recall that $R = P - T = P - \nu(\cdot)1_S$ satisfies: $\forall k \ge 0,\ \|R^k\|_V \le \delta^k$ (see (11)).

Lemma 3.1 *Let $z \in \mathbb{C}$ be such that $|z| > \delta$. Then $zI - R$ is invertible on $\mathcal{B}_V$ with*

$$(zI-R)^{-1} = \sum_{k=0}^{+\infty} z^{-(k+1)} R^k. \tag{23}$$

Moreover, with $\mu_z \in \mathcal{B}_V'$ defined in (17), we have

$$\forall f \in \mathcal{B}_V, \quad \nu\big((zI-R)^{-1}f\big) = \mu_z(f). \tag{24}$$

Lemma 3.2 *Let $z \in \mathbb{C} \setminus \{1\}$ be such that $|z| \in \big(\max(\delta, \varrho), 1\big]$. Then $zI - P$ is invertible on $\mathcal{B}_V$, and*

$$\forall f \in \mathcal{B}_V, \quad (zI-P)^{-1}f = (zI-R)^{-1}f + \frac{\mu_z(f)}{1-\mu_z(1_S)}(zI-R)^{-1}1_S. \tag{25}$$

Moreover we have

$$\|(zI-P)^{-1}\|_V \le \frac{1}{|z|-\delta}\left(1 + \frac{\nu(V)\|1_S\|_V}{\big|1-\mu_z(1_S)\big|\big(|z|-\delta\big)}\right). \tag{26}$$

Proof of Theorem 3.1. Property (21) is established in Lemma 3.2. Now let $\rho \in \big(\max(\delta, \varrho), 1\big)$. We deduce from Lemma 2.1 and from Theorem 2.2 that

$$\forall z \in \mathbb{C},\ |z| = \rho, \quad \mu_z(1_S) := \sum_{k=1}^{+\infty} z^{-k}\beta_k(1_S) \neq 1.$$

This gives (22b) since $z \mapsto \mu_z(1_S) := \sum_{k=1}^{+\infty} z^{-k}\beta_k(1_S)$ is continuous on the compact set $\{z \in \mathbb{C} : |z| = \rho\}$. Finally let us prove (22a). It follows from the spectral decomposition (16) applied here to $\gamma = \rho$ that

$$\forall n \geq 1, \quad \|P^n - \pi(\cdot)1_{\mathbb{X}}\|_V \leq \frac{\rho^{n+1}}{2\pi} \max_{z \in \mathbb{C}:|z|=\rho} \big\|(zI - P)^{-1}\big\|_V.$$

Consequently the constant C_ρ in (2) can be chosen as

$$\begin{aligned} C_\rho &:= \frac{\rho}{2\pi} \max_{z\in\mathbb{C}:|z|=\rho} \big\|(zI-P)^{-1}\big\|_V \\ &\leq \frac{\rho}{2\pi} \times \frac{1}{\rho-\delta}\left(1 + \frac{\nu(V)\|1_S\|_V}{m_\rho(\rho-\delta)}\right) \qquad \text{(from (26) and (22b))}. \end{aligned}$$

This provides (22a).

□

Proof of Lemma 3.1. Let $z \in \mathbb{C}$ be such that $|z| > \delta$. Then $zI - R$ is invertible on $\mathcal{B}_V$ since the spectral radius of R is less than δ from $\|R^k\|_V \leq \delta^k$. Then Formula (23) is classical. Moreover note that for every $f \in \mathcal{B}_V$

$$\sum_{k=0}^{+\infty} \int_{\mathbb{X}} |z|^{-(k+1)} |R^k f|\, d\nu \leq |z|^{-1}\nu(V)\, \|f\|_V \sum_{k=0}^{+\infty} \big(|z|^{-1}\delta\big)^k < \infty$$

from $\|R^k\|_V \leq \delta^k$ and from $\delta < |z|$. Therefore the permutation of the integral and the series in the following equality is allowed:

$$\nu((zI-R)^{-1}f) = \nu(\sum_{k=0}^{+\infty} z^{-(k+1)} R^k f) = \sum_{k=0}^{+\infty} z^{-(k+1)}\nu(R^k f).$$

This gives (24) due to (12) and (17).

□

Proof of Lemma 3.2. Let $z \in \mathbb{C}\setminus\{1\}$ such that $|z| \in \big(\max(\delta,\varrho), 1\big]$. If $zI - P$ is not invertible on $\mathcal{B}_V$, then z is an eigenvalue of P from Theorem 2.1, which is impossible from Lemma 2.1. Thus $zI - P$ is invertible on $\mathcal{B}_V$. Next we have

$$zI - P = zI - R - T = U_z \circ (zI - R) \quad \text{with} \quad U_z := I - T \circ (zI - R)^{-1}. \tag{27}$$

We deduce from $T = \nu(\cdot)1_S$ and from (24) that

$$\forall f \in \mathcal{B}_V, \quad U_z f = f - \mu_z(f)1_S \quad \text{or} \quad f = U_z f + \mu_z(f)1_S.$$

Next

$$U_z f = \big(U_z \circ (zI - R)\big) \circ (zI - R)^{-1} f = (zI - P) \circ (zI - R)^{-1} f$$

using (27), so that

$$f = (zI - P) \circ (zI - R)^{-1} f + \mu_z(f) 1_S$$

and

$$(zI - P)^{-1} f = (zI - R)^{-1} f + \mu_z(f)(zI - P)^{-1} 1_S.$$

The last equality applied to $f = 1_S$ gives

$$(zI - P)^{-1} 1_S = \frac{1}{1 - \mu_z(1_S)} (zI - R)^{-1} 1_S$$

where $\mu_z(1_S) \neq 1$ from Corollary 2.1. This provides (25).

Next we have

$$\|(zI - R)^{-1}\|_V \leq \frac{1}{|z| - \delta}, \quad \text{in particular} \quad \|(zI - R)^{-1} 1_S\|_V \leq \frac{\|1_S\|_V}{|z| - \delta}$$

from (23) and $\|R^k\|_V \leq \delta^k$. Moreover we have

$$\forall f \in \mathcal{B}_V, \quad |\mu_z(f)| \leq \frac{\nu(V)}{|z| - \delta} \|f\|_V$$

from (12) and $\|R^k\|_V \leq \delta^k$. Then (26) follows from (25) and the previous inequalities. □

4 A bound for the norm of solutions to Poisson's equation

Recall that the existence of Poisson's equation is studied under weak drift condition in [GM96] (also see [MT93, Th. 17.4.2]). In this section the solutions to Poisson's equation are more easily obtained since we assume that P satisfies Assumptions ($\boldsymbol{A}$) of Section 1 which include the geometric drift condition ($\boldsymbol{D_{S^c}}$). Indeed assume that Assumptions ($\boldsymbol{A}$) holds and let $\alpha_0 \in (0, 1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Then we know that P satisfies Inequality (2), from which we deduce that the operator $(I - P + \Pi)$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$ with

$$(I - P + \Pi)^{-1} = \sum_{n=0}^{+\infty} (P - \Pi)^n = \sum_{n=0}^{+\infty} (P^n - \Pi) \tag{28}$$

where $\Pi := \pi(\cdot) 1_{\mathbb{X}}$. Then, for any $f \in \mathcal{B}_{V^{\alpha_0}}$, it is easily checked that $\widehat{f} := (I - P + \Pi)^{-1} f$ is a solution to Poisson's equation on $\mathcal{B}_{V^{\alpha_0}}$

$$(I - P)\widehat{f} = f - \Pi f. \tag{29}$$

Note that $E_1 := \{h \in \mathcal{B}_{V^{\alpha_0}}, Ph = h\} = \mathbb{C} \cdot 1_{\mathbb{X}}$ from (2) (i.e. 1 is a simple eigenvalue of P) and that the difference of two solutions to Poisson's equation on $\mathcal{B}_{V^{\alpha_0}}$ belongs to E_1. Hence two solutions to Poisson's equation on $\mathcal{B}_{V^{\alpha_0}}$ differ by a constant function. Now, let $f \in \mathcal{B}_{V^{\alpha_0}}$ be such that $\pi(f) = 0$. In Theorem 4.1 below we prove that the function

$$\widetilde{f} := (I - R)^{-1} f = \sum_{n=0}^{+\infty} R^n f$$

is a solution in $\mathcal{B}_{V^{\alpha_0}}$ to Poisson's equation, where R is the non-negative operator of Section 2. Next, since $\pi(f) = 0$, the function

$$\widehat{f} = (I - P + \Pi)^{-1} f = \sum_{n=0}^{+\infty} P^n f$$

satisfies $\pi(\widehat{f}) = \pi(f) = 0$. In fact $\widehat{f}$ is the unique solution in $\mathcal{B}_{V^{\alpha_0}}$ to Poisson's equation which has a null $\pi-$integral. Finally we have $\widehat{f} = \widetilde{f} - \pi(\widetilde{f})1_{\mathbb{X}}$ since $\widetilde{f}$ and $\widehat{f}$ only differ by a constant function.

In Theorem 4.1 below we give a simple and explicit bound for $\|\widetilde{f}\|_{V^{\alpha_0}}$, which is relevant for the perturbation issue of Section 6. This allows us to derive an explicit bound for $\|\widehat{f}\|_{V^{\alpha_0}}$, that will be relevant in Section 5.

Theorem 4.1 *Assume that P satisfies Assumptions ($\boldsymbol{A}$). Let $\alpha_0 \in (0, 1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Then, for any $f \in \mathcal{B}_{V^{\alpha_0}}$ such that $\pi(f) = 0$, the following assertions hold.*

1. *$\widetilde{f} := (I - R)^{-1} f$ is a solution in $\mathcal{B}_{V^{\alpha_0}}$ to the Poisson equation (29), and*

$$\|\widetilde{f}\|_{V^{\alpha_0}} \leq \frac{1}{1 - \delta^{\alpha_0}} \|f\|_{V^{\alpha_0}}. \tag{30}$$

2. *$\widehat{f} := (I - P + \Pi)^{-1} f$ is the unique solution in $\mathcal{B}_{V^{\alpha_0}}$ to Poisson's equation (29) which has a null $\pi-$integral, and*

$$\|\widehat{f}\|_{V^{\alpha_0}} \leq \frac{1 + \pi(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}{1 - \delta^{\alpha_0}} \|f\|_{V^{\alpha_0}} \tag{31a}$$

$$\leq \frac{1 - \delta^{\alpha_0} + \nu(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}{(1 - \delta^{\alpha_0})^2} \|f\|_{V^{\alpha_0}}. \tag{31b}$$

Again, for the proof below, we may assume that $\alpha_0 = 1$ in ($\boldsymbol{D^{\alpha_0}}$). If $\alpha_0 < 1$, then replace V and δ with V^{α_0} and δ^{α_0} respectively in the proof below.

