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PHILIPPE GILLE, ERHARD NEHER, AND CAMERON RUETHER

Abstract: We investigate quadratic pairs for Azumaya algebras with involutions over a base scheme S as defined by Calmès and Fasel, generalizing the case of quadratic pairs on central simple algebras over a field (Knus, Merkurjev, Rost, Tignol). We describe a cohomological obstruction for an Azumaya algebra over S with orthogonal involution to admit a quadratic pair. When S is affine this obstruction vanishes, however it is non-trivial in general. In particular, we construct explicit examples with non-trivial obstructions.

Keywords: Azumaya algebras, involutions, quadratic forms, quadratic pairs. *MSC 2000: 16H05, 20G35*

Introduction

In the classical theory of quadratic forms over a field $\mathbb{F}$ of characteristic not 2, there is an equivalence between symmetric bilinear forms and quadratic forms. Given a quadratic form q one considers its polar symmetric bilinear form $b_q(x,y) = q(x+y) - q(x) - q(y)$, and given a symmetric bilinear form b one considers a quadratic form $q_b(x) = b(x,x)$. Then, $b(x,y) = \frac{1}{2}b_{q_b}(x,y)$ and so every symmetric bilinear form arises as the polar of a quadratic form. This correspondence between quadratic forms and symmetric bilinear forms breaks down when the base field is of characteristic 2. There exist symmetric bilinear forms which are not the polar of a quadratic form. For example, on the $\mathbb{F}$–vector space $\mathbb{F}$ the polar of any quadratic form vanishes, while $b(x,y) = xy$ is a non-zero symmetric bilinear form. Quirks of characteristic 2 rear their head in other places as well. Consider a central simple algebra with $\mathbb{F}$–linear involution (A,σ). There exists a splitting field extension $\mathbb{F} \hookrightarrow \mathbb{K}$ such that $A \otimes_{\mathbb{F}} \mathbb{K} \cong \mathrm{M}_n(\mathbb{K})$. The involution $\sigma \otimes 1$ on $\mathrm{M}_n(\mathbb{K})$ is then the adjoint involution of some bilinear form b. We follow the convention of Calmès and Fasel [CF] and say that σ is *orthogonal* if b is symmetric, σ is *weakly symplectic* if b is skew-symmetric, and σ is *symplectic* if b is alternating, i.e.,

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$b(x,x) = 0$ for all x. These notions do not depend on the splitting field. In characteristic 2, symmetric and skew-symmetric are the same notion, and so an orthogonal involution is simultaneously a weakly-symplectic involution, and vice versa. Another quirk of characteristic 2 is the following. If σ is an orthogonal involution, the associated orthogonal group scheme $\mathbf{O}(A,\sigma)$ is not smooth, [KMRT, Remark 23.5]. The authors of The Book of Involutions [KMRT] addressed these discrepancies by introducing the notion of *quadratic pairs*.

Definition. Let A be a central simple $\mathbb{F}$–algebra. A *quadratic pair* on A is a pair (σ, f) where σ is an orthogonal involution and f is a linear function $f\colon \mathrm{Sym}(A,\sigma) \to \mathbb{F}$ on the symmetric elements of A (those $a \in A$ satisfying $\sigma(a) = a$), such that for all $a \in A$,

$$f(a + \sigma(a)) = \mathrm{Trd}_A(a).$$

Here Trd_A denotes the reduced trace of A, a notation we will also use later for Azumaya algebras over a scheme. If b is a regular symmetric bilinear form on a vector space V with adjoint involution σ, then each way $(\mathrm{End}(V),\sigma)$ can be extended to a quadratic pair $(\mathrm{End}(V),\sigma,f)$ corresponds to a quadratic form whose polar form is b, [KMRT, 5.11]. Further, for any central simple algebra with quadratic pair, the orthogonal group $\mathbf{O}(A,\sigma,f)$ ([KMRT, 23.B]) is smooth.

The notion of quadratic pairs was extended by Calmès and Fasel [CF] from the setting of central simple algebras over a field, to the setting of Azumaya algebras over a scheme S. In their setting, objects such as algebras, groups, bilinear forms, etc., are sheaves or maps of sheaves on the ringed site $(\mathfrak{Sch}_S, \mathcal{O})$ where $\mathfrak{Sch}_S$ is the (big) fppf site of schemes over S, and $\mathcal{O}$ is the global sections functor. The spirit of the original theory persists in this generalization. Assume b is a regular symmetric bilinear form on a locally free $\mathcal{O}$–module of finite positive rank $\mathcal{M}$, and let σ be its adjoint involution on $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$. Then, each way $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$ can be equipped with a quadratic pair $(\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}),\sigma,f)$ corresponds to a quadratic form q_f whose polar form is b. The groups associated to quadratic pairs, discussed in [CF, 4.4], are also the appropriate notion of semisimple groups of type D over a scheme.

In this paper we continue the study of Azumaya algebras and quadratic pairs over an arbitrary base scheme S. In particular, we do not assume that 2 is invertible over S. We begin with a preliminary section which reviews the technicalities of our sheaf point of view. Then, we discuss a globalized version of the Skolem-Noether theorem which describes isomorphisms of Azumaya algebras. Starting in section 3 our philosophy comes to the forefront. We are interested in how global phenomenon may differ from local behaviour in this context, and in providing descriptions which are as global as possible. For example, given an Azumaya algebra with involution $(\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}),\sigma)$, there may not exist a bilinear form $b\colon \mathcal{M}\times\mathcal{M} \to \mathcal{O}$ whose adjoint involution is σ, despite this being the case locally. The global statement is Lemma 3.3, which states that σ is adjoint to a line bundle-valued bilinear form $b\colon \mathcal{M}\times\mathcal{M}\to\mathcal{L}$. Towards this global point of view, section 3 reviews the theory of line bundle-valued forms as in [A] or [BK], and connects them to involutions.

Section 4 defines quadratic pairs, gives examples, and describes procedures for constructing quadratic pairs. Proposition 4.6 connects the existence of a quadratic pair on $(\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}),\sigma)$ to the existence of a quadratic form whose polar is a regular symmetric bilinear form b with adjoint involution σ, only now in the more global the context of line bundle-valued forms. Section 5 is the generalization of [KMRT, 5.18, 5.20], which deals with quadratic pairs on tensor products of algebras with involution.

The final two sections, 6 and 7, address cohomological obstructions which prevent local quadratic pairs from being extended to global objects. For an Azumaya algbera with orthogonal involution $(\mathcal{A},\sigma)$, we call σ *locally quadratic* if it can be equipped with quadratic pairs locally with respect to the fppf topology on $\mathfrak{Sch}_S$. Then, for such an involution we define the *weak obstruction* $\omega(\mathcal{A},\sigma)\in H^1(S,\mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma})$, and the *strong obstruction* $\Omega(\mathcal{A},\sigma)\in H^1(S,\mathcal{S}kew_{\mathcal{A},\sigma})$. These obstructions encode whether $(\mathcal{A},\sigma)$ can be equipped with a quadratic pair, and if it can, whether that pair can be described by a single element in $\mathcal{A}(S)$. More precisely, we show in Theorem 6.8:

Theorem. *Let $(\mathcal{A},\sigma)$ be an Azumaya $\mathcal{O}$–algebra with a locally quadratic involution.*

(i) *There exists a linear map $f\colon \mathcal{A}\to\mathcal{O}$ such that $(\mathcal{A},\sigma,f|_{\mathcal{S}ym_{\mathcal{A},\sigma}})$ is a quadratic triple if and only if $\Omega(\mathcal{A},\sigma)=0$. In this case $f=\mathrm{Trd}_{\mathcal{A}}(\ell\_\_)$ for an element $\ell\in\mathcal{A}(S)$ with $\ell+\sigma(\ell)=1$.*

(ii) *There exists a linear map $f\colon \mathcal{S}ym_{\mathcal{A},\sigma}\to\mathcal{O}$ such that $(\mathcal{A},\sigma,f)$ is a quadratic triple if and only if $\omega(\mathcal{A},\sigma)=0$.*

These obstructions capture global phenomenon, since locally, for instance over an affine scheme, both obstructions are zero. Section 6 concludes by discussing whether a given quadratic triple $(\mathcal{A},\sigma,f)$ can be described as $f=\mathrm{Trd}_{\mathcal{A}}(\ell\_\_)$ for an element $\ell\in\mathcal{A}(S)$. This is also detected with a cohomological obstruction $c(f)\in H^1(S,\mathcal{A}lt_{\mathcal{A},\sigma})$, and the element ℓ exists when $c(f)=0$. Finally, section 7 contains various examples of Azumaya algebras with locally quadratic involutions such that their obstructions are non-trivial. In section 7.1 we construct an Azumaya algebra with quadratic pair, and therefore trivial weak obstruction, which we show in Lemma 7.2 has non-trivial strong obstruction. Additionally, in section 7.4 we construct a quaternion algebra with orthogonal involution which by Lemma 7.5 has non-trivial weak obstruction, and therefore also non-trivial strong obstruction.

1. Preliminaries

Throughout this paper S is a scheme with structure sheaf $\mathcal{O}_S$. Following the style of [CF], we consider "objects over S" as sheaves on the category of schemes over S equipped with the fppf topology. Below, we explain this viewpoint for the notions that are most important for this paper.

1.1. **The Ringed Site $(\mathfrak{Sch}_S, \mathcal{O})$.** Let $\mathfrak{Sch}_S$ be the big fppf site of S as in [SGA3, Exposé IV]. By a *cover* $\{T_i \to T\}_{i\in I}$ of some $T \in \mathfrak{Sch}_S$, we mean a cover of the fppf site. When considering a cover, we will use the notation $T_{ij} = T_i \times_T T_j$ for $i, j \in I$. When a scheme is affine we will usually denote it by U, and so an affine cover will usually be written $\{U_i \to T\}_{i\in I}$ with $U_{ij} = U_i \times_T U_j$ for $i, j \in I$. We warn that U_{ij} may not be affine since T need not be separated.

1.2. **Remark.** In [St, Tag 021S], the Stacks project authors introduce "a" big fppf site of S, instead of "the". They do so because of set theoretic nuances avoided in [SGA3] through the use of universes. We also avoid such difficulties and work with "the" big fppf site.

If $\mathcal{F}$ is any sheaf on $\mathfrak{Sch}_S$ and $T \in \mathfrak{Sch}_S$, we denote by $\mathcal{F}|_T$ the restriction of the sheaf to the site $\mathfrak{Sch}_S/T = \mathfrak{Sch}_T$. We use the same notation for the restriction of elements, namely if $f\colon V \to T$ is a morphism in $\mathfrak{Sch}_S$ and $t \in \mathcal{F}(T)$, then we write $t|_V = \mathcal{F}(f)(t)$. This overlap of notation is borrowed from [St]. Since these operations apply only to sheaves or sections respectively, the meaning of restriction will be clear from the context.

By [St, Tags 03O4, 03O5], the contravariant functor

$$\mathcal{O}\colon \mathfrak{Sch}_S \to \mathfrak{Rings}$$
$$T \mapsto \mathcal{O}_T(T)$$

where $\mathfrak{Rings}$ is the category of commutative rings, is an fppf-sheaf on $\mathfrak{Sch}_S$, making $(\mathfrak{Sch}_S, \mathcal{O})$ a ringed site in the sense of [St, Tag 03AD]. We call $\mathcal{O}$ the *structure sheaf.* If $T \to S$ is an open immersion, then $\mathcal{O}(T) = \mathcal{O}_S(T)$.