Proof. Recall that $\|R^k\|_V \leq \delta^k$ (see (11)), so that $I - R$ is invertible on $\mathcal{B}_V$ with (see (23))

$$(I - R)^{-1} = \sum_{k=0}^{+\infty} R^k. \tag{32}$$

Next, we have

$$I - P = I - R - T = U \circ (I - R) \quad \text{with} \quad U := I - T \circ (I - R)^{-1}. \tag{33}$$

Let $f \in \mathcal{B}_{V^{\alpha_0}}$ be such that $\pi(f) = 0$ and let $\widetilde{f} := (I - R)^{-1} f$. Then we obtain from (33)

$$(I - P)\widetilde{f} = \big(U \circ (I - R) \circ (I - R)^{-1}\big) f = Uf.$$

From $T = \nu(\cdot)1_S$ and from (24) applied to $z = 1$, we obtain that

$$Uf = f - \mu_1(f)1_S$$

where μ_1 is defined in (17). Moreover we know from Theorem 2.2 that μ_1 is a $P-$invariant positive finite measure, more precisely $\mu_1 = \mu_1(1_{\mathbb{X}})\pi$ (see also [HL20, HL22]). Hence we have $\mu_1(f) = 0$ since $\pi(f) = 0$, so that $Uf = f$. Thus $\widetilde{f}$ is a solution to the Poisson equation on $\mathcal{B}_V$. Moreover, it follows from (32) and $\|R^k\|_V \leq \delta^k$ that

$$\|\widetilde{f}\|_V = \|(I-R)^{-1}f\|_V = \Big\|\sum_{k=0}^{+\infty} R^k f\Big\|_V \leq \frac{1}{1-\delta}\,\|f\|_V. \tag{34}$$

The proof of the first assertion is complete.

Now let $f \in \mathcal{B}_{V^{\alpha_0}}$ be such that $\pi(f) = 0$ and let $\widehat{f} := (I-P+\Pi)^{-1}f$. Since $\widehat{f}$ satisfies Poisson's equation on $\mathcal{B}_{V^{\alpha_0}}$, we know that $\widetilde{f}$ and $\widehat{f}$ differ by a constant function, so that $\widehat{f} = \widetilde{f} - \pi(\widetilde{f})1_{\mathbb{X}}$ due to $\pi(\widehat{f}) = \pi(f) = 0$. Hence

$$\begin{aligned}
\|\widehat{f}\|_V &\leq \|\widetilde{f}\|_V + |\pi(\widetilde{f})| \times \|1_{\mathbb{X}}\|_V \qquad \text{(triangular inequality)} \\
&\leq \big(1+\pi(V)\|1_{\mathbb{X}}\|_V\big)\|\widetilde{f}\|_V \qquad (\text{since } |\widetilde{f}| \leq \|\widetilde{f}\|_V\, V) \\
&\leq \frac{1+\pi(V)\|1_{\mathbb{X}}\|_V}{1-\delta}\,\|f\|_V \qquad \text{(from (34))}.
\end{aligned}$$

This gives (31a). Finally (31b) follows from the inequality $\pi(V) \leq \nu(V)/(1-\delta)$ which can be easily derived from (18). The second assertion is proved. □

5 Bounds for the second eigenvalue of P

Using the results of Sections 2-3-4, we first present some results on the location of the eigenvalues of P on $B_{V^{\alpha_0}}$. For any $a \in \mathbb{C}$ and for any $r > 0$, define

$$D(a,r) := \{\lambda \in \mathbb{C} : |\lambda - a| < r\}, \;\; C(a,r) := \{\lambda \in \mathbb{C} : |\lambda - a| = r\}, \;\; \overline{D}(a,r) = D(a,r) \cup C(a,r).$$

Proposition 5.1 *Suppose that P satisfies Assumptions (**A**). Let $\alpha_0 \in (0,1]$ be given in (**D^{α_0}**), and define*

$$\widehat{r}_1 := \frac{1-\delta^{\alpha_0}}{1+\pi(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}. \tag{35}$$

Then $\lambda = 1$ is the single spectral value of P on $\mathcal{B}_{V^{\alpha_0}}$ in the open disk $D(1,\widehat{r}_1)$, that is: for every $\lambda \in D(1,\widehat{r}_1) \setminus \{1\}$, the operator $\lambda I - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$.

The real number $\widehat{r}_1$ in (35) satisfies

$$\widehat{r}_1 \geq \widetilde{r}_1 := \frac{(1-\delta^{\alpha_0})^2}{1-\delta^{\alpha_0}+\nu(V^{\alpha_0})}$$

since $\pi(V^{\alpha_0}) \leq \nu(V^{\alpha_0})/(1-\delta^{\alpha_0})$. Therefore $\widehat{r}_1$ may be replaced with $\widetilde{r}_1$ in the conclusion of Proposition 5.1 when π is unknown.

Proof. Note that $\widehat{r}_1 < 1-\delta^{\alpha_0}$. Therefore, if $\lambda \in D(1,\widehat{r}_1)$, then $|\lambda| > \delta^{\alpha_0}$. Thus it follows from (15) that any spectral value of P on $\mathcal{B}_{V^{\alpha_0}}$ in $D(1,\widehat{r}_1)$ is actually an eigenvalue. Consequently we have to prove that $\lambda = 1$ is the single eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ in $D(1, \widehat{r}_1)$. Let $\lambda \in \mathbb{C}\setminus\{1\}$, be an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$, and let $f_\lambda \in \mathcal{B}_{V^{\alpha_0}}$ be such that $f_\lambda \neq 0$ and $Pf_\lambda = \lambda f_\lambda$. Then

$$(1-\lambda)f_\lambda = (I-P)f_\lambda. \tag{36}$$

Since $\lambda \neq 1$, we have $\pi(f_\lambda) = 0$. It follows that

$$(I-P+\Pi)^{-1} \circ (I-P)f_\lambda = (I-P+\Pi)^{-1} \circ (I-P+\Pi)f_\lambda = f_\lambda.$$

Then we obtain by composing to the left of (36) by $(I-P+\Pi)^{-1}$ that

$$(1-\lambda)\widehat{f_\lambda} = f_\lambda \quad \text{where} \quad \widehat{f_\lambda} := (I-P+\Pi)^{-1} f_\lambda$$

so that $\widehat{f_\lambda} = (1-\lambda)^{-1} f_\lambda$. It follows from (31a) applied to $f = f_\lambda$ that $|1-\lambda|^{-1} \leq \widehat{r}_1^{\,-1}$, thus $|1-\lambda| \geq \widehat{r}_1$. This proves the expected statement. □

Proposition 5.2 *Assume that P satisfies Assumptions ($\boldsymbol{A}$). Let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Let $z \in \mathbb{C}$ be such that $|z| = 1$, $z \neq 1$, and define*

$$r_z := \big(1-\delta^{\alpha_0}\big)\left(1 + \frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{\big|1-\mu_z(1_S)\big|\big(1-\delta^{\alpha_0}\big)}\right)^{-1}. \tag{37}$$

Then there is no spectral value of P on $\mathcal{B}_{V^{\alpha_0}}$ in the open disk $D(z, r_z)$, that is: $\forall \lambda \in D(z, r_z)$, the operator $\lambda I - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$.

Proof. Let $z \in \mathbb{C}$ be such that $|z| = 1$, $z \neq 1$. Then $zI - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$ from Theorem 3.1, so that $\mu_z(1_S) \neq 1$ due to Theorem 2.2. Since $r_z < 1 - \delta^{\alpha_0}$ we have to prove that there is no eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ in $D(z, r_z)$ (as in the proof of Proposition 5.1). Let $\lambda \in \mathbb{C}$ be an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ (thus $\lambda \neq z$), and let $f_\lambda \in \mathcal{B}_{V^{\alpha_0}}$ be such that $Pf_\lambda = \lambda f_\lambda$ and $\|f_\lambda\|_{V^{\alpha_0}} = 1$. We have $(zI-P)f_\lambda = (z-\lambda)f_\lambda$, so that $(zI-P)^{-1}f_\lambda = (z-\lambda)^{-1}f_\lambda$. Using $|z| = 1$, it follows from (21) applied to f_λ that $|z-\lambda|^{-1} \leq r_z^{-1}$, thus $|z-\lambda| \geq r_z$. This proves the desired statement.

□

Under Assumptions ($\boldsymbol{A}$) let $z_0 = e^{i\vartheta_0} \in \mathbb{C}$, $\vartheta_0 \in (0, \pi/2)$, be defined by

$$C(0,1) \cap C(1, \widehat{r}_1) = \{e^{i\vartheta_0}, e^{-i\vartheta_0}\} \tag{38}$$

with $\widehat{r}_1$ defined in (35), and let Γ_0 be the following closed subset of $C(0,1)$:

$$\Gamma_0 := \big\{z \in \mathbb{C} : z = e^{i\vartheta},\ \vartheta \in [\vartheta_0, 2\pi - \vartheta_0]\big\}.$$

Note that

$$m_0 := \min_{z \in \Gamma_0} \big|1 - \mu_z(1_S)\big| > 0.$$

Indeed let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Then for every $z \in \Gamma_0$ we know that $zI - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$ (Theorem 3.1), so that $\mu_z(1_S) \neq 1$ (Theorem 2.2). Then the positivity of m_0 follows from the continuity of the function $z \mapsto \mu_z(1_S) := \sum_{k=1}^{+\infty} z^{-k}\beta_k(1_S)$ on the compact set Γ_0. Finally let

$$\widehat{r}_0 := \big(1-\delta^{\alpha_0}\big)\left(1 + \frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{m_0\big(1-\delta^{\alpha_0}\big)}\right)^{-1}. \tag{39}$$

Note that $\widehat{r}_0 \leq r_{z_0}$ from the definition of m_0 and that $r_{z_0} \leq \widehat{r}_1$ since $|z_0 - 1| = \widehat{r}_1$ and the eigenvalue 1 cannot belong to $D(z_0, r_{z_0})$ from Proposition 5.2. Consequently $\widehat{r}_0 \leq \widehat{r}_1$, and we can define ξ_0 as the unique complex number such that

$$|\xi_0| < 1 \quad \text{and} \quad \xi_0 \in C(1, \widehat{r}_1) \cap C(z_0, \widehat{r}_0). \tag{40}$$

Corollary 5.1 *Assume that P satisfies Assumptions ($\boldsymbol{A}$). Let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$) and let ξ_0 be defined in (40). Then the second eigenvalue ϱ_{α_0} of P on $\mathcal{B}_{V^{\alpha_0}}$ (see (2)) is such that $\varrho_{\alpha_0} \leq |\xi_0|$.*

Proof. From the definition of m_0 and from Proposition 5.2 we deduce that, for every $z \in \Gamma_0$, there is no spectral value of P on $\mathcal{B}_{V^{\alpha_0}}$ in the open disk $D(z, \widehat{r}_0)$, that is: $\forall z \in \Gamma_0$, $\forall \lambda \in D(z, \widehat{r}_0)$, the operator $\lambda I - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$. Then Corollary 5.1 follows from Propositions 5.1 and from the spectral properties of Section 2. □

Note that the series $\sum_{k=1}^{+\infty} \beta_k(1_S)$ is convergent (see (17)) and that $\mu_1(1_S) = \sum_{k=1}^{+\infty} \beta_k(1_S) = 1$ since 1 is an eigenvalue of P (Theorem 2.2). Thus

$$1 - \mu_{-1}(1_S) = \sum_{k=1}^{+\infty} \left(1 - (-1)^k\right) \beta_k(1_S) = 2 \sum_{k=1}^{+\infty} \beta_{2k-1}(1_S) \tag{41}$$

and $1 - \mu_{-1}(1_S) \in [2\nu(1_S), 2]$ since we have $\beta_1(1_S) = \nu(1_S)$ and $\sum_{k=1}^{+\infty} \beta_k(1_S) = 1$. Recall that P is said to be reversible with respect to π if $\pi(dx)P(x, dy) = \pi(dy)P(y, dx)$. Under Assumptions ($\boldsymbol{A}$) and with $\alpha_0 \in (0,1]$ given in ($\boldsymbol{D^{\alpha_0}}$), define

$$\forall t > 0, \quad \psi(t) := \frac{\delta^{\alpha_0}(1 - \delta^{\alpha_0})t + \nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{(1 - \delta^{\alpha_0})t + \nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}, \tag{42}$$

$$\forall n \geq 1, \quad \eta_n := 2 \sum_{k=1}^{n} \beta_{2k-1}(1_S) \quad \text{and} \quad \eta_\infty := 2 \sum_{k=1}^{+\infty} \beta_{2k-1}(1_S) = 1 - \mu_{-1}(1_S). \tag{43}$$

Note that for any $n \geq 1$, $\eta_\infty \geq \eta_n \geq 2\beta_1(1_S) = 2\nu(1_S) > 0$.

Corollary 5.2 *Assume that P satisfies Assumptions ($\boldsymbol{A}$), and let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Moreover assume that P is reversible with respect to π, that $\pi(V^{2\alpha_0}) < \infty$, and that the following implication holds for every $\lambda \in \mathbb{C}$, $|\lambda| > \delta^{\alpha_0}$ and for every $f \in \mathcal{B}_{V^{\alpha_0}}$:*

$$Pf = \lambda f, \; f \neq 0 \implies \pi(|f|) \neq 0. \tag{44}$$

Then

$$\forall n \geq 1, \quad \varrho_{\alpha_0} \leq \psi(\eta_\infty) \leq \psi(\eta_n). \tag{45}$$

Recall that the bounds of ϱ_{α_0} in (45) can be used in the $\mathbb{L}^2(\pi)-$geometrical ergodicity (see (3)). Also recall that $\lim_n \eta_n = \eta_\infty$ and $\eta_\infty \in [2\nu(1_S), 2]$. The second bound in (45) applied to $n = 1$ gives $\varrho_{\alpha_0} \leq \psi(2\nu(1_S))$, but this bound is not accurate in general because $\nu(1_S)$ is small, so that the bound $\psi(\nu(1_S))$ is close to 1.

Proof. From reversibility we know that P is a self-adjoint bounded linear operator on $\mathbb{L}^2(\pi)$, and that the spectral values of P on $\mathbb{L}^2(\pi)$ are contained in $[-1, 1]$, e.g. see [RR97, Bax05].

Moreover note that every $f \in \mathcal{B}_{V^{\alpha_0}}$ is such that $\pi(|f|^2) < \infty$ from $\pi(V^{2\alpha_0}) < \infty$. Now let $\lambda \in \mathbb{C}$, $|\lambda| > \delta^{\alpha_0}$, be an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$, and let $f \in \mathcal{B}_{V^{\alpha_0}}$ be such that $Pf = \lambda f$ and $f \neq 0$. Then $\pi(|f|) \neq 0$ from (44), so that λ is an eigenvalue of P on $\mathbb{L}^2(\pi)$. Therefore every eigenvalue $\lambda \in \mathbb{C}$ of P on $\mathcal{B}_{V^{\alpha_0}}$ such that $|\lambda| > \delta^{\alpha_0}$ actually belongs to $(-1, -\delta^{\alpha_0}) \cup (\delta^{\alpha_0}, 1]$. Next, note that r_{-1} defined in (37) (with $z = -1$) satisfies $r_{-1} < 1 - \delta^{\alpha_0}$, thus $\delta^{\alpha_0} < 1 - r_{-1}$. Also observe that $1 - r_{-1} = \psi(\eta_\infty)$ from an easy computation, (41) and $1 - \mu_{-1}(1_S) > 0$. Thus, using Theorem 2.1, the first inequality in (45) holds if we establish that there is no eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ in $I_1 := (1 - r_{-1}, 1)$ and in $I_{-1} := [-1, -1 + r_{-1})$. This is true for I_1 since we know from Theorem 2.2 that $\lambda = 1$ is the unique solution to Equation $\sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k(1_S) = 1$ in the interval $(\delta^{\alpha_0}, 1]$. Moreover this is true for I_{-1} since we know from Proposition 5.2 applied to $z = -1$ that there is no spectral value of P on $\mathcal{B}_{V^{\alpha_0}}$ in the open disk $D(-1, r_{-1})$. Thus $\varrho_{\alpha_0} \leq |-1 + r_{-1}| = 1 - r_{-1}$. Finally easy computations show that ψ in (42) is decreasing on $(0, +\infty)$. This proves the second inequality in (45) since $0 < \eta_n \leq \eta_\infty$.

□

Remark 5.1 *Assumption (44) is used in the previous proof to ensure that every eigenvalue λ of P on $\mathcal{B}_{V^{\alpha_0}}$ such that $|\lambda| > \delta^{\alpha_0}$ is also an eigenvalue of P on $\mathbb{L}^2(\pi)$. If P is of the form $P(x, dy) = p(x, y) d\mu(y)$ where μ is a positive measure on $(\mathbb{X}, \mathcal{X})$ and if P admits an invariant probability measure $\pi(dx)$, then $\pi(dx)$ is absolutely continuous with respect to μ (i.e. $\pi(dx) = \pi(x)\, \mu(dx)$). If moreover the density function π is positive on $\mathbb{X}$, then Condition (44) holds. Indeed, if $f \in \mathcal{B}_{V^{\alpha_0}}$ is such that $\pi(|f|) = 0$, then $f = 0$ μ−a.s., so that $\forall x \in \mathbb{X}$, $(Pf)(x) = 0$. This proves (44). More generally note that Condition (44) is fulfilled when for every $f \in \mathcal{B}_{V^{\alpha_0}}$ we have:*

$$f = 0 \ \ \pi\text{-almost surely} \implies \forall x \in \mathbb{X},\ \exists n = n_x \geq 1,\ (P^n f)(x) = 0. \tag{46}$$

Remark 5.2 *In the atomic case, that is when **(S)** holds with $S \in \mathcal{X}$ such that $\forall (a, a') \in S^2$, $P(a, \cdot) = P(a', \cdot)$ and with $\nu(\cdot) := P(s_0, \cdot)$ for some (any) $s_0 \in S$, then*

$$\forall n \geq 1, \quad \beta_n(1_S) = \mathbb{P}_{s_0}\big(R_S = n\big) \tag{47}$$

where $R_S := \inf\{n \geq 1 : X_n \in S\}$ is the first return time in S. Then Equality $\sum_{k=1}^{+\infty} \beta_k(1_S) = 1$ reads as $\mathbb{P}_{s_0}(R_S < \infty) = 1$, and $\eta_\infty = 2\mathbb{P}_{s_0}\big(R_S \in 2\mathbb{N} + 1\big)$. Moreover Conditions ($\boldsymbol{D^{\alpha_0}}$) holds with $\alpha_0 = 1$ (see Appendix A). Consequently, when the assumptions of Corollary 5.2 hold with an atom S, then we have the following upper bound for the second eigenvalue ϱ_1 of P on $\mathcal{B}_V$:

$$\varrho_1 \leq \psi(2p_1) = \frac{2\delta(1-\delta)p_1 + \nu(V)\|1_S\|_V}{2(1-\delta)p_1 + \nu(V)\|1_S\|_V} \quad \textit{with} \quad p_1 := \mathbb{P}_{s_0}\big(R_S \in 2\mathbb{N} + 1\big).$$

Recall that a reversible Markov kernel P with respect to π is said to be positive if the following condition holds

$$\forall f \in \mathbb{L}^2(\pi), \quad \langle Pf, f \rangle_{\mathbb{L}^2(\pi)} = \int_{\mathbb{X}} (Pf)(x) f(x) \pi(dx) \geq 0. \tag{48}$$

Under this condition, every eigenvalue λ of P on $\mathbb{L}^2(\pi)$ is non-negative from (48). Consequently, if P satisfies the assumptions of Corollary 5.2 and if P is positive, then every eigenvalue λ of P on $\mathcal{B}_{V^{\alpha_0}}$ such that $|\lambda| > \delta^{\alpha_0}$ is actually positive. However, as already mentioned in the proof of Corollary 5.2, $\lambda \in (\delta^{\alpha_0}, 1)$ is not an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ since $\sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k(1_S) > 1$ (Theorem 2.2). Thus the following statement holds.

Corollary 5.3 *Assume that P is a positive reversible Markov kernel with respect to π satisfying Assumptions ($\boldsymbol{A}$) and (44). Let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$), and assume that $\pi(V^{2\alpha_0}) < \infty$. Then we have $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$.*

Recall that $\alpha_0 = 1$ in the atomic case. If S is an atom, then the conclusion of Corollary 5.3 has been proved in [Bax05, Th. 1.3] (Condition (44) is not assumed in [Bax05, Th. 1.3]). Therefore, the previous corollary extends this result to the non-atomic case, provided that Condition (44) is assumed and that the space $\mathcal{B}_V$ is replaced with $\mathcal{B}_{V^{\alpha_0}}$. The bound $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$ can be used in (3) too.