1.3. **$\mathcal{O}$–Modules.** From [St, Tag 03CW], an $\mathcal{O}$–module is then an fppf-sheaf $\mathcal{M}\colon \mathfrak{Sch}_S \to \mathfrak{Ab}$ together with a map of sheaves

$$\mathcal{O} \times \mathcal{M} \to \mathcal{M}$$

such that for each $T \in \mathfrak{Sch}_S$, $\mathcal{O}(T) \times \mathcal{M}(T) \to \mathcal{M}(T)$ makes $\mathcal{M}(T)$ a usual $\mathcal{O}(T)$–module. Given two $\mathcal{O}$–modules $\mathcal{M}$ and $\mathcal{N}$, their internal homomorphism functor

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N})\colon \mathfrak{Sch}_S \to \mathfrak{Ab}$$
$$T \mapsto \mathrm{Hom}_{\mathcal{O}|_T}(\mathcal{M}|_T, \mathcal{N}|_T)$$

is another $\mathcal{O}$–module by [St, 03EM].

We refer to [St, Tags 03DE, 03DK] for definitions of various properties of $\mathcal{O}$–modules. In particular, we will make use of modules which are locally free and/or of finite type, therefore we present these definitions for the convenience of the reader.

An $\mathcal{O}$–module $\mathcal{M}$ is called *locally free* if for all $T \in \mathfrak{Sch}_S$, there is a covering $\{T_i \to T\}_{i\in I}$ such that for each $i \in I$, the restriction $\mathcal{M}|_{T_i}$ is a free

$\mathcal{O}|_{T_i}$–module. Explicitly, $\mathcal{M}|_{T_i} \cong \bigoplus_{j\in J_i} \mathcal{O}|_{T_i}$ for some index set J_i. If all J_i have the same cardinality then $\mathcal{M}$ has *constant rank* $|J_i|$. A locally free $\mathcal{O}$–module of constant rank 1 is called a *line bundle*. Isomorphism classes of line bundles form a group under tensor product, denoted $\mathrm{Pic}(S)$. Since the (additive) category of line bundles is equivalent to the (additive) category of $\mathbb{G}_m$–torsors over S [CF, prop. 2.4.3.1], the group $\mathrm{Pic}(S)$ is isomorphic to the group $H^1_{\mathrm{fppf}}(S,\mathbb{G}_m)$ which is also isomorphic to the group $H^1_{\mathrm{Zar}}(S,\mathbb{G}_m)$ because $\mathbb{G}_m$–torsors are locally trivial for the Zariski topology.

An $\mathcal{O}$–module $\mathcal{M}$ is called *of finite type* if for all $T \in \mathfrak{Sch}_S$, there is a covering $\{T_i \to T\}_{i\in I}$ such that for each $i \in I$, the restriction $\mathcal{M}|_{T_i}$ is an $\mathcal{O}|_{T_i}$–module which is generated by finitely many global sections. That is, there exists a surjection of $\mathcal{O}|_{T_i}$–modules $\mathcal{O}^{\oplus n_i} \to \mathcal{M}$ for some $n_i \in \mathbb{N}$.

An $\mathcal{O}$–module $\mathcal{M}$ is called *of finite presentation* if for all $T \in \mathfrak{Sch}_S$, there is a covering $\{T_i \to T\}_{i\in I}$ such that for each $i \in I$, the restriction $\mathcal{M}|_{T_i}$ is an $\mathcal{O}|_{T_i}$–module which has a finite global presentation. That is, there exists an exact sequence of $\mathcal{O}|_{T_i}$–modules

$$\bigoplus_{j\in J_i} \mathcal{O}|_{T_i} \to \bigoplus_{k\in K_i} \mathcal{O}|_{T_i} \to \mathcal{E}|_{T_i} \to 0$$

for some finite index sets J_i and K_i.

We will use the terminology *finite locally free* to mean locally free and of finite type. Since $S \in \mathfrak{Sch}_S$ is a final object, [St, Tag 03DN] applies and it suffices for us to check local conditions, such as the three detailed above or quasi-coherence detailed below, for an fppf-covering of S.

We associate with an endomorphism σ of an $\mathcal{O}$–module $\mathcal{M}$ satisfying $\sigma^2 = \mathrm{Id}_{\mathcal{M}}$ the following $\mathcal{O}$–modules:

$$\begin{aligned} \mathcal{S}ym_{\mathcal{M},\sigma} &= \mathrm{Ker}(\mathrm{Id} - \sigma) \quad (\textit{symmetric elements}),\\ \mathcal{A}lt_{\mathcal{M},\sigma} &= \mathrm{Im}(\mathrm{Id} - \sigma) \quad (\textit{alternating elements}),\\ \mathcal{S}kew_{\mathcal{M},\sigma} &= \mathrm{Ker}(\mathrm{Id} + \sigma) \quad (\textit{skew-symmetric elements}),\\ \mathcal{S}ymd_{\mathcal{M},\sigma} &= \mathrm{Im}(\mathrm{Id} + \sigma) \quad (\textit{symmetrized elements}), \end{aligned}$$

where $\mathrm{Im}(\_\_)$ is the image fppf-sheaf.

1.4. **Quasi-coherent Modules.** From [St, Tag 03DK], an $\mathcal{O}$–module $\mathcal{E}$ is called *quasi-coherent* if for all $T \in \mathfrak{Sch}_S$ there is a covering $\{T_i \to T\}_{i\in I}$ such that for each $i \in I$, $\mathcal{E}|_{T_i}$ has a global presentation. That is there is an exact sequence of $\mathcal{O}|_{T_i}$–modules

$$\bigoplus_{j\in J_i} \mathcal{O}|_{T_i} \to \bigoplus_{k\in K_i} \mathcal{O}|_{T_i} \to \mathcal{E}|_{T_i} \to 0$$

for some index sets J_i and K_i. Thanks to [St, Tag 03OJ], in our context the classical notion of a quasi-coherent sheaf on S and the notion of a quasi-coherent $\mathcal{O}$–module are essentially equivalent: given a quasi-coherent $\mathcal{O}$–module $\mathcal{E}$, there exists a classical quasi-coherent sheaf E on S such that for $T \in \mathfrak{Sch}_S$ with structure morphism $f\colon T \to S$, we have $\mathcal{E}(T) = f^*(E)(T)$, and conversely, any quasi-coherent sheaf E on S gives rise to a unique quasi-coherent $\mathcal{O}$–module $\mathcal{E}$ satisfying $\mathcal{E}(T) = f^*(E)(T)$.

We also consider the dual of the quasi-coherent $\mathcal{O}$–module $\mathcal{E}$, defined to be $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O})$. The functor $\mathcal{E}^\vee$ is represented by the affine S–scheme $\mathbf{V}(\mathcal{E}) = \operatorname{Spec}(\operatorname{Sym}^\bullet(E))$ [EGA-I, I, 9.4.9]. If E is of finite type, then $\mathbf{V}(\mathcal{E})$ is of finite type as an S–scheme (ibid, 9.4.11).

If $\mathcal{E}$ is finite locally free, then so is $\mathcal{E}nd(\mathcal{E}) = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{E}) \cong \mathcal{E}^\vee \otimes_{\mathcal{O}} \mathcal{E}$. As in [EGA-I, I, 9.6.2] we view its associated scheme $\mathbf{V}(\mathcal{E}nd(\mathcal{E}))$ as an S–scheme of unital associative algebras. Moreover, the functor

$$\begin{aligned} \operatorname{GL}(\mathcal{E})\colon \mathfrak{Sch}_S &\to \mathfrak{Grp} \\ T &\mapsto \operatorname{Aut}_{\mathcal{O}|_T}(\mathcal{E}|_T) \end{aligned}$$

is representable by an open S–subscheme of $\mathbf{V}(\mathcal{E}nd(\mathcal{E}))$, denoted $\mathbf{GL}(\mathcal{E})$ (*loc. cit.*, 9.6.4).

Given an $\mathcal{O}$–module $\mathcal{M}$ and $\sigma \in \mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$ of order 2, the $\mathcal{O}$–modules $\mathcal{S}ym_{\mathcal{M},\sigma}, \mathcal{A}lt_{\mathcal{M},\sigma}, \mathcal{S}kew_{\mathcal{M},\sigma}$, and $\mathcal{S}ymd_{\mathcal{M},\sigma}$ are quasi-coherent whenever $\mathcal{M}$ is because then they are kernels and images of a map between quasi-coherent sheaves. This is shown in [GW, 7.19] for quasi-coherent sheaves on S, and the result generalizes to quasi-coherent $\mathcal{O}$–modules by considering an affine cover of S and using the "$\mathcal{E} \sim E$" correspondence above.

1.5. **$\mathcal{O}$–Algebras.** An $\mathcal{O}$–module $\mathcal{B}\colon \mathfrak{Sch}_S \to \mathfrak{nc}\text{-}\mathfrak{Rings}$ from $\mathfrak{Sch}_S$ to the category of all (not necessarily commutative) rings such that each $\mathcal{B}(T)$ is a $\mathcal{O}(T)$–algebra will be called an $\mathcal{O}$–algebra. We call it unital, associative, commutative, etc., if each $\mathcal{B}(T)$ is so. If $\mathcal{B}$ is a unital associative $\mathcal{O}$–algebra which is finite locally free, then the functor of invertible elements

$$\begin{aligned} \mathbf{GL}_{1,\mathcal{B}}\colon \mathfrak{Sch}_S &\to \mathfrak{Grp} \\ T &\mapsto \mathcal{B}(T)^\times \end{aligned}$$

is representable by an affine S-group scheme [CF, 2.4.2.1]. A section $u \in \mathbf{GL}_{1,\mathcal{B}}(S)$ induces an inner automorphism of $\mathcal{B}$, denoted $\operatorname{Inn}(u)$ which is given on $\mathcal{B}(T)$ by $b \mapsto u|_T \cdot b \cdot u|_T^{-1}$. If $\mathcal{B}$ is a separable $\mathcal{O}$–algebra which is locally free of finite type, then $\mathbf{GL}_{1,\mathcal{B}}$ is a reductive S–group scheme [CF, 3.1.0.50]. We also use the notation $\mathbf{GL}_{1,\mathcal{O}} = \mathcal{O}^\times = \mathbb{G}_m$. In addition, we have the functor of norm 1 elements

$$\begin{aligned} \mathbf{SL}_{1,\mathcal{B}}\colon \mathfrak{Sch}_S &\to \mathfrak{Grp} \\ T &\mapsto \{x \in \mathcal{B}(T)^\times \mid \operatorname{Nrd}_{\mathcal{B}}(x) = 1\}, \end{aligned}$$

which is representable by a closed subgroup scheme of $\mathbf{GL}_{1,\mathcal{B}}$.

1.6. **Azumaya Algebras.** A key object of interest is that of an Azumaya $\mathcal{O}$–algebra. First, we recall that over a commutative ring R, an Azumaya

R–algebra is a central separable R–algebra. Equivalently, the sandwich map

$$\begin{aligned} \mathrm{Sand}\colon A \otimes_R A &\to \mathrm{End}_R(A) \\ a \otimes b &\mapsto (x \mapsto axb) \end{aligned}$$

is an isomorphism. For separable and Azumaya algebras over rings (equivalently, over affine schemes) we refer to [Fo, KO]. Following [Gro, 5.1], we consider an *Azumaya $\mathcal{O}$–algebra*, or simply an *Azumaya algebra*, to be a finite locally free $\mathcal{O}$–algebra $\mathcal{A}$ such that the sandwich morphism

$$\mathrm{Sand}\colon\ \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{O}}(\mathcal{A})$$

is an isomorphism. This is equivalent to the definition of [CF, 2.5.3.4], which asks that $\mathcal{A}$ be finite locally free and that for all affine schemes $U \in \mathfrak{Sch}_S$, we have that $\mathcal{A}(U)$ is an Azumaya $\mathcal{O}(U)$–algebra. Therefore, regarding Azumaya algebras over arbitrary schemes, we may follow [CF, 2.5.3]. By [CF, 2.5.3.6] the rank of an Azumaya algebra is always square. Following [CF, 2.5.3.7] we call the square root of an Azumaya algebra's rank the *degree* of the algebra. It is a locally constant integer valued function.

Azumaya algebras of the form $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$, where $\mathcal{M}$ is a locally free $\mathcal{O}$–module of finite positive rank, will be called *neutral* algebras.

1.7. **Morita Correspondence.** We review Morita theory in the setting of this paper, following [KS, §19.5]. Over affine schemes, this is classical, see for example [Bas, II] or [B:A3, §6].

The following are equivalent for $\mathcal{O}$–algebras $\mathcal{A}$ and $\mathcal{B}$ and an $(\mathcal{A},\mathcal{B})$–bimodule $\mathcal{P}$:

(i) $\mathcal{P}$ is faithfully flat and of finite presentation as an $\mathcal{A}$–module and $\mathcal{B}^{\mathrm{op}} \xrightarrow{\sim} \mathcal{E}nd_{\mathcal{A}}(\mathcal{P})$ under the canonical map.
(ii) $\mathcal{P}$ is faithfully flat and of finite presentation as a $\mathcal{B}$–module and $\mathcal{A} \xrightarrow{\sim} \mathcal{E}nd_{\mathcal{B}^{\mathrm{op}}}(\mathcal{P})$ under the canonical map.
(iii) There exists a $(\mathcal{B},\mathcal{A})$–bimodule $\mathcal{Q}$ such that $\mathcal{P} \otimes_{\mathcal{O}} \mathcal{Q} \cong \mathcal{A}$ and $\mathcal{Q} \otimes_{\mathcal{O}} \mathcal{P} \cong \mathcal{B}$ as $(\mathcal{A},\mathcal{A})$– respectively $(\mathcal{B},\mathcal{B})$–bimodule.
(iv) For the $(\mathcal{B},\mathcal{A})$–bimodule $\mathcal{Q}_0 = \mathcal{H}om_{\mathcal{A}}(\mathcal{P},\mathcal{A})$, the canonical morphisms $\mathcal{P} \otimes_{\mathcal{O}} \mathcal{Q}_0 \xrightarrow{\sim} \mathcal{A}$ and $\mathcal{Q}_0 \otimes_{\mathcal{O}} \mathcal{P} \xrightarrow{\sim} \mathcal{B}$ are $(\mathcal{A},\mathcal{A})$– respectively $(\mathcal{B},\mathcal{B})$–bimodule isomorphisms.
(v) $\mathcal{M} \mapsto \mathcal{P} \otimes_{\mathcal{B}} \mathcal{M}$ induces an equivalence $\mathcal{B}$-mod $\xrightarrow{\sim}$ $\mathcal{A}$-mod of stacks.
(vi) $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{B}} \mathcal{P}$ induces an equivalence mod-$\mathcal{A}$ $\xrightarrow{\sim}$ mod-$\mathcal{B}$ of stacks.
(vii) $\mathcal{M} \mapsto \mathcal{H}om_{\mathcal{A}}(\mathcal{P},\mathcal{M})$ induces an equivalence $\mathcal{A}$-mod $\xrightarrow{\sim}$ $\mathcal{B}$-mod of stacks.

In case (i), we have $\mathcal{Q} \cong \mathcal{H}om_{\mathcal{A}}(\mathcal{P},\mathcal{A}) \cong \mathcal{H}om_{\mathcal{B}}(\mathcal{P},\mathcal{B})$ as $(\mathcal{A},\mathcal{B})$–bimodules. Moreover, any equivalence mod-$\mathcal{A}$ $\xrightarrow{\sim}$ mod-$\mathcal{B}$ of stacks has the from (vi) for some $(\mathcal{A},\mathcal{B})$–bimodule satisfying (i); similarly for (vii). Following the classical terminology, we call $\mathcal{A}$ and $\mathcal{B}$ *Morita-equivalent* if the conditions (i)– (vii) hold, and we call $\mathcal{P}$ satisfying (i) an *invertible* $(\mathcal{A},\mathcal{B})$–bimodule; analogously for $\mathcal{Q}$.

Example. Let $\mathcal{P}$ be a faithfully flat $\mathcal{O}$–module of finite presentation. Then $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}}(\mathcal{P})$ and $\mathcal{B} = \mathcal{O}$ are Morita-equivalent, for example by (ii).

2. Isomorphisms of Azumaya Algebras

2.1. **The Skolem-Noether Theorem.** We globalize here [KO, §IV.3]. Let $\mathcal{A}$ be an Azumaya $\mathcal{O}$–algebra, the map $\mathbf{GL}_{1,\mathcal{A}} \to \mathbf{Aut}_{\mathcal{A}}$ defined by inner automorphisms induce an isomorphism $\mathbf{GL}_{1,\mathcal{A}}/\mathbb{G}_m \xrightarrow{\sim} \mathbf{Aut}_{\mathcal{A}}$ [CF, prop. 2.4.4.3] hence an exact sequence of S–group schemes

$$1 \to \mathbb{G}_m \to \mathbf{GL}_{1,\mathcal{A}} \to \mathbf{Aut}_{\mathcal{A}} \to 1.$$

2.2. **Lemma.** *Assume that $\mathcal{A}$ is of degree d. Then the sequence above fits in an exact commutative diagram of exact sequences of S–group schemes*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu}_d & \longrightarrow & \mathbf{SL}_{1,\mathcal{A}} & \longrightarrow & \mathbf{Aut}_{\mathcal{A}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbf{GL}_{1,\mathcal{A}} & \longrightarrow & \mathbf{Aut}_{\mathcal{A}} & \longrightarrow & 1. \end{array}$$

Proof. The commutativity is clear so the statement is the fact that the top horizontal sequence is exact as sequence of flat sheaves over S. This is established in [CF, Rem. 3.5.0.94]. □

By taking the S–points we get an exact sequence of pointed sets

$$1 \to \mathbb{G}_m(S) \to \mathbf{GL}_{1,\mathcal{A}}(S) \to \mathbf{Aut}_{\mathcal{A}}(S) \xrightarrow{\mathfrak{I}} \mathrm{Pic}(S) \to H^1_{\mathrm{fppf}}(S, \mathbf{GL}_{1,\mathcal{A}})$$

(note $\mathrm{Pic}(S) = H^1_{\mathrm{fppf}}(S, \mathbb{G}_m)$) where $\mathfrak{I}$ is the characteristic map [Gir, III, 3.4.3], described in [RZ, Theorem 7] in a concrete way. This description is as follows.

2.3. **Proposition** (Skolem-Noether)**.** *Let $\mathcal{A}$ be an Azumaya $\mathcal{O}$–algebra, and let φ be an automorphism of $\mathcal{A}$. We define an $\mathcal{O}$–module $\mathcal{I}_\varphi$ by setting*

$$\mathcal{I}_\varphi(T) = \{x \in \mathcal{A}(T) \mid \varphi(a)x|_V = x|_V a \text{ for all } V \in \mathfrak{Sch}_T \text{ and } a \in \mathcal{A}(V)\}$$

for $T \in \mathfrak{Sch}_S$, with the natural restriction maps. Then $\mathcal{I}_\varphi$ is an invertible $\mathcal{O}$–module, i.e., a locally projective $\mathcal{O}$–module of rank 1. Moreover, the following hold.

(i) *φ is an inner automorphism if and only if $\mathcal{I}_\varphi$ is free.*
(ii) *If ψ is another automorphism of $\mathcal{A}$, then $\mathcal{I}_{\psi\circ\varphi} \cong \mathcal{I}_\psi \otimes_{\mathcal{O}} \mathcal{I}_\varphi$.*
(iii) *The characteristic map $\mathfrak{I}\colon \mathbf{Aut}_{\mathcal{A}}(S) \to \mathrm{Pic}(S)$ is given by $\mathfrak{I}(\psi) = \mathcal{I}_\psi$.*
(iv) *Denoting by $\mathrm{Inn}(\mathcal{A})$ the inner automorphism group sheaf of $\mathcal{A}$, the characteristic map $\mathfrak{I} : \varphi \mapsto \mathcal{I}_\varphi$ gives rise to an embedding*

$$\mathbf{Aut}_{\mathcal{A}}(S)/\mathrm{Inn}_{\mathcal{A}}(S) \xrightarrow{\bar{\mathfrak{I}}} \mathrm{Pic}(S)$$

and $\mathrm{Im}(\bar{\mathfrak{I}}) = \{[I] \in \mathrm{Pic}(S) : \mathcal{A} \otimes_{\mathcal{O}} I \cong \mathcal{A}$ as left $\mathcal{A}$–modules$\}$.
(v) *If $\mathcal{A}$ has constant degree d, the automorphism φ^d is inner.*

Proof. Since $\mathcal{A}$ is quasi-coherent, over an affine scheme $U \in \mathfrak{Sch}_S$ we have that

$$\mathcal{I}_\varphi(U) = \{x \in \mathcal{A}(U) \mid \varphi(a)x = xa \text{ for all } a \in \mathcal{A}(U)\}.$$

It is shown in [RZ, §3] that $\mathcal{I}_\varphi(U)$ is a line bundle over U, and therefore by taking an affine cover of S we see that $\mathcal{I}_\varphi$ is indeed a line bundle.

To see (i) we adapt the proof of [RZ, §3 Lemma 5] to our context. Not many changes are needed. First, assume that φ is an inner automorphism corresponding to a section $u \in \mathcal{A}(S)^\times$. Then, for some $T \in \mathfrak{Sch}_S$ we have

$$\begin{aligned}\mathcal{I}_\varphi(T) &= \{x \in \mathcal{A}(T) \mid u|_V a u|_V^{-1} x|_V = x|_V a,\ \forall\, V \in \mathfrak{Sch}_T, a \in \mathcal{A}(V)\}\\ &= \{x \in \mathcal{A}(T) \mid a(u^{-1}x)|_V = (u^{-1}x)|_V a,\ \forall\, V \in \mathfrak{Sch}_T, a \in \mathcal{A}(V)\}\end{aligned}$$

Hence if $x \in \mathcal{I}_\varphi(T)$, then $u^{-1}x \in \mathcal{O}(T)$. In particular, this gives an isomorphism $\mathcal{I}_\varphi \xrightarrow{\sim} \mathcal{O}$ by sending $x \mapsto u^{-1}x$, and so we see $\mathcal{I}_\varphi$ is a free $\mathcal{O}$–module. Conversely, assume that $\mathcal{I}_\varphi$ is a free $\mathcal{O}$–module, generated over $\mathcal{O}$ by an element $u \in \mathcal{I}_\varphi(S) \subset \mathcal{A}(S)$. The map $\mathcal{I}_\varphi \otimes_{\mathcal{O}} \mathcal{A} \to \mathcal{A}$ given by $i \otimes a \mapsto ia$ is locally an isomorphism as per [RZ], and therefore is an isomorphism. Hence we have

$$u \cdot \mathcal{A}(S) = \mathcal{I}_\varphi(S)\,\mathcal{A}(S) = \mathcal{A}(S),$$

and so $u \in \mathcal{A}(S)^\times$. Then, by the definition of $\mathcal{I}_\varphi$, for $T \in \mathfrak{Sch}_S$ and a section $a \in \mathcal{A}(T)$,

$$\varphi(a)u = ua \;\Rightarrow\; \varphi(a) = uau^{-1}$$

and so φ is the inner automorphism associated to u.