If P is reversible with respect to π and if $\ell \geq 2$ is any even integer, then the $\ell-$th iterate P^ℓ of P is a positive reversible Markov kernel with respect to π. Moreover, if $\varrho(P^\ell)$ is the second eigenvalue of P^ℓ on $\mathcal{B}_W$ for some $W \geq 1$, then $\varrho(P^\ell)^{1/\ell}$ is the second eigenvalue of P on $\mathcal{B}_W$. Indeed, writing $n = k\ell + r$ (euclidean division) and defining $\Pi f = \pi(f)1_{\mathbb{X}}$, we have

$$P^n - \Pi = P^{k\ell+r} - \Pi = (P - \Pi)^r \left((P^\ell)^k - \Pi\right)$$

from which we easily deduce the desired result. Then the following statement follows from Corollary 5.3 applied to P^ℓ.

Corollary 5.4 *Assume that P is reversible with respect to π. Moreover assume that, for some even integer $\ell \geq 2$, the Markov kernel P^ℓ satisfies Assumptions ($\boldsymbol{A}$), so that P^ℓ satisfies Conditions ($\boldsymbol{D_{S^c}}$) and ($\boldsymbol{D^{\alpha_0}}$) with some $\delta(P^\ell) \in (0,1)$ and some $\alpha_0(P^\ell) \in (0,1]$. Finally suppose that P^ℓ satisfies (44) and that $\pi(V^{2\alpha_0(P^\ell)}) < \infty$. Then we have $\varrho_{\alpha_0} \leq \delta^{\alpha_0(P^\ell)/\ell}$.*

Remark 5.3 *For the standard (non-atomic) examples of positive reversible Markov kernels provided by the Metropolis-Hastings algorithm for simulating the Gaussian distribution and by the Gaussian autoregressive Markov chain, the numerical values of α_0 are reported in [HL22, Sec. 6.3-6.4]. These numerical findings show that the bound $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$ proved in Corollary 5.3 is of the same order (even sometimes better) as the bounds of the V-geometrical rate of convergence obtained in [Bax05, Sec. 8.2-8.3], see [HL22, Tables 1-2] for details. Consequently the rate of convergence in the $\mathbb{L}^2(\pi)-$geometrical ergodicity (3) deduced from the bound $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$ is of the same order as those addressed in [Bax05, Sec. 8.2-8.3].*

Now, let us give a simple example for which the exact value of the second eigenvalue of P is known and compared with the bound provided by Corollary 5.4 (applied here with $\ell = 2$). Let $P = (P(i,j))_{(i,j)\in\mathbb{N}^2}$ be the reversible Markov kernel defined on $\mathbb{X} = \mathbb{N}$ by

$$P(0,0) = 0.1,\ P(0,1) = 0.9 \quad \text{and} \quad \forall n \geq 1,\ P(n,n-1) := 0.6,\ P(n,n+1) := 0.4. \tag{49}$$

Define $\forall n \in \mathbb{N},\ V(n) = (0.6/0.4)^{n/2}$. We know that the second eigenvalue ϱ_1 of P on $\mathcal{B}_V$ is given by $\varrho_1 = 0.98$, see [Bax05, Sec. 8.4], [HL14b, Prop. 4.1] where this exact value is obtained from algebraic computations. Note that P is not positive but P^2 is. Moreover P^2 satisfies the assumptions of Corollary 5.4. Indeed, P^2 satisfies ($\boldsymbol{S}$) with $S = \{0,1\}$, $\nu = \nu(0)\delta_0 + \nu(1)\delta_1$ with

$$\nu(0) = \min\left(P^2(0,0), P^2(1,0)\right) = 0.06, \quad \nu(1) = \min\left(P^2(0,1), P(1,1)\right) = 0.09,$$

and P^2 satisfies ($\boldsymbol{D_{S^c}}$) with V as above defined and with $\delta(P^2) = 4 \times 0.6 \times 0.4 = 0.96$. Finally the real number $\alpha_0(P^2) \in (0,1]$ has to be chosen such that P^2 satisfies ($\boldsymbol{D^{\alpha_0}}$), that is

$$\forall i \in \{0,1\},\ (P^2 V^{\alpha_0(P^2)})(i) = 0.96^{\alpha_0(P^2)}\,(0.6/0.4)^{i\,\alpha_0(P^2)/2} + 0.06 + 0.09 \times (0.6/0.4)^{\alpha_0(P^2)/2}.$$

We find $\alpha_0(P^2) = 0.71$*. Consequently we deduce from Corollary 5.4 that*

$$\varrho_{\alpha_0} \leq \sqrt{0.96^{\alpha_{0,2}}} \leq \sqrt{0.9714} = 0.9856.$$

This bound is not very far from the exact value 0.98.

We conclude this section with an application to Markov chain Monte Carlo (MCMC) algorithms. Let π (the target density) be a positive distribution density function on $\mathbb{X} = \mathbb{R}^d$, and let $q(x, \cdot)$ be a proposal density on $\mathbb{X} = \mathbb{R}^d$ for any $x \in \mathbb{R}^d$. Define

$$\forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad p(x, y) := \begin{cases} \min\left(1\,,\ \frac{\pi(y)\,q(y,x)}{\pi(x)\,q(x,y)}\right) & \text{if} \quad \pi(x)\,q(x,y) > 0 \\ 1 & \text{if} \quad \pi(x)\,q(x,y) = 0. \end{cases}$$

Recall that the associated Metropolis-Hastings kernel is defined by

$$P(x, dy) := r(x)\,\delta_x(dy) + p(x,y)q(x,y)\,dy \quad \text{ with } \ r(x) := 1 - \int_{\mathbb{R}^d} p(x,z)q(x,z)\,dz, \tag{50}$$

where $\delta_x(dy)$ denotes the Dirac distribution at x, and that P is reversible with respect to π. For any $x_0 \in \mathbb{R}^d$, let f_{x_0} be the Dirac function at x_0, that is: $f_{x_0}(x_0) = 1$ and $\forall x \neq x_0,\ f_{x_0}(x) = 0$. Then $Pf_{x_0} = r(x_0)f_{x_0}$, thus $r(x_0)$ is an eigenvalue of P on the space $\mathcal{B}_W$ for any function $W \geq 1$. Therefore a necessary condition for P to be $W-$geometrically ergodic is that $r_\infty := \sup_{x\in\mathbb{R}^d} r(x) < 1$.

Corollary 5.5 *Assume that the target density function* π *is positive on* $\mathbb{R}^d$ *and that* $r_\infty < 1$*. Moreover assume that the Metropolis-Hastings Markov kernel* P *defined in (50) satisfies Assumptions* $(\boldsymbol{A})$*, and let* $\alpha_0 \in (0, 1]$ *be given in* $(\boldsymbol{D}^{\boldsymbol{\alpha_0}})$*. Finally suppose that* $\delta^{\alpha_0} \geq r_\infty$ *and that* $\pi(V^{2\alpha_0}) < \infty$*. Then the second eigenvalue* ϱ_{α_0} *of* P *on* $\mathcal{B}_{V^{\alpha_0}}$ *satisfies (45).*

Proof. Corollary 5.5 follows from Corollary 5.2, provided that P in (50) satisfies Condition (44). Let $\lambda \in \mathbb{C}$, $|\lambda| > \delta^{\alpha_0}$, and let $f \in \mathcal{B}_{V^{\alpha_0}}$, $f \neq 0$, be such that $Pf = \lambda f$. We must prove that $\pi(|f|) \neq 0$. Suppose that $\pi(|f|) = 0$. Since $\pi > 0$, we have $f(y) = 0$ for almost every $y \in \mathbb{R}^d$ with respect to Lebesgue's measure on $\mathbb{R}^d$. Then

$$\forall x \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} f(y)\,p(x,y)q(x,y)\,dy = 0,$$

hence: $\forall x \in \mathbb{R}^d,\ \lambda f(x) = (Pf)(x) = r(x)\,f(x)$. Since $f \neq 0$, there exists $x_0 \in \mathbb{R}^d$ such that $f(x_0) \neq 0$, so that $r(x_0) = \lambda$. But this is impossible since $\forall x \in \mathbb{R}^d,\ r(x) \leq \delta^{\alpha_0} < |\lambda|$. □

Remark 5.4 *Note that, under the conditions* $(\boldsymbol{D_{S^c}})$ *and* $r_\infty < 1$*, the real number* δ *in* $(\boldsymbol{D_{S^c}})$ *may always be chosen such that* $\delta^{\alpha_0} \geq r_\infty$*. For instance choose* δ *in* $(\boldsymbol{D_{S^c}})$ *such that* $r_\infty \leq \delta$*, and compute the associated real number* α_0 *of* $(\boldsymbol{D}^{\boldsymbol{\alpha_0}})$*: then we have* $r_\infty \leq \delta \leq \delta^{\alpha_0}$*. Moreover observe that, if* $(\boldsymbol{D}^{\boldsymbol{\alpha_0}})$ *holds for some* $\delta \in (0, 1)$ *and* $\alpha_0 \in (0, 1]$*, then we automatically have* $\forall x \in S^c,\ r(x) \leq \delta^{\alpha_0}$ *since*

$$\forall x \in S^c, \quad r(x)V(x)^{\alpha_0} \leq (PV^{\alpha_0})(x) \leq \delta^{\alpha_0}V(x)^{\alpha_0}$$

from the definition of P *in (50) and from* $(\boldsymbol{D}^{\boldsymbol{\alpha_0}})$*. Hence, under Assumption* $(\boldsymbol{D}^{\boldsymbol{\alpha_0}})$*, the condition* $\delta^{\alpha_0} \geq r_\infty$ *is in fact equivalent to:* $\forall x \in S,\ \delta^{\alpha_0} \geq r(x)$*. In practice* δ *and* α_0 *of* $(\boldsymbol{D}^{\boldsymbol{\alpha_0}})$ *have to be chosen so that* $\delta^{\alpha_0} \geq r_\infty$ *and that the first or the second bound used in (45) is minimal.*

6 Applications to perturbed Markov kernels

Let $\{P_\theta\}_{\theta\in\Theta}$ be a family of transition kernels on $(\mathbb{X},\mathcal{X})$, where Θ is an open subset of some metric space. We assume that the family $\{P_\theta\}_{\theta\in\Theta}$ satisfies the following conditions: there exist $S\in\mathcal{X}$ and $\nu\in\mathcal{M}^+$ such that

$$\nu(1_S)>0 \quad\text{and}\quad \forall\theta\in\Theta,\ \forall x\in\mathbb{X},\ \forall A\in\mathcal{X},\quad P_\theta(x,A)\geq \nu(1_A)\,1_S(x) \tag{$\boldsymbol{S_\Theta}$}$$

and there exists a Lyapunov function $V:\mathbb{X}\to[1,+\infty)$ such that

$$\exists\delta\in(0,1),\ \forall\theta\in\Theta,\ \forall x\in S^c,\quad (P_\theta V)(x)\leq \delta\, V(x) \tag{$\boldsymbol{D_{\Theta,S^c}}$}$$

$$K:=\sup_{\theta\in\Theta}\sup_{x\in S}(P_\theta V)(x)<\infty. \tag{$\boldsymbol{K_\Theta}$}$$