The statements (ii), (iii), and the first claim of (iv) are shown over an affine scheme in [KO, IV,§1,1.2], and therefore they hold globally as well. To check the second claim of (iv), we note that as shown above there is an isomorphism $\mathcal{I}_\varphi \otimes_{\mathcal{O}} \mathcal{A} \to \mathcal{A}$ and so

$$\mathrm{Im}(\bar{\mathfrak{I}}) \subseteq \{[I] \in \mathrm{Pic}(S) : \mathcal{A} \otimes_{\mathcal{O}} I \cong \mathcal{A} \text{ as left } \mathcal{A}\text{–modules}\}.$$

To show the reverse inclusion, we adapt the construction from [KO, IV,§1,1.2] to our setting. Again, no fundamental changes are needed. Let I be a line bundle such that there is an isomorphism $f\colon \mathcal{A} \otimes_{\mathcal{O}} I \to \mathcal{A}$ of left $\mathcal{A}$–modules. Since there is an isomorphism

$$\begin{aligned}\gamma\colon \mathcal{A} &\xrightarrow{\sim} \mathcal{E}nd_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{O}} I, \mathcal{A} \otimes_{\mathcal{O}} I)^{\mathrm{op}}\\ a &\mapsto (\gamma_a\colon x \otimes i \mapsto xa \otimes i),\end{aligned}$$

we may make the following definition. For $T \in \mathfrak{Sch}_S$ and $b \in \mathcal{A}(T)$, the map

$$\begin{aligned}(\mathcal{A} \otimes_{\mathcal{O}} I)|_T &\to (\mathcal{A} \otimes_{\mathcal{O}} I)|_T\\ x &\mapsto f^{-1}(f(x)b)\end{aligned}$$

is left $\mathcal{A}\,|_T$–linear, and so there exists $\varphi(b) \in \mathcal{A}(T)$ such that $f^{-1}(f(x)b) = \gamma_{\varphi(b)}(x)$. This defines an automorphism $\varphi\colon \mathcal{A} \xrightarrow{\sim} \mathcal{A}$. It is then straightforward to check that f induces an inclusion

$$f\colon I \cong 1 \otimes_{\mathcal{O}} I \to \mathcal{I}_\varphi,$$

which by [KO] is an isomorphism over affine schemes, and hence is an isomorphism. Therefore $[I] = [\mathcal{I}_\varphi] \in \operatorname{Im}(\bar{\mathfrak{I}})$ as desired.

Over an affine scheme, (v) is proven in [KO, IV, 3.1]. Here we give a general cohomological proof. Assume furthermore that $\mathcal{A}$ is of degree d and consider the commutative diagram of exact sequences of S–group schemes from Lemma 2.2,

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\mu}_d & \longrightarrow & \mathbf{SL}_{1,\mathcal{A}} & \longrightarrow & \mathbf{Aut}_{\mathcal{A}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbf{GL}_{1,\mathcal{A}} & \longrightarrow & \mathbf{Aut}_{\mathcal{A}} & \longrightarrow & 1.
\end{array}$$

For $f \in \mathbf{Aut}_{\mathcal{A}}(S)$, we get an obstruction $\varphi_d(f) \in H^1_{\mathrm{fppf}}(S, \boldsymbol{\mu}_d)$ such that $\varphi(f)$ is the image of $\varphi_d(f)$ under the group homomorphism $H^1_{\mathrm{fppf}}(S, \boldsymbol{\mu}_d) \to H^1_{\mathrm{fppf}}(S, \mathbb{G}_m)$. We then obtain an embedding

$$\mathbf{Aut}_{\mathcal{A}}(S)/\operatorname{Inn}_{\mathcal{A}}(S) \hookrightarrow {}_d\operatorname{Pic}(S),$$

where ${}_d\operatorname{Pic}(S)$ denotes the d–torsion elements in $\operatorname{Pic}(S)$. Therefore $\varphi^d \mapsto 0 \in {}_d\operatorname{Pic}(S)$ and hence is an inner automorphism as desired. □

In particular, if $\mathcal{A}$ is of constant degree d and ${}_d\operatorname{Pic}(S) = 0$, we see that all automorphisms of $\mathcal{A}$ are inner. In the general case, since any Azumaya algebra is locally of constant degree, we have the following corollary.

2.4. **Corollary.** *If* $\operatorname{Pic}(S)$ *is torsion free, then all automorphisms of* $\mathcal{A}$ *are inner.*

For neutral Azumaya algebras, Proposition 2.3 can be generalized to a description of isomorphisms.

2.5. **Lemma** (Isomorphisms of neutral Azumaya algebras)**.** *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be locally free* $\mathcal{O}$*–modules of finite positive rank. Put* $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$*, a neutral Azumaya* $\mathcal{O}$*–algebra.*

(a) *Let* $\mathcal{L}$ *be an invertible* $\mathcal{O}$*–module and let* $\rho\colon \mathcal{M} \otimes_{\mathcal{O}} \mathcal{L} \xrightarrow{\sim} \mathcal{N}$ *be an* $\mathcal{O}$*–module isomorphism.*

(i) *Then the map* $\varphi\colon \mathcal{E}nd_{\mathcal{O}}(\mathcal{M}) \xrightarrow{\sim} \mathcal{E}nd_{\mathcal{O}}(\mathcal{N})$*, given by*

$$\varphi(a) = \rho \circ (a \otimes \mathrm{Id}_{\mathcal{L}}) \circ \rho^{-1}, \tag{2.5.1}$$

is an isomorphism of Azumaya $\mathcal{O}$*–algebras.*

(ii) *Letting* $\mathcal{A}$ *act canonically on the first factor of* $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}$ *and on* $\mathcal{N}$ *through* φ*, the map* ρ *becomes an isomorphism of* $\mathcal{A}$*–algebras.*

(iii) *If $(\mathcal{L}', \rho')$ is another pair as in the assumption of (a), then $\mathcal{L} \cong \mathcal{L}'$ as $\mathcal{O}$–modules.*

(b) *Conversely, let $\varphi\colon \mathcal{E}nd_{\mathcal{O}}(\mathcal{M}) \xrightarrow{\sim} \mathcal{E}nd_{\mathcal{O}}(\mathcal{N})$ be an isomorphism of Azumaya $\mathcal{O}$–algebras.*

(i) *Denoting by ${}_{\varphi}\mathcal{N}$ the left $\mathcal{A}$–module obtained from $\mathcal{N}$ by letting $\mathcal{A}$ act through φ, the $\mathcal{O}$–module $\mathcal{L}_{\varphi} = \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, {}_{\varphi}\mathcal{N})$ is invertible.*

(ii) *Letting $\mathcal{A}$ act on the first factor of $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}$, the natural evaluation map*

$$\rho_{\varphi}\colon\ \mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}_{\varphi} \xrightarrow{\sim} {}_{\varphi}\mathcal{N}$$

is an isomorphism of $\mathcal{A}$–modules.

(iii) *The isomorphism φ satisfies (2.5.1) with respect to the pair $(\mathcal{L}_{\varphi}, \rho_{\varphi})$.*

Proof. In (a) the parts (i) and (ii) are obvious. In (iii) observe that $\rho^{-1} \circ \rho'$ is an $\mathcal{A}$–module isomorphism of $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}$. Hence, by Morita correspondence as in section 1.7, the $\mathcal{O}$–modules $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic.

As before in Proposition 2.3, the proof of (b) easily reduces to the case $S = \operatorname{Spec}(R)$ and faithfully flat and finitely presented R–modules M and N. In this case, the lemma is for example shown in [Knu2, III, 8.2.1]. We include a proof with some more details than in [Knu2].

Morita theory implies that $\mathcal{M}$ and ${}_{\varphi}\mathcal{N}$ are invertible $(\mathcal{A}, \mathcal{O})$–bimodules, while $\mathcal{M}^* = \operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ is an invertible $(\mathcal{O}, \mathcal{A})$–bimodule. Hence $\mathcal{L}_{\varphi} \cong \mathcal{M}^* \otimes_{\mathcal{A}} ({}_{\varphi}\mathcal{N})$ is an invertible $(\mathcal{O}, \mathcal{O})$–bimodule, which is the same as an invertible $\mathcal{O}$–module. The map ρ_{φ} is the composition of the canonical isomorphisms $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}^* \otimes_{\mathcal{A}} ({}_{\varphi}\mathcal{N}) \cong \mathcal{A} \otimes_{\mathcal{A}} ({}_{\varphi}\mathcal{N}) \cong {}_{\varphi}\mathcal{N}$. □

Remark. In case $\mathcal{M} = \mathcal{N}$, the invertible $\mathcal{O}$–module $\mathcal{L}_{\varphi}$ becomes the invertible $\mathcal{O}$–module $\mathcal{I}_{\varphi}$ of Proposition 2.3.

3. Line Bundle-valued Bilinear and Quadratic Forms

Here we recount the needed pieces of the theory of bilinear and quadratic forms with values in line bundles from [A]. Readers will recognize the usual theory of bilinear forms and quadratic forms in what follows in the case that the line bundle involved is simply $\mathcal{O}$.

3.1. **$\mathcal{L}$–duality.** Let $\mathcal{L}$ be a line bundle on $\mathcal{O}$, i.e., an invertible $\mathcal{O}$–module or, equivalently, a locally projective $\mathcal{O}$–module of constant rank 1.

Given a quasi-coherent $\mathcal{O}$–module $\mathcal{M}$, its *$\mathcal{L}$–dual* is $\mathcal{M}^{\vee\mathcal{L}} = \mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{L})$, while the $\mathcal{L}$–dual of a morphism $f\colon\ \mathcal{M} \to \mathcal{N}$ of quasi-coherent $\mathcal{O}$–modules is $f^{\vee\mathcal{L}}\colon\ \mathcal{N}^{\vee\mathcal{L}} \to \mathcal{M}^{\vee\mathcal{L}}$, given on sections by $g \mapsto g \circ f$. The assignments $\mathcal{M} \mapsto \mathcal{M}^{\vee\mathcal{L}}$ and $f \mapsto f^{\vee\mathcal{L}}$ define a contravariant functor $\mathcal{H}om_{\mathcal{O}}(\_\_, \mathcal{L})$ on the category of quasi-coherent $\mathcal{O}$–modules, called *$\mathcal{L}$–duality*.

The canonical evaluation map $\operatorname{ev}^{\mathcal{L}}\colon\ \mathcal{M} \to (\mathcal{M}^{\vee\mathcal{L}})^{\vee\mathcal{L}}$ gives rise to a natural transformation of functors, $\operatorname{ev}^{\mathcal{L}}\colon \operatorname{Id} \longrightarrow ((\_\_)^{\vee\mathcal{L}})^{\vee\mathcal{L}}$ which is an auto-equivalence on the subcategory of finite locally free $\mathcal{O}$–modules. That this is an auto-equivalence can be justified locally where $\mathcal{L}$ becomes trivial and the $\mathcal{L}$–dual becomes the usual dual.

3.2. **Line Bundle-valued Bilinear Forms.** A *line bundle-valued bilinear form* is a triple $(\mathcal{M}, b, \mathcal{L})$ consisting of a locally projective $\mathcal{O}$–module $\mathcal{M}$ of finite type, a line bundle $\mathcal{L}$ over $\mathcal{O}$, and a bilinear $\mathcal{O}$–module morphism $b\colon \mathcal{M} \times \mathcal{M} \to \mathcal{L}$. If a confusion is not possible, we will refer to $(\mathcal{M}, b, \mathcal{L})$ as a *bilinear form*. If the line bundle is omitted it is assumed to be $\mathcal{O}$, i.e. a bilinear form $(\mathcal{M}, b)$ is a bilinear $\mathcal{O}$–module morphism $b\colon \mathcal{M} \times \mathcal{M} \to \mathcal{O}$. For $T \in \mathfrak{Sch}_S$ and sections $m_1, m_2 \in \mathcal{M}(T)$, we will usually write $b(m_1, m_2)$ instead of $b(T)(m_1, m_2)$, nor will we specify T.

We associate with a bilinear from $(\mathcal{M}, b, \mathcal{L})$ its *adjoint*

$$\widehat{b}\colon \mathcal{M} \to \mathcal{M}^{\vee\mathcal{L}}, \quad m \mapsto b(m, \_\_).$$

Conversely, any $\mathcal{O}$–linear map $\beta\colon \mathcal{M} \to \mathcal{M}^{\vee\mathcal{L}}$ gives rise to a bilinear map $b\colon \mathcal{M} \times \mathcal{M} \to \mathcal{L}$, defined on sections by $b(m_1, m_2) = \beta(m_1)(m_2)$. We call $(\mathcal{M}, b, \mathcal{L})$ *regular* if $\widehat{b}$ is an isomorphism.

The *opposite* of a bilinear form $(\mathcal{M}, b, \mathcal{L})$ is $(\mathcal{M}, b, \mathcal{L})^{\mathrm{op}} = (\mathcal{M}, b^{\mathrm{op}}, \mathcal{L})$ where b^{op} is defined on sections by $b^{\mathrm{op}}(m_1, m_2) = b(m_2, m_1)$. The adjoints of b and b^{op} are related by

$$\widehat{b^{\mathrm{op}}} = \widehat{b}^{\,\vee\mathcal{L}} \circ \mathrm{ev}^{\vee\mathcal{L}}.$$

In particular, b is regular if and only if b^{op} is regular.

Let $(\mathcal{M}, b, \mathcal{L})$ be a regular bilinear form. It gives rise to an *adjoint anti-homomorphism* η_b of the neutral Azumaya algebra $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$, defined on sections a of $\mathcal{A}$ and m_1, m_2 of $\mathcal{M}$ by

$$b(\eta_b(a)(m_1),\, m_2) = b(m_1,\, a(m_2)).$$

Equivalently, if $a \in \mathcal{A}(T)$ for $T \in \mathfrak{Sch}_S$, then $\eta_b(a)$ is the composition of the maps

$$\mathcal{M}\,|_T \xrightarrow{\widehat{b}|_T} \mathcal{M}^{\vee\mathcal{L}}\,|_T \xrightarrow{a^{\vee\mathcal{L}}} \mathcal{M}^{\vee\mathcal{L}}\,|_T \xrightarrow{\widehat{b}|_T^{-1}} \mathcal{M}\,|_T$$

where $a^{\vee\mathcal{L}}$ is the $\mathcal{L}$–dual endomorphism of a. It satisfies

$$\eta_{b^{\mathrm{op}}} = \eta_b^{\;-1} \tag{3.2.1}$$

and so η_b is an adjoint anti-automorphism.

In fact, there is a bijective correspondence between anti-automorphisms of a neutral Azumaya $\mathcal{O}$–algebra and line bundle-valued regular bilinear forms modulo invertible scalar factors. This is an immediate application of Lemma 2.5, globalizing Saltman's result [Sa, 4.2.(a)] for the affine case, cf. [KPS, Lem. 9] or [Knu2, III, 8.2.2]. We include a proof which is slightly different from the published ones.

3.3. **Lemma** (Anti-automorphisms of neutral Azumaya algebras)**.** *Suppose that $\mathcal{M}$ is a locally free $\mathcal{O}$–module of finite positive rank and set $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$, a neutral Azumaya algebra.*

(i) *Let $\mathcal{L}$ be an invertible $\mathcal{O}$–module and let $(\mathcal{M}, b, \mathcal{L})$ be a regular bilinear form. Then η_b is an anti-automorphism of $\mathcal{A}$. If $(\mathcal{M}, b', \mathcal{L})$*

is another regular bilinear form, then $\eta_b = \eta_{b'}$ if and only if there exists $c \in \mathbb{G}_m(S)$ such that $b = cb'$.

(ii) *Conversely, supposed that σ is a $\mathcal{O}$–linear anti-automorphism of $\mathcal{A}$. Then there exists an invertible $\mathcal{O}$–module $\mathcal{L}$ and a regular bilinear form $(\mathcal{M}, b, \mathcal{L})$ such that $\sigma = \eta_b$ is the adjoint anti-automorphism of b. That is,*

$$b(\sigma(a)(m_1), m_2) = b(m_1, a(m_2)) \tag{3.3.1}$$

holds for all sections $a \in \mathcal{A}$ and $m_1, m_2 \in \mathcal{M}$. Morever, the invertible $\mathcal{O}$–module $\mathcal{L}$ is uniquely determined by (3.3.1) up to $\mathcal{O}$–module isomorphism.

(iii) *The constructions of (i) and (ii) induce a bijection between line bundle-valued regular bilinear forms on $\mathcal{M}$ module invertible scalars and anti-automorphisms of $\mathcal{A}$. Under this bijection, involutions correspond to regular bilinear forms which are symmetric up to a factor in $\boldsymbol{\mu}_2(S)$.*

Proof. (i): We know that η_b is an anti-automorphism from the discussion preceding the lemma. Now, consider the two regular bilinear forms $(\mathcal{M}, b, \mathcal{L})$ and $(\mathcal{M}, b', \mathcal{L})$. Their adjoint anti-automorphisms η_b and $\eta_{b'}$ are related by

$$\eta_b = \operatorname{Inn}(g) \circ \eta_{b'}$$

where $g = \widehat{b}^{-1} \circ \widehat{b}' \in \mathbf{GL}_{1,\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})}(S)$. Thus, for the appropriate sections we have $b'(m_1, m_2) = b(g(m_1), m_2)$. It follows that

$$\eta_b = \eta_{b'} \iff b' = cb \text{ for some } c \in \mathbb{G}_m(S). \tag{3.3.2}$$

(ii): Let $\mathcal{M}^* = \mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{O})$ and denote by $a^* \in \mathcal{E}nd_{\mathcal{O}}(\mathcal{M}^*)$ the dual of $a \in \mathcal{A}$. Since $\tau\colon \mathcal{E}nd_{\mathcal{O}}(\mathcal{M}) \to \mathcal{E}nd_{\mathcal{O}}(\mathcal{M}^*)$, $a \mapsto a^*$, is an anti-isomorphism, the composition $\tau\sigma\colon \mathcal{E}nd_{\mathcal{O}}(\mathcal{M}) \to \mathcal{E}nd_{\mathcal{O}}(\mathcal{M}^*)$ is an isomorphism of Azumaya $\mathcal{O}$–algebras. Applying Lemma 2.5, there exists an invertible $\mathcal{O}$–module $\mathcal{I}$ and an isomorphism $h\colon \mathcal{M} \otimes_{\mathcal{O}} \mathcal{I} \xrightarrow{\sim} \mathcal{M}^*$ of $\mathcal{O}$–modules satisfying

$$\sigma(a) \otimes \operatorname{Id}_{\mathcal{I}} = h^{-1} \circ a^* \circ h$$

for all sections a of $\mathcal{A}$. Tensoring h with $\operatorname{Id}_{\mathcal{I}^*}$, we get the following commutative diagram

$$\begin{array}{ccccccc}
\mathcal{M} & \xleftarrow[\cong]{} & \mathcal{M}\otimes\mathcal{I}\otimes\mathcal{I}^* & \xrightarrow[\cong]{h\otimes \operatorname{Id}_{\mathcal{I}^*}} & \mathcal{M}^*\otimes\mathcal{I}^* & \xrightarrow[\cong]{} & \mathcal{H}om(\mathcal{M},\mathcal{I}^*) \\
\downarrow{\scriptstyle \sigma(a)} & & \downarrow{\scriptstyle \sigma(a)\otimes\operatorname{Id}_{\mathcal{I}}\otimes\operatorname{Id}_{\mathcal{I}^*}} & & \downarrow{\scriptstyle a^*\otimes\operatorname{Id}_{\mathcal{I}^*}} & & \downarrow{\scriptstyle \mathcal{H}om(a,\operatorname{Id}_{\mathcal{I}^*})} \\
\mathcal{M} & \xleftarrow{\cong} & \mathcal{M}\otimes\mathcal{I}\otimes\mathcal{I}^* & \xleftarrow[h^{-1}\otimes\operatorname{Id}_{\mathcal{I}^*}]{\cong} & \mathcal{M}^*\otimes\mathcal{I}^* & \xrightarrow{\cong} & \mathcal{H}om(\mathcal{M},\mathcal{I}^*)
\end{array}$$

where $\otimes = \otimes_{\mathcal{O}}$ and $\mathcal{H}om(a, \operatorname{Id}_{\mathcal{I}^*})\colon f \mapsto f \circ a$. Thus the composition $\beta\colon \mathcal{M} \to \mathcal{H}om(\mathcal{M}, \mathcal{I}^*)$ of the top and second row is an isomorphism of $\mathcal{O}$–modules satisfying $\beta \circ \sigma(a) = \mathcal{H}om(a, \operatorname{Id}_{\mathcal{I}^*}) \circ \beta$. Finally, putting $\mathcal{L} = \mathcal{I}^*$, the bilinear form $b\colon \mathcal{M} \times \mathcal{M} \to \mathcal{L}$, defined on sections m_1, m_2 by $b(m_1, m_2) = \big(\beta(m_1)\big)(m_2)$,

satisfies (3.3.1). The uniqueness assertion follows from uniqueness of $\mathcal{I}$ in Lemma 2.5.
(iii): The bijection induced by (i) and (ii) is clear. Regarding involutions, when we combine (3.2.1) and (3.3.2) we see that

$$\begin{array}{ccc} \eta_b \text{ is an involution,} & \iff & \text{there exists } \varepsilon \in \boldsymbol{\mu}_2(S) \\ \text{i.e., } \eta_b^{-1} = \eta_b & & \text{such that } b^{\mathrm{op}} = \varepsilon b. \end{array} \tag{3.3.3}$$

□

If η_b is an involution, the element $\varepsilon \in \boldsymbol{\mu}_2(S)$ is uniquely determined by η_b, or equivalently by b itself, and called the *type* of b and η_b. Bilinear forms of type 1 are called *symmetric*, and those of type -1 are called *skew-symmetric*. Finally, if $b(m, m) = 0$ for all sections $m \in \mathcal{M}$, then b is called *alternating* and is necessarily also skew-symmetric. See section 3.5 for an extension to involutions on arbitrary Azumaya $\mathcal{O}$–algebras.