This means that the whole family $\{P_\theta\}_{\theta\in\Theta}$ has is a small-set S with the same positive measure ν and satisfies the geometric drift conditions ($\boldsymbol{D_{S^c}}$)-($\boldsymbol{K}$) in a uniform way in $\theta\in\Theta$. Throughout this section, Assumptions ($\boldsymbol{A_\Theta}$) will stand for the set of Assumptions ($\boldsymbol{S_\Theta}$)-($\boldsymbol{D_{\Theta,S^c}}$)-($\boldsymbol{K_\Theta}$). Then for every $\theta\in\Theta$ there exists a unique $P_\theta-$invariant probability measure π_θ on $(\mathbb{X},\mathcal{X})$ such that $\pi_\theta(V)<\infty$. Moreover, under Assumptions ($\boldsymbol{A_\Theta}$), there exists $\alpha_0\in(0,1]$ such that

$$\forall\theta\in\Theta,\quad P_\theta V^{\alpha_0}\leq \delta^{\alpha_0}\,V^{\alpha_0}+\nu(V^{\alpha_0})1_S. \tag{$\boldsymbol{D^{\alpha_0}_\Theta}$}$$

In fact Property ($\boldsymbol{D^{\alpha_0}_\Theta}$) can be proved as for ($\boldsymbol{D^{\alpha_0}}$) (see Appendix A) since the data of Assumptions ($\boldsymbol{A_\Theta}$) are the same for every $\theta\in\Theta$. Now, let $\theta_0\in\Theta$ be fixed, and define

$$\forall\theta\in\Theta,\quad \Delta_{\theta,\alpha_0}(x):=\|P_\theta(x,\cdot)-P_{\theta_0}(x,\cdot)\|'_{V^{\alpha_0}}, \tag{51}$$

that is: $\Delta_{\theta,\alpha_0}(x)$ is the V^{α_0}-weighted total variation norm of $P_\theta(x,\cdot)-P_{\theta_0}(x,\cdot)$. Next let us introduce the following condition:

$$\forall x\in\mathbb{X},\quad \lim_{\theta\to\theta_0}\Delta_{\theta,\alpha_0}(x)=0. \tag{$\boldsymbol{\Delta^{\alpha_0}_\Theta}$}$$

The stationary distribution π_{θ_0} of P_{θ_0} is supposed to be unknown and not directly computable, and P_θ for $\theta\neq\theta_0$ must be thought of as a perturbed Markov kernel of P_{θ_0}. Then, if the stationary distribution π_θ of P_θ is computable for $\theta\neq\theta_0$, Theorem 6.1 below provides an explicit control for the V^{α_0}-weighted total variation norm $\|\pi_\theta-\pi_{\theta_0}\|'_{V^{\alpha_0}}$, provided that the function Δ_{θ,α_0} in (51) is computable, so that the real number $\pi_\theta(\Delta_{\theta,\alpha_0})$ in (52a)-(52b) below is available.

Provided that V is replaced with V^{α_0}, Inequalities (52a)-(52b) below extend to the non-atomic case the statement [LL18, Th. 2] obtained in the context of truncation of discrete Markov chains with an atom (see Remark 6.1).

Theorem 6.1 *Suppose that the family $\{P_\theta\}_{\theta\in\Theta}$ satisfies Assumptions ($\boldsymbol{A_\Theta}$). Let $\alpha_0\in(0,1]$ be given in ($\boldsymbol{D^{\alpha_0}_\Theta}$), let $\theta_0\in\Theta$ and assume that Condition ($\boldsymbol{\Delta^{\alpha_0}_\Theta}$) holds. Then*

$$\lim_{\theta\to\theta_0}\|\pi_\theta-\pi_{\theta_0}\|'_{V^{\alpha_0}}=0 \quad\text{and}\quad \lim_{\theta\to\theta_0}\pi_\theta\big(\Delta_{\theta,\alpha_0}\big)=0.$$

Moreover we have for every $\theta\in\Theta$

$$\big\|\pi_\theta-\pi_{\theta_0}\big\|'_{TV}\ \leq\ \big\|\pi_\theta-\pi_{\theta_0}\big\|'_{V^{\alpha_0}}\ \leq\ \frac{1+\pi_{\theta_0}(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}{1-\delta^{\alpha_0}}\times\pi_\theta\big(\Delta_{\theta,\alpha_0}\big) \tag{52a}$$

$$\leq\ \frac{1-\delta^{\alpha_0}+\nu(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}{(1-\delta^{\alpha_0})^2}\times\pi_\theta\big(\Delta_{\theta,\alpha_0}\big). \tag{52b}$$

Remark 6.1 (Comparison with [LL18]) *Let $P := (P(i,j))_{(i,j)\in\mathbb{N}^2}$ be a stochastic infinite matrix and for every $k \geq 1$ let P_k be the linear augmentation (e.g. in the last column) of the $(k+1)\times(k+1)$ northwest corner truncation of P. Hence P_k is a stochastic matrix of order $k+1$. Assume that P satisfies (**S**) with an atom $S \subset \mathbb{N}$, and that there exist $b > 0$ and a Lyapunov function $V = (V(n))_{n\in\mathbb{N}}$ such that $PV \leq \delta V + b1_S$ and $\|1_{\mathbb{X}}\|_V = 1$. Let π (resp. π_k) be the invariant probability measure of P (resp. of P_k). For the sake of simplicity we also denote by P_k and π_k the natural extensions to $\mathbb{N}$ of P_k and π_k respectively. For every $k \in \mathbb{N}$ define*

$$\forall i = 0,\ldots,k, \quad \Delta_k(i,V) = \sum_{j>k} P(i,j)\big(V(k)+V(j)\big) \quad \text{and} \quad \delta_k := \sum_{i=0}^{k} \pi_k(i)\,\Delta_k(i,V). \tag{53}$$

It is proved in [LL18, Th. 2] that

$$\big\|\pi - \pi_k\big\|'_V \leq \frac{1+\pi(V)}{1-\delta} \times \delta_k \tag{54a}$$

$$\leq \frac{1-\delta+b}{(1-\delta)^2} \times \delta_k. \tag{54b}$$

Let us show that Properties (54a)-(54b) and (52a)-(52b) coincide in this truncation and atomic context. Note that $\Theta = \mathbb{N}\cup\{\infty\}$ here, with $P_\infty = P$ and $P_\theta = P_k$ for $\theta = k \in \mathbb{N}$. Moreover recall that in the atomic case we have $\alpha_0 = 1$ in ($\boldsymbol{D_\Theta^{\alpha_0}}$). Hence $\Delta_k(i,V)$ and δ_k in (53) are nothing else but the error term in (51) and the real number $\pi_\theta(\Delta_{\theta,\alpha_0})$ of Theorem 6.1. Moreover the constants in (54a)-(54b) are exactly those in (52a)-(52b) since we can choose $b = \nu(V)$ in the atomic case. This proves the claimed fact. Finally mention that the proof of the property $\lim_k \delta_k = 0$ in [LL18] is not complete because of the incorrect statement [LL18, lem. 1].

Again, without lost of generality we may suppose that $\alpha_0 = 1$ for the proof of Theorem 6.1. If $\alpha_0 < 1$, replace V and δ with V^{α_0} and δ^{α_0} respectively in the proof below. We use the next lemmas where the assumptions of Theorem 6.1 are supposed to hold. These lemmas extend results in [Twe98, Sect. 3] and [LL18, Eq. (3)] proved for truncated discrete Markov kernels.

Lemma 6.1 *We have: $\lim_{\theta\to\theta_0} \|\pi_\theta - \pi_{\theta_0}\|'_V = 0$.*

Proof. Note that

$$\forall\theta\in\Theta, \quad P_\theta V \leq cV \quad \text{with} \quad c := \delta + K \tag{55}$$

from ($\boldsymbol{K_\Theta}$)-($\boldsymbol{D_{\Theta,S^c}}$). Moreover we have for every $f \in \mathcal{B}_V$ such that $\|f\|_V \leq 1$

$$\forall n \geq 1,\ \forall x\in\mathbb{X}, \quad \big|(P_{\theta_0}^n f)(x) - (P_\theta^n f)(x)\big| \leq \sum_{j=0}^{n-1} c^{n-1-j}\big(P_{\theta_0}^j \Delta_{\theta,1}\big)(x). \tag{56}$$

Indeed, for $n = 1$ Inequality (56) holds since we have for every $f \in \mathcal{B}_V$ such that $\|f\|_V \leq 1$

$$\big|(P_{\theta_0} f)(x) - (P_\theta f)(x)\big| \leq \Delta_{\theta,1}(x)$$

from the definition of the V-weighted total variation norm. Next proceed by induction. Assume that (56) holds for some $n \geq 1$. Let $g \in \mathcal{B}_V$ be such that $\|g\|_V \leq 1$. Then

$$\begin{aligned}
\big|(P_{\theta_0}^{n+1} g)(x) - (P_{\theta}^{n+1} g)(x)\big| &\leq \big|\big(P_{\theta_0}^{n}(P_{\theta_0} - P_{\theta})g\big)(x)\big| + \big|\big((P_{\theta_0}^{n} - P_{\theta}^{n})P_{\theta} g\big)(x)\big| \\
&\leq \int_{\mathbb{X}} \big|(P_{\theta_0} g)(y) - (P_{\theta} g)(y)\big|\, P_{\theta_0}^{n}(x, dy) + \sum_{j=0}^{n-1} c^{n-j}\big(P_{\theta_0}^{j}\Delta_{\theta_0,1}\big)(x) \\
&\leq \int_{\mathbb{X}} \Delta_{\theta_0,1}(y)\, P_{\theta_0}^{n}(x, dy) + \sum_{j=0}^{n-1} c^{n-j}\big(P_{\theta_0}^{j}\Delta_{\theta_0,1}\big)(x)
\end{aligned}$$

using the triangular inequality, the fact that $\|P_\theta g\|_V \leq c$ by (55) and the induction assumption, and finally the definition of $\Delta_{\theta,1}$. This gives (56) at order $n+1$. Now let $x_0 \in \mathbb{X}$ be fixed and define

$$\varepsilon_{n,\Theta} := \sup_{\theta \in \Theta} \big\|P_{\theta}^{n}(x_0, \cdot) - \pi_\theta\big\|_V' . \tag{57}$$