3.4. **Line Bundle-valued Quadratic Forms.** A *line bundle-valued quadratic form* is a triple $(\mathcal{M}, q, \mathcal{L})$ consisting of a locally projective $\mathcal{O}$–module $\mathcal{M}$ of finite type, a line bundle $\mathcal{L}$ over $\mathcal{O}$, and a natural transformation $q\colon \mathcal{M} \to \mathcal{L}$ such that

(i) $q(cm) = c^2 q(m)$ for sections $m \in M$ and $c \in \mathcal{O}$,
(ii) $b_q(m_1, m_2) = q(m_1 + m_2) - q(m_1) - q(m_2)$ for sections $m_1, m_2 \in \mathcal{M}$ defines a bilinear form $(\mathcal{M}, b_q, \mathcal{L})$.

The bilinear form b_q is called the *polar* of q, and is a symmetric bilinear form. Again, if confusion is not possible we refer to $(\mathcal{M}, q, \mathcal{L})$ as a *quadratic form*, and we assume $\mathcal{L} = \mathcal{O}$ if the line bundle is omitted. A quadratic form is called *regular* if its polar b_q is regular.

3.5. **Involutions of Azumaya algebras.** We use the conventions of [CF], which are slightly different from those of [KMRT] in the case of base fields. The advantage of Calmès-Fasel's definitions are that they behave well under arbitrary base change.

Let $\mathcal{A}$ be an Azumaya $\mathcal{O}$–algebra. An *involution of the first kind* is an isomorphism $\sigma : \mathcal{A} \to \mathcal{A}^{op}$ of Azumaya algebras which satisfies $\sigma^2 = \mathrm{Id}_{\mathcal{A}}$. We say that two involutions σ and σ' are *conjugate* if there exists $\phi \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{A})$ such that $\sigma' = \phi \circ \sigma \circ \phi^{-1}$. In that case we have $\mathcal{S}ym_{\mathcal{A},\sigma'} = \phi(\mathcal{S}ym_{\mathcal{A},\sigma})$ and similarly for the other $\mathcal{O}$–modules $\mathcal{A}lt_{\mathcal{A},\sigma}$, $\mathcal{S}kew_{\mathcal{A},\sigma}$ and $\mathcal{S}ymd_{\mathcal{A},\sigma}$.

An Azumaya algebra with involution of the first kind $(\mathcal{A}, \sigma)$ is étale-locally isomorphic to $(\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}), \eta_b)$ for some regular bilinear form $(\mathcal{M}, b)$ [CF, 2.7.0.25]. Thus, by (3.3.3), the bilinear form b is ε–symmetric for some $\varepsilon \in \boldsymbol{\mu}_2(S)$. It is known that ε only depends on σ. Following [CF, 2.7.0.26] (and not [KMRT]), we call σ *orthogonal* if $\varepsilon = 1$ (so b is symmetric), we call it *weakly symplectic* if $\varepsilon = -1$ (so b is skew-symmetric), and we call it *symplectic* if $\varepsilon = -1$ and b is alternating. These notions are well-defined, stable under base change, and are local for the fppf topology on $\mathfrak{Sch}_S$.

When $(\mathcal{A},\sigma)$ is a neutral Azumaya algebra, any involution of the first kind induces a decomposition of $\mathcal{A}$ as an $\mathcal{O}$–module with respect to which the involution appears as an exchange of tensor factors.

3.6. **Lemma.** *Consider the neutral Azumaya algebra $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$ associated to a locally free $\mathcal{O}$–module $\mathcal{M}$ of finite positive rank. Assume that $\mathcal{A}$ has an involution σ of the first kind. By Lemma 3.3, σ is the adjoint involution of a regular bilinear form $b\colon \mathcal{M}\times\mathcal{M}\to\mathcal{L}$ for an $\mathcal{O}$–line bundle $\mathcal{L}$. Furthermore, for sections $x,y\in\mathcal{M}$ we have*

$$b(x,y) = \varepsilon b(y,x)$$

where $\varepsilon\in\boldsymbol{\mu}_2(S)$ is the type of σ. Then, there is an isomorphism of $\mathcal{O}$–modules with involution,

$$\begin{aligned}\varphi_b\colon (\mathcal{M}\otimes_{\mathcal{O}}\mathcal{M}\otimes_{\mathcal{O}}\mathcal{L}^*,\varepsilon\tau\otimes 1) &\xrightarrow{\sim} (\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}),\sigma)\\ m_1\otimes m_2\otimes \ell^* &\longmapsto \ell^*(b(m_1,\_\_))\cdot m_2.\end{aligned}$$

where $\tau\colon \mathcal{M}\otimes_{\mathcal{O}}\mathcal{M}\to\mathcal{M}\otimes_{\mathcal{O}}\mathcal{M}$ is the switch map $m_1\otimes m_2\mapsto m_2\otimes m_1$.

Proof. We begin with the isomorphism which is the composition of canonical isomorphisms

$$\begin{aligned}\mathcal{M}\otimes_{\mathcal{O}}\mathcal{M}\otimes_{\mathcal{O}}\mathcal{L}^* &\xrightarrow{1\otimes\hat{b}\otimes 1} \mathcal{M}\otimes_{\mathcal{O}}\mathcal{M}^{\vee\mathcal{L}}\otimes_{\mathcal{O}}\mathcal{L}^* \xrightarrow{\sim} \mathcal{M}\otimes_{\mathcal{O}}\mathcal{M}^*\otimes_{\mathcal{O}}\mathcal{L}\otimes_{\mathcal{O}}\mathcal{L}^*\\ &\xrightarrow{\sim} \mathcal{M}\otimes_{\mathcal{O}}\mathcal{M}^*\xrightarrow{\sim}\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}).\end{aligned}$$

Consider a suitably fine fppf cover $\{T_i\to S\}_{i\in I}$ such that $\mathcal{M}|_{T_i}$ and $\mathcal{L}|_{T_i}$ are free. Let $\mathcal{M}|_{T_i}$ have free basis $\{m_1,\dots,m_n\}$ and denote by $\{m_1^*,\dots,m_n^*\}$ the corresponding dual elements. Then, over T_i we can compute

$$\begin{aligned}m_1\otimes m_2\otimes\ell^* &\mapsto m_1\otimes b(m_2,\_\_)\otimes\ell^*\mapsto\sum_{k=1}^n m_1\otimes m_k^*\otimes b(m_2,m_k)\otimes\ell^*\\ &\mapsto m_1\otimes\left(\sum_{k=1}^n \ell^*(b(m_2,m_k))m_k^*\right)\mapsto \ell^*(b(m_2,\_\_))\cdot m_1.\end{aligned}$$

Therefore this map is $m_1\otimes m_2\otimes\ell^*\mapsto\ell^*(b(m_2,\_\_))\cdot m_1$ globally as well. Now we will compose this with σ. Since σ is adjoint to b, we know that for $a\in\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$,

$$\sigma(a) = \hat{b}^{-1}\circ a^{\vee\mathcal{L}}\circ\hat{b}.$$

Hence, for $x\in\mathcal{M}$ we compute

$$\begin{aligned}\sigma\big(\ell^*(b(m_2,\_\_))\cdot m_1\big)(x) &= \hat{b}^{-1}\big(b(x,\_\_)\circ\ell^*(b(m_2,\_\_))\cdot m_1\big)\\ &= \hat{b}^{-1}\big(b(x,\ell^*(b(m_2,\_\_))\cdot m_1)\big)\\ &= \hat{b}^{-1}\big(b(x,m_1)\ell^*(b(m_2,\_\_))\big)\\ &= \hat{b}^{-1}\big(\ell^*(b(x,m_1))b(m_2,\_\_)\big)\end{aligned}$$

since over T_i the action of ℓ^* is scalar multiplication,

$$
\begin{aligned}
&= \hat{b}^{-1}\big(b(\ell^*(b(x,m_1))\cdot m_2, \_\_)\big)\\
&= \ell^*(b(x,m_1))\cdot m_2\\
&= \varepsilon\ell^*(b(m_1,x))\cdot m_2.
\end{aligned}
$$

Thus $\sigma\big(\ell^*(b(m_2,\_\_))\cdot m_1\big) = \varepsilon\ell^*(b(m_1,\_\_))\cdot m_2$. Then composing the above isomorphism with $\varepsilon\sigma$ produces φ_b of the statement, and the calculation above shows that the involutions are also compatible via φ_b as claimed. $\square$

3.7. **Remark.** When the line bundle $\mathcal{L}$ in Lemma 3.6 is trivial, the isomorphism of $\mathcal{O}$–modules with involution is simply

$$
\begin{aligned}
\varphi_b\colon (\mathcal{M}\otimes_{\mathcal{O}}\mathcal{M}, \varepsilon\tau) &\xrightarrow{\sim} (\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}), \sigma)\\
m_1\otimes m_2 &\mapsto b(m_1,\_\_)m_2.
\end{aligned}
$$

Over affine schemes the lemma becomes a statement regarding Azumaya algebras over rings. In this setting, it has for example been proven in [Knu2, III,(8.2.2)] and over fields in [KMRT, 5.1].

Let $(\mathcal{A},\sigma)$ be an Azumaya $\mathcal{O}$–algebra with orthogonal involution. We mention some basic facts about the submodules of $\mathcal{A}$ related to σ. We call a submodule $\mathcal{M}\subset\mathcal{A}$ a *direct summand* of $\mathcal{A}$ if there exists another module $\mathcal{N}$ such that $\mathcal{A}=\mathcal{M}\oplus\mathcal{N}$. We say that $\mathcal{M}$ is *locally a direct summand* if there exists a cover $\{T_i\to S\}_{i\in I}$ such that each $\mathcal{M}|_{T_i}$ is a direct summand of $\mathcal{A}|_{T_i}$. Some of the results of Lemma 3.8 below are proven in [CF, 2.7.0.29(1), (2)]) under the assumption that b is the polar of a regular quadratic form, an assumption that is not needed for the result below.

3.8. **Lemma.** *Let $(\mathcal{A},\sigma)$ be an Azumaya $\mathcal{O}$–algebra with orthogonal involution σ.*

(i) *If S is affine, then $\mathcal{S}ym_{\mathcal{A},\sigma}$ and $\mathcal{A}lt_{\mathcal{A},\sigma}$ are direct summands of $\mathcal{A}$. In particular, $\mathcal{S}ym_{\mathcal{A},\sigma}(S)$ and $\mathcal{A}lt_{\mathcal{A},\sigma}(S)$ are direct summands of $\mathcal{A}(S)$ and hence are finite projective $\mathcal{O}(S)$–modules.*

(ii) *$\mathcal{S}ym_{\mathcal{A},\sigma}$ and $\mathcal{A}lt_{\mathcal{A},\sigma}$ are locally direct summands of $\mathcal{A}$ and hence are finite locally free $\mathcal{O}$–modules.*

(iii) *If $\mathcal{A}$ is locally free of constant rank $n^2\in\mathbb{N}_+$, then $\mathcal{S}ym_{\mathcal{A},\sigma}$ and $\mathcal{A}lt_{\mathcal{A},\sigma}$ have rank $n(n+1)/2$ and $n(n-1)/2$ respectively.*