Let $f \in \mathcal{B}_V$ be such that $\|f\|_V \leq 1$. Then we have

$$\begin{aligned}
\big|\pi_{\theta_0}(f) - \pi_\theta(f)\big| &\leq \big|\pi_{\theta_0}(f) - (P_{\theta_0}^{n} f)(x_0)\big| + \big|(P_{\theta_0}^{n} f)(x_0) - (P_{\theta}^{n} f)(x_0)\big| + \big|(P_{\theta}^{n} f)(x_0) - \pi_\theta(f)\big| \\
&\leq 2\,\varepsilon_{n,\Theta} + \sum_{j=0}^{n-1} c^{n-1-j}\big(P_{\theta_0}^{j}\Delta_{\theta,1}\big)(x_0)
\end{aligned}$$

from the definition of $\varepsilon_{n,\Theta}$ and from (56). Next fix $n \geq 1$. We have

$$\forall j = 0, \ldots, n-1, \quad \lim_{\theta \to \theta_0} \big(P_{\theta_0}^{j}\Delta_{\theta,1}\big)(x_0) = 0$$

from Lebesgue's theorem applied to the probability measure $P_{\theta_0}^{j}(x_0, \cdot)$ using Assumption $(\boldsymbol{\Delta}_{\boldsymbol{\Theta}}^{\boldsymbol{\alpha_0}})$ (with $\alpha_0 = 1$ here) and

$$\forall \theta \in \Theta, \quad \Delta_{\theta,1} \leq 2cV \tag{58}$$

with c defined in (55). Hence

$$\forall n \geq 1, \quad \limsup_{\theta \to \theta_0} \big\|\pi_{\theta_0} - \pi_\theta\big\|_V' \leq 2\,\varepsilon_{n,\Theta}.$$

Moreover we have

$$\lim_{n} \varepsilon_{n,\Theta} = 0 \tag{59}$$

from [Bax05, GP14] since Assumptions $(\boldsymbol{A_\Theta})$ are stated in a uniform way in $\theta \in \Theta$. Property (59) can be also derived from the results of Sections 2-3 when the parameter set Θ is assumed to be locally compact, see Appendix C. It follows that $\limsup_{\theta \to \theta_0} \big\|\pi_{\theta_0} - \pi_\theta\big\|_V' = 0$, hence the assertion of Lemma 6.1 holds. □

Lemma 6.2 *For any $f \in \mathcal{B}_V$, let us introduce $f_0 := f - \pi_{\theta_0}(f)1_{\mathbb{X}}$. Set $\widetilde{f_0} := (I - R_{\theta_0})^{-1} f_0$ with $R_{\theta_0} := P_{\theta_0} - \nu(\cdot)1_S$. Then*

$$\pi_\theta(f) - \pi_{\theta_0}(f) = \pi_\theta\big(\Delta_\theta \widetilde{f_0}\big) \quad \textit{with} \quad \Delta_\theta := P_\theta - P_{\theta_0}.$$

Proof. Since $\pi_{\theta_0}(f_0) = 0$, we know from Theorem 4.1 applied to P_{θ_0} that $\widetilde{f}_0$ is a solution to Poisson's equation, that is $\widetilde{f}_0$ satisfies $\widetilde{f}_0 - P_{\theta_0}\widetilde{f}_0 = f_0$, or $P_{\theta_0}\widetilde{f}_0 = \widetilde{f}_0 - f_0$. Then, it follows that

$$\begin{aligned}\pi_\theta\big(\Delta_\theta \widetilde{f}_0\big) &= \pi_\theta\big(P_\theta \widetilde{f}_0 - P_{\theta_0}\widetilde{f}_0\big) = \pi_\theta(\widetilde{f}_0) + \pi_\theta\big(-\widetilde{f}_0 + f_0\big) \\ &= \pi_\theta(f_0) = \pi_\theta(f) - \pi_{\theta_0}(f) \qquad \text{(from the definition of } f_0).\end{aligned}$$

□

Proof of Theorem 6.1. That $\lim_{\theta\to\theta_0}\|\pi_\theta - \pi_{\theta_0}\|'_V = 0$ is proved in Lemma 6.1.

Next we have

$$\pi_\theta(\Delta_{\theta,1}) \le \big|\pi_\theta(\Delta_{\theta,1}) - \pi_{\theta_0}(\Delta_{\theta,1})\big| + \pi_{\theta_0}(\Delta_{\theta,1}) \le 2c\|\pi_\theta - \pi_{\theta_0}\|'_V + \pi_{\theta_0}(\Delta_{\theta,1})$$

from (58). Moreover we obtain that $\lim_{\theta\to\theta_0}\pi_{\theta_0}(\Delta_{\theta,1}) = 0$ from Lebesgue's dominated convergence theorem with respect to the probability measure π_{θ_0} using Assumption ($\boldsymbol{\Delta}_{\boldsymbol{\Theta}}^{\boldsymbol{\alpha_0}}$) (with $\alpha_0 = 1$ here), (58) and $\pi_{\theta_0}(V) < \infty$. We have proved that $\lim_{\theta\to\theta_0}\pi_\theta\big(\Delta_{\theta,1}\big) = 0$. Now let $f \in \mathcal{B}_V$ be such that $\|f\|_V \le 1$. Define $f_0 := f - \pi_{\theta_0}(f)1_{\mathbb{X}}$ and $\widetilde{f}_0 := (I - R_{\theta_0})^{-1} f_0$ as in Lemma 6.2. Then by using Theorem 4.1 applied to the Markov kernel P_{θ_0} we obtain that

$$\begin{aligned}\big|\pi_\theta(f) - \pi_{\theta_0}(f)\big| &\le \int_{\mathbb{X}} \big|(P_\theta\widetilde{f}_0)(x) - (P_{\theta_0}\widetilde{f}_0)(x)\big|\,\pi_\theta(dx) \qquad \text{(from Lemma 6.2)} \\ &\le \|\widetilde{f}_0\|_V \int_{\mathbb{X}} \Delta_{\theta,1}(x)\,\pi_\theta(dx) \qquad \text{(from the definition of } \Delta_{\theta,1}) \\ &\le \frac{1}{1-\delta}\|f_0\|_V \times \pi_\theta\big(\Delta_{\theta,1}\big) \qquad \text{(from (30))} \\ &\le \frac{1 + \pi_{\theta_0}(V)||1_X||_V}{1-\delta} \times \pi_\theta\big(\Delta_{\theta,1}\big) \qquad \text{(from the definition of } f_0) \\ &\le \frac{1 - \delta + \nu(V)\|1_{\mathbb{X}}\|_V}{(1-\delta)^2} \times \pi_\theta\big(\Delta_{\theta,1}\big) \qquad \text{(from } \pi_{\theta_0}(V) \le \nu(V)/(1-\delta)).\end{aligned}$$

The proof of Theorem 6.1 is complete. □

Remark 6.2 *As introduced in [Twe98] for discrete set $\mathbb{X}$, Condition ($\boldsymbol{\Delta}_{\boldsymbol{\Theta}}^{\boldsymbol{\alpha_0}}$) is the expected continuity assumption in order to study the V^{α_0}-weighted total variation distance between π_θ and π_{θ_0}. When this condition is satisfied, not only the bound (52a) in Theorem 6.1 has the expected form, but also the constant in (52a) is simple (and moreover explicit in (52b)). Let us compare Condition ($\boldsymbol{\Delta}_{\boldsymbol{\Theta}}^{\boldsymbol{\alpha_0}}$) with alternative assumptions used in prior works.*

- *The standard operator norm continuity assumption introduced in [Kar86] writes as $\lim_{\theta\to\theta_0}\|P_\theta - P_{\theta_0}\|_{V^{\alpha_0}} = 0$, namely*

  $$\lim_{\theta\to\theta_0}\sup_{x\in\mathbb{X}} \frac{\Delta_{\theta,\alpha_0}(x)}{V(x)^{\alpha_0}} = 0.$$

  This condition is clearly much more restrictive than Condition ($\boldsymbol{\Delta}_{\boldsymbol{\Theta}}^{\boldsymbol{\alpha_0}}$).

- *The weak operator norm continuity assumptions used in [SS00, FHL13, HL14a, RS18, MARS20] requires that*

  $$\lim_{\theta\to\theta_0}\sup_{x\in\mathbb{X}} \frac{\|P_\theta(x,\cdot) - P_{\theta_0}(x,\cdot)\|'_{TV}}{V(x)^{\alpha_0}} = 0. \qquad (60)$$

To understand the difference between Conditions ($\boldsymbol{\Delta_\Theta^{\alpha_0}}$) and (60), consider the following simple example derived from perturbed linear autoregressive models:

$$\forall \theta \in (0,1),\ \forall x \in \mathbb{X} = \mathbb{R},\ \forall A \in \mathcal{X}, \quad P_\theta(x, A) := \int_{\mathbb{R}} 1_A(y)\, \nu(y - \theta x)\, dy,$$

where $\mathcal{X}$ is here the Borel $\sigma-$algebra on $\mathbb{R}$ and where ν is some probability density function (p.d.f.) with respect to Lebesgue's measure on $\mathbb{R}$. Let $\widehat{\theta} \in (0,1)$. It is well-known that, under moment conditions on the p.d.f. ν, the family $\{P_\theta\}_{\theta \in (0,\widehat{\theta})}$ satisfies Assumptions ($\boldsymbol{A_\Theta}$) (e.g. see [RS18, HL23]). Here we only focus on Conditions ($\boldsymbol{\Delta_\Theta^{\alpha_0}}$) and (60). Let $\theta_0 \in (0, \widehat{\theta})$ be fixed. Condition ($\boldsymbol{\Delta_\Theta^{\alpha_0}}$) writes as follows

$$\forall x \in \mathbb{R}, \quad \lim_{\theta \to \theta_0} \int_{\mathbb{X}} V(y)^{\alpha_0} \big|\nu(y - \theta x) - \nu(y - \theta_0 x)\big| dy, \tag{61}$$

while Condition (60) is:

$$\lim_{\theta \to \theta_0} \sup_{x \in \mathbb{R}} \frac{\int_{\mathbb{X}} \big|\nu(z - \theta x) - \nu(z - \theta_0 x)\big| dz}{V(x)^{\alpha_0}} = 0. \tag{62}$$

Actually Conditions (61) and (62) are quite different. In (61) the convergence is simple in $x \in \mathbb{R}$, but the presence of $V(y)$ in the integral may be problematic. In (62) the absence of the function V in the integral is of course an advantage, but the convergence has to be uniform on $\mathbb{R}$ (actually it has to be uniform on every compact of $\mathbb{R}$ thanks to the division by $V(x)$). In this example Condition (62) is satisfied thanks to the continuity of $t \mapsto \nu(\cdot - t)$ from $\mathbb{R}$ to the Lebesgue space $\mathbb{L}^1(\mathbb{R})$ (see [HL23]), so that the bounds obtained in [RS18, HL23] for $\|\pi_\theta - \pi_{\theta_0}\|'_{TV}$ hold. However, if the p.d.f. ν satisfies Condition (61) (thanks to Lebesgue's theorem for instance), then the bound (52a) and (52b) are simpler and more explicit than those in [RS18, HL23].

A Complement on the real number α_0

Let $\alpha \in (0, 1]$. If $x \in \mathbb{X} \setminus S$, then we have $(PV^\alpha)(x) \le \delta^\alpha V(x)^\alpha$ from ($\boldsymbol{D_{S^c}}$) and Jensen's inequality. Recall that $K := \sup_{x \in S}(PV)(x)$ (see ($\boldsymbol{K}$)). We have $1 \le \sup_{x\in S}(PV^\alpha)(x) \le K^\alpha$ from $1_{\mathbb{X}} \le V^\alpha$ and $PV^\alpha \le (PV)^\alpha$ using again Jensen's inequality. Finally we have

$$\forall x \in S, \quad (PV^\alpha)(x) - \delta^\alpha V(x)^\alpha - \nu(V^\alpha) \le K^\alpha - \delta^\alpha - \nu(1_{\mathbb{X}})$$

from $1_{\mathbb{X}} \le V$. Passing to the limit when $\alpha \to 0$ provides the existence of $\alpha_0 \in (0, 1]$ such that ($\boldsymbol{D^{\alpha_0}}$) holds since $\nu(1_{\mathbb{X}}) > 0$. Note that, if Condition ($\boldsymbol{S}$) is fulfilled with an atom S and with $\nu(\cdot) := P(a_0, \cdot)$ for some (any) $a_0 \in S$, then ($\boldsymbol{D^{\alpha_0}}$) holds with $\alpha_0 = 1$. Indeed we then have

$$\forall x \in S, \quad PV(x) - \delta\, V(x) - \nu(V) = -\delta\, V(x) \le 0.$$

Since under Assumption ($\boldsymbol{D_{S^c}}$) we have $PV^\alpha \le \delta^\alpha V^\alpha$ on $\mathbb{X} \setminus S$ for any $\alpha \in (0, 1]$, the computation of α_0 in ($\boldsymbol{D^{\alpha_0}}$) only concerns the elements $x \in S$. Under Assumption ($\boldsymbol{S}$) define $\sigma := 1 - \nu(1_{\mathbb{X}}) \in [0, 1)$. The value $\sigma = 0$ corresponds to the atomic case for which $\alpha_0 = 1$. If $\alpha_0 = 1$ does not work, the following statement is useful to find an explicit value for $\alpha_0 \in (0, 1)$ in ($\boldsymbol{D^{\alpha_0}}$).

Proposition A.1 *Assume that P satisfies Condition ($\boldsymbol{S}$) with S that is not an atom, so that $\sigma \in (0,1)$. Then we have for any Lyapunov function V:*

$$\forall \alpha \in (0,1],\ \forall x \in S, \quad (PV^\alpha)(x) - \nu(V^\alpha) \leq \frac{\sigma}{\sigma^\alpha}\big[(PV)(x) - \nu(V)\big]^\alpha. \tag{63}$$

Proof. Let $x \in S$. Note that $\sigma_x(\cdot) := P(x,\cdot) - \nu(\cdot)$ is a non-negative measure on $(\mathbb{X}, \mathcal{X})$ from Assumption ($\boldsymbol{S}$), and that $\sigma_x(1_{\mathbb{X}}) = 1 - \nu(1_{\mathbb{X}}) = \sigma$ does not depend on x. Define the following probability measure on $(\mathbb{X}, \mathcal{X})$: $\widehat{\sigma}_x(\cdot) = \sigma_x(\cdot)/\sigma$. Let $\alpha \in (0,1]$. It follows from Jensen's inequality that

$$\frac{(PV^\alpha)(x) - \nu(V^\alpha)}{\sigma} = \widetilde{\sigma}_x(V^\alpha) \leq \big[\widetilde{\sigma}_x(V)\big]^\alpha = \frac{\big[(PV)(x) - \nu(V)\big]^\alpha}{\sigma^\alpha},$$

from which we deduce (63).

□

The real number α_0 can be computed as follows thanks to Proposition A.1. Let $M := K - \nu(V)$ with K given in ($\boldsymbol{K}$). Then

$$\forall \alpha \in (0,1],\ \forall x \in S, \quad (PV^\alpha)(x) - \nu(V^\alpha) - \delta^\alpha V(x)^\alpha \leq \frac{\sigma}{\sigma^\alpha} M^\alpha - \delta^\alpha$$

since $V \geq 1_{\mathbb{X}}$. Then $\alpha_0 \in (0,1]$ can be chosen such that $\frac{\sigma}{\sigma^{\alpha_0}} M^{\alpha_0} - \delta^{\alpha_0} \leq 0$ since

$$\lim_{\alpha \to 0} \Big[\frac{\sigma}{\sigma^\alpha} M^\alpha - \delta^\alpha\Big] = \sigma - 1 < 0.$$

B Order of the eigenvalues of P

Under Assumptions ($\boldsymbol{A}$) we deduce from Property (11) that $z \mapsto \mu_z$ given in (17) is derivable on the domain $D_0 = \{z \in \mathbb{C} : |z| > \delta^{\alpha_0}\}$ with $\alpha_0 \in (0,1]$ given in ($\boldsymbol{D^{\alpha_0}}$), and that its derivative is given by

$$\forall z \in D_0, \quad \mu'_z := -\sum_{k=1}^{+\infty} k\, z^{-(k+1)}\, \beta_k \tag{64}$$

which is absolutely convergent in $\mathcal{B}'_{V^{\alpha_0}}$.

Proposition B.1 *Assume that P satisfies ($\boldsymbol{A}$), and let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Let $\lambda \in D_0$ be an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ (equivalently $\mu_\lambda(1_S) = 1$ from Theorem 2.2). Then the two following assertions are equivalent:*

(i) λ is of order one, that is $\mathrm{Ker}(P - \lambda I)^2 = \mathrm{Ker}(P - \lambda I)$ or equivalently $\mathrm{Ker}(P^* - \lambda I)^2 = \mathrm{Ker}(P^* - \lambda I)$;

(ii) $\mu'_\lambda(1_S) \neq 0$.

Moreover, if we have $\mu'_\lambda(1_S) = 0$, then the system $\{\mu_\lambda, \mu'_\lambda\}$ form a basis of the subspace $\mathrm{Ker}(P^* - \lambda I)^2 := \{\psi \in \mathcal{B}'_{V^{\alpha_0}} : \psi \circ (P - \lambda I)^2 = 0\}$.

Proof. Again we suppose that $\alpha_0 = 1$ in ($\boldsymbol{D^{\alpha_0}}$). Let $\lambda \in \mathbb{C}$ be an eigenvalue of P on $\mathcal{B}_V$ such that $|\lambda| > \delta$. Assume that $\mu'_\lambda(1_S) = 0$. From (19) we obtain the following equality in $\mathcal{B}'_V$

$$\mu'_\lambda \circ P = \mu_\lambda + \lambda\mu'_\lambda + \mu'_\lambda \circ T$$

$$\text{with} \quad \forall f \in \mathcal{B}_V, \quad (\mu'_\lambda \circ T)(f) = -\sum_{k=1}^{+\infty} k\,\lambda^{-(k+1)}\,\beta_k(Tf) \;=\; -\nu(f)\sum_{k=1}^{+\infty} k\,\lambda^{-(k+1)}\,\beta_k(1_S) \;=\; -\nu(f)\,\mu'_\lambda(1_S) = \;0.$$

Hence $\mu'_\lambda \circ P = \lambda\mu'_\lambda + \mu_\lambda$. Recall that $\mu_\lambda \in \mathrm{Ker}(P^* - \lambda I)$ (Theorem 2.2) with $\mu_\lambda \neq 0$ since $\mu_\lambda(1_S) = 1$. Thus μ'_λ is nonzero and satisfies $\mu'_\lambda \circ (P - \lambda I) = \mu_\lambda \in \mathrm{Ker}(P^* - \lambda I)$, so that $\mu'_\lambda \in \mathrm{Ker}(P^* - \lambda I)^2 \setminus \mathrm{Ker}(P^* - \lambda I)$. We have proved the implication $(i) \Rightarrow (ii)$. Conversely, assume that there exists $\psi \in \mathcal{B}'_V$, $\psi \neq 0$, such that $\psi \circ (P - \lambda I)^2 = 0$ and $\psi \circ (P - \lambda I) \neq 0$. Since $\phi := \psi \circ (P - \lambda I) \in \mathrm{Ker}(P^* - \lambda I)$, we deduce from the last assertion of Theorem 2.2 that $\phi = c\,\mu_\lambda$ for some $c \in \mathbb{C}$. Obviously we may suppose that $c = 1$ (replacing ψ with ψ/c). Hence $\psi \circ P = \lambda\psi + \mu_\lambda$, and an easy induction gives

$$\forall n \geq 0, \quad \psi \circ P^n = \lambda^n\psi + n\lambda^{n-1}\mu_\lambda.$$

Next, composing on the left by ψ in (20), we obtain the following equalities in $\mathcal{B}'_V$

$$\lambda^n\psi + n\lambda^{n-1}\mu_\lambda - \psi(1_S)\sum_{k=1}^{n}\lambda^{n-k}\,\beta_k - \mu_\lambda(1_S)\sum_{k=1}^{n}(n-k)\lambda^{n-k-1}\,\beta_k = \mathrm{O}(\delta^n).$$

Using $\mu_\lambda(1_S) = 1$ we deduce that

$$\psi - \psi(1_S)\sum_{k=1}^{n}\lambda^{-k}\,\beta_k + \sum_{k=1}^{n}k\lambda^{-(k+1)}\,\beta_k + n\lambda^{-1}\left(\mu_\lambda - \sum_{k=1}^{n}\lambda^{-k}\,\beta_k\right) = \mathrm{o}(1).$$

When $n \to +\infty$ we obtain that

$$\psi = \psi(1_S)\mu_\lambda + \mu'_\lambda$$

since $\mu_\lambda - \sum_{k=1}^{n}\lambda^{-k}\,\beta_k = \mathrm{O}((\delta/|\lambda|)^n)$ with $|\lambda| > \delta$. Applying the above equality to the function 1_S gives $\mu'_\lambda(1_S) = 0$ since $\mu_\lambda(1_S) = 1$. We have proved the implication $(ii) \Rightarrow (i)$, as well as the last assertion of Proposition B.1.