Proof. (i): Over an affine scheme we are considering the case of an Azumaya algebra with orthogonal involution (A,σ) over a ring R. Being finite projective can be checked after a faithfully flat extension, and by [EGA-I, 0, 6.7.5] this is also true for the property of being a direct summand. Therefore, by [CF, 2.7.0.25]) we can assume that $(A,\sigma)\cong(\mathrm{End}_R(M),\eta_b)$ where M is free of finite rank and $b\colon M\times M\to R$ is a regular symmetric bilinear form. Thus, as R–modules with involution we have $(A,\sigma)\cong(M^{\otimes 2},\tau)$ where τ is the switch. Let $\{m_1,\dots,m_n\}$ be a basis of M. Then $m_{ij}=m_i\otimes m_j$ is a basis of $M^{\otimes 2}$. It follows that

$$\{m_{ii} : 1 \le i \le n\} \cup \{m_{ij} + m_{ji} : 1 \le i < j \le n\}$$

is a basis of $\mathrm{Sym}(M^{\otimes 2}, \tau)$, while

$$\{m_{ij} - m_{ji} : 1 \le i < j \le n\}$$

is a basis for $\mathrm{Alt}(M^{\otimes 2}, \tau)$. Now, observe that $\mathrm{Sym}(M^{\otimes 2}, \tau)$ is complemented by $\mathrm{Span}_R(\{m_{ij} : 1 \le i < j \le n\})$, while $\mathrm{Alt}(M^{\otimes 2}, \tau)$ is complemented by

$$\mathrm{Span}_R(\{m_{ii} : 1 \le i \le n\} \cup \{m_{ij} : 1 \le i < j \le n\}),$$

and so both are direct summands of A.
(ii): This follows from (i) by considering an affine cover of S.
(iii): If $\mathcal{A}$ is of constant rank n^2, then locally we are in the case of the proof of (i) above where the basis $\{m_1, \dots, m_n\}$ of M has n elements. Therefore, the basis given for $\mathrm{Sym}(A, \sigma)$ has $n(n+1)/2$ elements, and the basis given for $\mathrm{Alt}(A, \sigma)$ has $n(n-1)/2$ elements. Thus the ranks of $\mathcal{S}ym_{\mathcal{A},\sigma}$ and $\mathcal{A}lt_{\mathcal{A},\sigma}$ are as claimed. □

In particular, since the above result tells us they are finite locally free, $\mathcal{S}ym_{\mathcal{A},\sigma}$ and $\mathcal{A}lt_{\mathcal{A},\sigma}$ are represented by vector group schemes. The reader will have noticed that the proof of Lemma 3.8 works for an involution of any type $\varepsilon \in \boldsymbol{\mu}_2(S)$. For example, when σ is weakly symplectic and $\varepsilon = -1$, the regular bilinear form $b\colon M \times M \to R$ will be skew symmetric and induce an isomorphism $(A, \sigma) \cong (M^{\otimes 2}, -\tau)$ of R–modules with involution. In this case, the same roles are played by $\mathcal{S}kew_{\mathcal{A},\sigma}$ and $\mathcal{S}ymd_{\mathcal{A},\sigma}$ instead.

3.9. **Corollary.** *Let $(\mathcal{A}, \sigma)$ be an Azumaya $\mathcal{O}$–algebra with weakly symplectic involution σ.*

(i) *If S is affine, then $\mathcal{S}kew_{\mathcal{A},\sigma}$ and $\mathcal{S}ymd_{\mathcal{A},\sigma}$ are direct summands of $\mathcal{A}$. In particular, $\mathcal{S}kew_{\mathcal{A},\sigma}(S)$ and $\mathcal{S}ymd_{\mathcal{A},\sigma}(S)$ are direct summands of $\mathcal{A}(S)$ and hence are finite projective $\mathcal{O}(S)$–modules.*
(ii) *$\mathcal{S}kew_{\mathcal{A},\sigma}$ and $\mathcal{S}ymd_{\mathcal{A},\sigma}$ are locally direct summands of $\mathcal{A}$ and hence are finite locally free $\mathcal{O}$–modules.*
(iii) *If $\mathcal{A}$ is locally free of constant rank $n^2 \in \mathbb{N}_+$, then $\mathcal{S}kew_{\mathcal{A},\sigma}$ and $\mathcal{S}ymd_{\mathcal{A},\sigma}$ have rank $n(n+1)/2$ and $n(n-1)/2$ respectively.*

These submodule pairs, $\mathcal{S}ym_{\mathcal{A},\sigma}$ and $\mathcal{A}lt_{\mathcal{A},\sigma}$ when σ is orthogonal, and $\mathcal{S}kew_{\mathcal{A},\sigma}$ and $\mathcal{S}ymd_{\mathcal{A},\sigma}$ when σ is weakly symplectic, enjoy convenient orthogonality properties with respect to the trace form. For a submodule $\mathcal{M} \subset \mathcal{A}$, its orthogonal compliment is another $\mathcal{O}$–module $\mathcal{M}^\perp\colon \mathfrak{Sch}_S \to \mathfrak{Ab}$ given over $T \in \mathfrak{Sch}_S$ by

$$\mathcal{M}^\perp(T) = \{a \in \mathcal{A}(T) \mid \mathrm{Trd}_{\mathcal{A}}((a|_V)m) = 0,\ \forall\, V \in \mathfrak{Sch}_T,\ \forall\, m \in \mathcal{M}(V)\}.$$

3.10. **Lemma.** *Let $\mathcal{A}$ be an Azumaya algebra and let $\mathcal{M} \subseteq \mathcal{A}$ be an $\mathcal{O}$–submodule which is finite locally free. Then,*

(i) *$\mathcal{M}^\perp$ is quasi-coherent, and*
(ii) *$\mathcal{M}^\perp$ is of finite type.*

Proof. First, choose a cover $\{T_i \to S\}_{i\in I}$ over which each $\mathcal{M}|_{T_i}$ is free of rank n_i. Over T_i, let $\{m_1, \ldots, m_{n_i}\} \in \mathcal{M}(T_i)$ be a free basis, then $\mathcal{M}^\perp|_{T_i}$ is the kernel of the $\mathcal{O}|_{T_i}$–module morphism

$$\begin{aligned} \mathcal{A}|_{T_i} &\to \mathcal{A}|_{T_i}^{n_i} \\ a &\mapsto (\mathrm{Trd}_{\mathcal{A}}(am_1), \ldots, \mathrm{Trd}_{\mathcal{A}}(am_{n_i})). \end{aligned}$$

Both claims then follow over T_i from [St, Tag 01BY (1), (2)], which applies here because $\mathcal{A}|_{T_i}$ is quasi-coherent, see section 1.4. Then, since quasi-coherence and being of finite type are local properties, we are done. □

3.11. **Lemma.** *Let $\mathcal{A}$ be an Azumaya $\mathcal{O}$–algebra. Then the following hold.*

(i) *The trace form $(x, y) \mapsto \mathrm{Trd}_{\mathcal{A}}(xy)$ is a regular symmetric bilinear form on $\mathcal{A}$.*

(ii) *Let σ be an involution of the 1^{st} kind on $\mathcal{A}$. Then,*

$$\mathrm{Trd}_{\mathcal{A}}(\mathcal{S}ym_{\mathcal{A},\sigma} \cdot \mathcal{A}lt_{\mathcal{A},\sigma}) = 0, \text{ and}$$
$$\mathrm{Trd}_{\mathcal{A}}(\mathcal{S}kew_{\mathcal{A},\sigma} \cdot \mathcal{S}ymd_{\mathcal{A},\sigma}) = 0,$$

by which we mean, for $T \in \mathfrak{Sch}_S$ and sections $x \in \mathcal{S}ym_{\mathcal{A},\sigma}(T)$, $y \in \mathcal{A}lt_{\mathcal{A},\sigma}(T)$ we have $\mathrm{Trd}_{\mathcal{A}}(xy) = 0$, and similarly for the second statement.

(iii) *If σ is an orthogonal involution on $\mathcal{A}$, then*

$$\mathcal{S}ym^\perp_{\mathcal{A},\sigma} = \mathcal{A}lt_{\mathcal{A},\sigma} \text{ and } \mathcal{A}lt^\perp_{\mathcal{A},\sigma} = \mathcal{S}ym_{\mathcal{A},\sigma}.$$

(iv) *If σ is a weakly symplectic involution on $\mathcal{A}$, then*

$$\mathcal{S}ymd^\perp_{\mathcal{A},\sigma} = \mathcal{S}kew_{\mathcal{A},\sigma} \text{ and } \mathcal{S}kew^\perp_{\mathcal{A},\sigma} = \mathcal{S}ymd_{\mathcal{A},\sigma}.$$

Proof. (i): This is [CF, 2.5.3.16] (the statement in [CF] is only for even degree algebras, but the proof holds for algebras of any degree).
(ii): Let $T \in \mathfrak{Sch}_S$ and consider $x \in \mathcal{S}ym_{\mathcal{A},\sigma}(T)$ and $y \in \mathcal{A}lt_{\mathcal{A},\sigma}(T)$. After passing to a cover of T if needed, we may assume $y = a - \sigma(a)$ for some $a \in \mathcal{A}(T)$. Thus, we have

$$\begin{aligned} \mathrm{Trd}_{\mathcal{A}}(x(a-\sigma(a)) = \mathrm{Trd}_{\mathcal{A}}(xa) - \mathrm{Trd}_{\mathcal{A}}(x\sigma(a)) &= \mathrm{Trd}_{\mathcal{A}}(xa) - \mathrm{Trd}_{\mathcal{A}}(a\sigma(x)) \\ &= \mathrm{Trd}_{\mathcal{A}}(xa) - \mathrm{Trd}_{\mathcal{A}}(ax) = 0. \end{aligned}$$

The second statement is seen similarly.
(iii): The first statement is [CF, 2.7.0.29(3)], and because of (ii), a symmetric argument produces the second statement.
(iv): In the case when σ is weakly symplectic, $\mathcal{S}kew_{\mathcal{A},\sigma}$ and $\mathcal{S}ymd_{\mathcal{A},\sigma}$ are finite locally free $\mathcal{O}$–modules by Corollary 3.9(ii), which also means that $\mathcal{S}kew^\perp_{\mathcal{A},\sigma}$ and $\mathcal{S}ymd^\perp_{\mathcal{A},\sigma}$ are quasi-coherent of finite type by Lemma 3.10. Therefore, we may argue as done in [CF] for the case above. From (ii) we have that

$$\mathcal{S}ymd_{\mathcal{A},\sigma} \subseteq \mathcal{S}kew^\perp_{\mathcal{A},\sigma}.$$

Over residue fields this will be an equality for dimension reasons because the dimensions of $\mathcal{S}kew_{\mathcal{A},\sigma}$ and $\mathcal{S}ymd_{\mathcal{A},\sigma}$ will be the same as the ranks given in

Corollary 3.9(iii). Thus by Nakayama's lemma, we will have equality over local rings, and thus

$$\mathcal{S}ymd_{\mathcal{A},\sigma} = \mathcal{S}kew^{\perp}_{\mathcal{A},\sigma}$$

globally. A symmetric argument shows the second claim. □

The trace can also be used to define pairings on $\mathcal{S}ymd_{\mathcal{A},\sigma}$ and $\mathcal{A}lt_{\mathcal{A},\sigma}$ by extending a construction provided by the exercise [KMRT, II, exercise 15].

3.12. **Lemma.** *Let $(\mathcal{A},\sigma)$ be an Azumaya $\mathcal{O}$–algebra with orthogonal involution.*

(i) *There is a unique symmetric bilinear form $b_+ : \mathcal{S}ymd_{\mathcal{A},\sigma} \times \mathcal{S}ymd_{\mathcal{A},\sigma} \to \mathcal{O}$ such that for each affine scheme $U \in \mathfrak{Sch}_S$ and $x, y \in \mathcal{A}(U)$, we have*

$$b_+\big((x+\sigma(x)),(y+\sigma(y))\big) = \mathrm{Trd}_{\mathcal{A}}\big((x+\sigma(x))y\big).$$

(ii) *There is a unique symmetric bilinear form $b_- : \mathcal{A}lt_{\mathcal{A},\sigma} \times \mathcal{A}lt_{\mathcal{A},\sigma} \to \mathcal{O}$ such that for each affine scheme $U \in \mathfrak{Sch}_S$ and $x, y \in \mathcal{A}(U)$, we have*

$$b_-\big((x-\sigma(x)),(y-\sigma(y))\big) = \mathrm{Trd}_{\mathcal{A}}\big((x-\sigma(x))y\big).$$

In addition, this bilinear form is regular.