□

Under Assumptions ($\boldsymbol{A}$) define for every $z \in \mathbb{C}$ such that $|z| > \delta^{\alpha_0}$

$$\chi_S(z) = \mu_z(1_S) - 1 = \sum_{k=1}^{+\infty} z^{-k}\,\beta_k(1_S) \;-1.$$

We know from Theorem 2.2 that $\lambda \in \mathbb{C}$ such that $\delta^{\alpha_0} < |\lambda| \leq 1$ is an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ if, and only if, $\chi_S(\lambda) = 0$. Moreover, from Proposition B.1, such an eigenvalue λ is of order one if, and only if, $\chi'_S(\lambda) \neq 0$. An easy extension of Proposition B.1 shows that, for every $p \geq 2$, λ is of order p if, and only if, $\forall i = 0, \ldots, p-1$, $\chi_S^{(i)}(\lambda) = 0$ and $\chi_S^{(p)}(\lambda) \neq 0$.

C Proof of (59) when Θ is locally compact

The following statement shows that, under Assumptions ($\boldsymbol{A_\Theta}$), the family $\{P_\theta\}_{\theta\in\Theta}$ satisfies (2) in a uniform way in θ when Θ is locally compact, so that Property (59) holds. Under Assumptions ($\boldsymbol{A_\Theta}$), we denote by $\varrho_{\alpha_0}^{(\theta)}$ the second eigenvalue of P_θ on $\mathcal{B}_{V^{\alpha_0}}$.

Proposition C.1 *Assume that $\{P_\theta\}_{\theta\in\Theta}$ satisfies Assumptions ($\boldsymbol{A_\Theta}$). Let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D_\Theta^{\alpha_0}}$), let $\theta_0 \in \Theta$ and suppose that Assumption ($\boldsymbol{\Delta_\Theta^{\alpha_0}}$) holds. Moreover suppose that Θ is locally compact. Then there exists a compact neighbourhood $\mathcal{V}_{\theta_0}$ of θ_0 in Θ such that*

$$\forall \theta \in \mathcal{V}_{\theta_0}, \quad \varrho_{\alpha_0}^{(\theta)} \le \max(\delta^{\alpha_0}, \varrho_{\alpha_0}^{(\theta_0)}). \tag{65}$$

Moreover, for every $\rho \in \big(\max(\delta^{\alpha_0}, \varrho_{\alpha_0}^{(\theta_0)}), 1\big)$, we have

$$\forall \theta \in \mathcal{V}_{\theta_0}, \quad \|P_\theta^n f - \pi_\theta(f)1_{\mathbb{X}}\|_{V^{\alpha_0}} \le \frac{\rho}{2\pi(\rho-\delta^{\alpha_0})}\left(1 + \frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{m_{\rho,\Theta}\big(\rho-\delta^{\alpha_0}\big)}\right)\rho^n \tag{66}$$

$$\textit{with} \qquad m_{\rho,\Theta} := \min\left\{\left|1-\mu_z^{(\theta)}(1_S)\right| \,:\, z\in\mathbb{C} : |z| = \rho,\ \theta \in \mathcal{V}_{\theta_0}\right\} > 0. \tag{67}$$

Proof. Again we suppose that $\alpha_0 = 1$. Note that under Assumptions ($\boldsymbol{A_\Theta}$) we have

$$\forall k \ge 1,\ \forall \theta \in \Theta, \quad \|R_\theta^k\|_V \le \delta^k \quad \text{with} \quad R_\theta := P_\theta - \nu(\cdot)1_S \tag{68}$$

from (11) and from the uniformity of ($\boldsymbol{A_\Theta}$) in $\theta \in \Theta$. Define

$$\forall \theta \in \Theta,\ \forall k \ge 1, \quad \beta_k^{(\theta)} = \nu \circ R_\theta^{k-1}. \tag{69}$$

Hence

$$\forall k \ge 1,\ \forall \theta \in \Theta, \quad \beta_k^{(\theta)}(1_S) \le \nu(V)\|1_S\|_V\, \delta^{k-1}. \tag{70}$$

For every $z \in \mathbb{C}$ such that $|z| > \delta$ and for every $\theta \in \Theta$ we define

$$\mu_z^{(\theta)}(1_S) := \sum_{k=1}^{+\infty} z^{-k}\beta_k^{(\theta)}(1_S). \tag{71}$$

Let $f \in \mathcal{B}_V$ such that $\|f\|_V \le 1$. Observing that $\Delta_{\theta,1}(x) := \|R_\theta(x,\cdot) - R_{\theta_0}(x,\cdot)\|_V$ and that $R_\theta V \le V$ from ($\boldsymbol{D_\Theta^{\alpha_0}}$) (with $\alpha_0 = 1$ here), we can prove as in (56) that

$$\forall k \ge 1,\ \forall x \in \mathbb{X}, \quad \big|(R_{\theta_0}^k f)(x) - (R_\theta^k f)(x)\big| \le \sum_{j=0}^{k-1}\big(R_{\theta_0}^j \Delta_{\theta,1}\big)(x) \le \sum_{j=0}^{k-1}\big(P_{\theta_0}^j \Delta_{\theta,1}\big)(x).$$

Note that $\beta_1^{(\theta)} = \nu$. Then, using the definition (69) of $\beta_k^{(\theta)}$, we have for every $k \ge 2$

$$\big|\beta_k^{(\theta)}(f) - \beta_k^{(\theta_0)}(f)\big| \le \int_{\mathbb{X}} \big|(R_\theta^{k-1}f)(x) - (R_{\theta_0}^{k-1}f)(x)\big)\big|\, d\nu(x) \le \sum_{j=0}^{k-2} \nu\big(P_{\theta_0}^j \Delta_{\theta,1}\big).$$

Moreover we have

$$\forall j = 0, \ldots, k-2, \quad \lim_{\theta\to\theta_0} \nu\big(P_{\theta_0}^j \Delta_{\theta,1}\big) = 0$$

from Lebesgue's dominated convergence theorem with respect to the positive measure $\nu P_{\theta_0}^j$ using Assumption $(\boldsymbol{\Delta}_{\boldsymbol{\Theta}}^{\boldsymbol{\alpha_0}})$, (58) and $\nu(P_{\theta_0}^j V) < \infty$ (use (55)). This proves that

$$\forall k \geq 1, \quad \lim_{\theta \to \theta_0} \|\beta_k^{(\theta)} - \beta_k^{(\theta_0)}\|_V' = 0. \tag{72}$$

To simplify, for every $\theta \in \Theta$ and for every $z \in \mathbb{C}$ such that $|z| > \delta$, we set $\phi(\theta, z) := \mu_z^{(\theta)}(1_S)$ (see (71)). We easily deduce from (70) and (72) that ϕ is continuous on $\Theta \times \{z \in \mathbb{C} : |z| > \delta\}$ (note that θ_0 has been arbitrarily chosen in Θ). Let $\rho \in \big(\max(\delta, \varrho^{(\theta_0)}), 1\big)$, where $\varrho^{(\theta_0)}$ denotes the second eigenvalue of P_{θ_0} on $\mathcal{B}_V$. Let $\gamma \in (0, 1-\rho)$, and finally let $\mathcal{D}_{\rho,\gamma}$ be the following compact subset of $\mathbb{C}$:

$$\mathcal{D}_{\rho,\gamma} := \big\{z \in \mathbb{C} : |z| \geq \rho,\ |z-1| \geq \gamma\big\}.$$

We know from the definition of $\varrho^{(\theta_0)}$ and from Theorem 2.2 that

$$\forall z \in \mathcal{D}_{\rho,\gamma}, \quad \phi(\theta_0, z) \neq 1. \tag{73}$$

Let us prove that there exists a neighbourhood $\mathcal{V}_{\theta_0} \equiv \mathcal{V}_{\theta_0}(\rho, \gamma)$ of θ_0 in Θ such that

$$\forall z \in \mathcal{D}_{\rho,\gamma},\ \forall \theta \in \mathcal{V}_{\theta_0}, \quad \phi(\theta, z) \neq 1. \tag{74}$$

Assume that such a neighbourhood does not exist. Then there exists a sequence $(\vartheta_n)_{n\geq 1} \in \Theta^{\mathbb{N}}$ and a sequence $(z_n)_{n\geq 1} \in \mathcal{D}_{\rho,\gamma}^{\mathbb{N}}$ such that $\lim \vartheta_n = \theta_0$ and $\forall n \geq 1,\ \phi(\vartheta_n, z_n) = 1$. Up to select a subsequence we can suppose that $\lim_n z_n = u$ for some u in the compact set $\mathcal{D}_{\rho,\gamma}$. Then we deduce from the continuity of ϕ that

$$\phi(\theta_0, u) = \lim_n \phi(\vartheta_n, z_n) = 1.$$

This contradicts Property (73). Hence (74) is proved. Next let $\widehat{r}_1$ be defined in (35) (with $\alpha_0 = 1$ here), let $\gamma \in (0, \min(1-\rho, \widehat{r}_1/2))$ and let $\mathcal{V}_{\theta_0} \equiv \mathcal{V}_{\theta_0}(\rho, \gamma)$ such that (74) holds. Let us prove that

$$\forall z \in \mathbb{C},\ |z| \geq \rho,\ z \neq 1,\ \forall \theta \in \mathcal{V}_{\theta_0}, \quad \phi(\theta, z) \neq 1. \tag{75}$$

First it follows from the uniformity in $\theta \in \Theta$ of Assumptions $(\boldsymbol{A_\Theta})$ and from Proposition 5.1 that, for every $\theta \in \Theta$, $\lambda = 1$ is the single spectral value of P_θ on $\mathcal{B}_V$ in the open disk $D(1, \widehat{r}_1)$. Thus we have

$$\forall \theta \in \Theta,\ \forall z \in D(1, \widehat{r}_1),\ z \neq 1, \quad \phi(\theta, z) \neq 1 \tag{76}$$

from Theorem 2.2. Then (75) follows from (74) and (76) since $\gamma < \widehat{r}_1/2$.

Now we can complete the proof of Proposition C.1. Let $\rho \in \big(\max(\delta, \varrho^{(\theta_0)}), 1\big)$. Using the spectral properties of Section 2 and Theorem 2.2 we deduce from (75) that, for every $\theta \in \mathcal{V}_{\theta_0}$, the spectral gap $\varrho^{(\theta)}$ of P_θ on $\mathcal{B}_V$ is less than ρ. In fact this gives (65) since ρ is arbitrarily close to $\max\big(\delta, \varrho^{(\theta_0)}\big)$. Next note that the neighbourhood $\mathcal{V}_{\theta_0}$ of θ_0 in (75) can be assumed to be compact since Θ is locally compact. Then (67) follows from the continuity of ϕ on the compact set $\mathcal{H} := \mathcal{V}_{\theta_0} \times \{z \in \mathbb{C} : |z| = \rho\}$ since we know from (75) that $\forall (\theta, z) \in H,\ \phi(\theta, z) \neq 1$. Finally (66) follows from Theorem 3.1 applied to P_θ, $\theta \in \mathcal{V}_{\theta_0}$. □

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