Proof. (i): We observe first that for each affine scheme U over S, $x, y \in \mathcal{A}(U)$, we have $\mathrm{Trd}_{\mathcal{A}}((x+\sigma(x))y) = \mathrm{Trd}_{\mathcal{A}}(x(y+\sigma(y)))$ for all $x, y \in \mathcal{A}(U)$ since $\mathrm{Trd}_{\mathcal{A}}$ is invariant under σ. This implies that b_+ is well-defined and also that it is symmetric.

(ii): The fact that b_- is well-defined and symmetric is similar to (i). To see that it is regular, we have to show that the induced map $\mathcal{A}lt_{\mathcal{A},\sigma} \to \mathcal{A}lt^{\vee}_{\mathcal{A},\sigma}$ is an isomorphism. Since $\mathcal{A}lt_{\mathcal{A},\sigma}$ is locally free of finite rank, the Nakayama lemma reduces to the field case. For all $x \in \mathrm{Alt}(A,\sigma)$ and $y \in A$, we we have $b_-\big(x(y-\sigma(y))\big) = \mathrm{Trd}_A(xy)$. Since Trd_A is regular, the kernel of b_- is zero. Thus b_- is regular. □

3.13. **Remark.** The proof of (ii) does not apply in case (i) because $\mathcal{S}ymd_{\mathcal{A},\sigma}$ is not locally free of finite rank. However, b_+ is regular over fields.

3.14. **Lemma.** *Let $(\mathcal{A},\sigma)$ be an Azumaya $\mathcal{O}$–algebra with an orthogonal involution.*

(i) *Let $u \in \mathbf{GL}_{1,\mathcal{A}}(S)$ and let $\mathrm{Inn}(u)$ be the inner automorphism of $\mathcal{A}$ induced by u. Then $\mathrm{Inn}(u)\circ\sigma$ is an involution of $\mathcal{A}$ of the first kind. It is orthogonal if and only if $u \in \mathcal{S}ym_{\mathcal{A},\sigma}(S)$.*

(ii) *Assume that the torsion subgroup $\mathrm{Pic}(S)_{tors} = 0$. Then for any orthogonal involution σ' of $\mathcal{A}$ there exists an invertible $u \in \mathcal{S}ym_{\mathcal{A},\sigma}(S)$ such that $\sigma' = \mathrm{Inn}(u)\circ\sigma$. Furthermore, u is unique up to an element of $\mathcal{O}^{\times}(S)$.*

Proof. (i): The first statement is obvious. For the proof of the second we can assume that S is affine. In this case our claim follows from [Knu2, III, (8.1.3)(2)].

(ii): Observe that $\sigma' \circ \sigma$ is an automorphism of $\mathcal{A}$, which is inner by the Knus-Skolem-Noether Theorem (Cor. 2.4), i.e., there exists $u \in \mathcal{A}(S)^\times$ such that $\sigma' = \mathrm{Inn}(u) \circ \sigma$. That $u \in \mathcal{S}ym_{\mathcal{A},\sigma}(S)$ follows from (i). If σ' can also be written as $\sigma' = \mathrm{Inn}(v) \circ \sigma$ for some invertible $v \in \mathcal{A}(S)^\times$, then $v^{-1}u$ is central, proving the last claim. □

4. Quadratic Pairs

4.1. **Definition** (Quadratic pairs)**.** Let $(\mathcal{A}, \sigma)$ be an Azumaya $\mathcal{O}$–algebra with orthogonal involution. Following [KMRT, §5B] and [CF, 2.7.0.30], we call (σ, f) a *quadratic pair* if $f : \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ is a linear form satisfying

$$f(a + \sigma(a)) = \mathrm{Trd}_{\mathcal{A}}(a)$$

for all $T \in \mathfrak{Sch}_S$ and every $a \in \mathcal{A}(T)$. In this case, we call $(\mathcal{A}, \sigma, f)$ a *quadratic triple.*

Two quadratic triples $(\mathcal{A}, \sigma, f)$ and $(\mathcal{A}', \sigma', f')$ are called *isomorphic* if there exists an isomorphism $\phi\colon \mathcal{A} \to \mathcal{A}'$ of $\mathcal{O}$–algebras such that $\sigma' = \phi \circ \sigma \circ \phi^{-1}$ and $f' = f \circ \phi^{-1}$ (this is well-defined since $\mathcal{S}ym_{\mathcal{A}',\sigma'} = \phi(\mathcal{S}ym_{\mathcal{A},\sigma})$). The notion of a quadratic triple is stable under arbitrary base change.

4.2. **Remark.** If $(\mathcal{A}, \sigma, f)$ is a quadratic triple, then $f(T) \neq 0$ for all $T \in \mathfrak{Sch}_S$, i.e. f is non-zero everywhere. This follows from Lemma 3.11(i), which guarantees that there is an element $a \in \mathcal{A}(T)$ with non-zero trace, and then

$$f(a + \sigma(a)) = \mathrm{Trd}_{\mathcal{A}}(a) \neq 0.$$

4.3. **Examples.** Let $(\mathcal{A}, \sigma)$ be an Azumaya $\mathcal{O}$–algebra with orthogonal involution.

(a) If (σ, f) is a quadratic pair, then for all $T \in \mathfrak{Sch}_S$ and $s \in \mathcal{S}ym_{\mathcal{A},\sigma}(T)$ we have that

$$2f(s) = f(s + \sigma(s)) = \mathrm{Trd}_{\mathcal{A}}(s).$$

Therefore, when $2 \in \mathcal{O}^\times$ there exists a unique f such that (σ, f) is a quadratic pair, namely $f(s) = \frac{1}{2}\,\mathrm{Trd}_{\mathcal{A}}(s)$.

(b) Given an element $\ell \in \mathcal{A}(S)$ with $\ell + \sigma(\ell) = 1$, we define $f\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ over $T \in \mathfrak{Sch}_S$ as

$$\begin{aligned} f(T)\colon \mathcal{S}ym_{\mathcal{A},\sigma}(T) &\to \mathcal{O}(T) \\ s &\mapsto \mathrm{Trd}_{\mathcal{A}}(\ell|_T \cdot s). \end{aligned}$$

This map is clearly linear, and for an element $a + \sigma(a) \in \mathcal{A}(T)$ we obtain

$$f(a + \sigma(a)) = \mathrm{Trd}_{\mathcal{A}}(\ell|_T \cdot (a + \sigma(a))) = \mathrm{Trd}_{\mathcal{A}}((\ell|_T + \sigma(\ell|_T)) \cdot a) = \mathrm{Trd}_{\mathcal{A}}(a),$$

and so $(\mathcal{A}, \sigma, f)$ is a quadratic triple. By Lemma 3.11(iii), the function f is determined by ℓ up to the addition of an element from $\mathcal{A}lt_{\mathcal{A},\sigma}(S)$, i.e.,

$\ell_1 - \ell_2 \in \mathcal{A}\!\ell t_{\mathcal{A},\sigma}(S)$ if and only if $\mathrm{Trd}_{\mathcal{A}}(\ell_1\_\_)$ agrees with $\mathrm{Trd}_{\mathcal{A}}(\ell_2\_\_)$ on $\mathcal{S}\!ym_{\mathcal{A},\sigma}$.
(c) Given an fppf cover $\{T_i \to S\}_{i\in I}$ and elements $\ell_i \in \mathcal{A}(T_i)$ satisfying $\ell_i + \sigma(\ell_i) = 1$ and $\ell_i|_{T_{ij}} - \ell_j|_{T_{ij}} \in \mathcal{A}\!\ell t_{\mathcal{A},\sigma}(T_{ij})$ for all $i, j \in I$, we can construct a unique quadratic triple $(\mathcal{A}, \sigma, f)$ such that $f|_{T_i} = \mathrm{Trd}_{\mathcal{A}}(\ell_i\_\_)$ holds for all $i \in I$. To do so, locally define maps $f_i \colon \mathcal{S}\!ym_{\mathcal{A},\sigma}|_{T_i} \to \mathcal{O}|_{T_i}$ by restricting the linear forms $\mathrm{Trd}_{\mathcal{A}}(\ell_i\_\_)$ to $\mathcal{S}\!ym_{\mathcal{A},\sigma}|_{T_i}$. By assumption and Lemma 3.11(ii), these local maps agree on overlaps of the cover $\{T_i \to S\}_{i\in I}$ and therefore glue together to define a global linear form $f \colon \mathcal{S}\!ym_{\mathcal{A},\sigma} \to \mathcal{O}$. That then $(\mathcal{A}, \sigma, f)$ is a quadratic triple, can be checked locally on the T_i, where it follows from Example 4.3(b).

More examples are be discussed below in Examples 4.8.

Given a quadratic triple $(\mathcal{A}, \sigma, f)$, it is of the form of Example 4.3(b) above exactly when f can be extended to a linear form on all of $\mathcal{A}$.

4.4. **Lemma.** *Let $(\mathcal{A}, \sigma, f)$ be a quadratic triple. Then, the following are equivalent.*

- (i) *There exists a linear form $f' \colon \mathcal{A} \to \mathcal{O}$ such that $f'|_{\mathcal{S}\!ym_{\mathcal{A},\sigma}} = f$,*
- (ii) *There exists $\ell \in \mathcal{A}(S)$ with $\ell + \sigma(\ell) = 1$ such that for all $T \in \mathfrak{Sch}_S$ and $s \in \mathcal{S}\!ym_{\mathcal{A},\sigma}(T)$, we have*

$$f(s) = \mathrm{Trd}_{\mathcal{A}}(\ell|_T \cdot s).$$

Proof. (ii)⇒(i): This implication is clear, see Example 4.3(b).
(i)⇒(ii): We follow the second half of the proof of [CF, 4.2.0.12]. The map f' is of the form $f' = \mathrm{Trd}_{\mathcal{A}}(\ell\_\_)$ for some $\ell \in \mathcal{A}(S)$. Then, for any $a \in \mathcal{A}(S)$ we have $\mathrm{Trd}_{\mathcal{A}}(\ell\sigma(a)) = \mathrm{Trd}_{\mathcal{A}}(a\sigma(\ell)) = \mathrm{Trd}_{\mathcal{A}}(\sigma(\ell)a)$ and so

$$\mathrm{Trd}_{\mathcal{A}}(a) = f'(a + \sigma(a)) = \mathrm{Trd}_{\mathcal{A}}(\ell(a + \sigma(a))) = \mathrm{Trd}_{\mathcal{A}}((\ell + \sigma(\ell))a).$$

Hence, $\ell + \sigma(\ell) = 1$ by regularity of the trace form (3.11(i)). □

If S is an affine scheme, then the construction of Example 4.3(b) produces all quadratic triples.

4.5. **Corollary.** *Assume S is an affine scheme and let $(\mathcal{A}, \sigma, f)$ be a quadratic triple. Then there exists $\ell \in \mathcal{A}(S)$ such that $\ell + \sigma(\ell) = 1$, and for all $T \in \mathfrak{Sch}_S$ and $s \in \mathcal{S}\!ym_{\mathcal{A},\sigma}(T)$, we have*

$$f(s) = \mathrm{Trd}_{\mathcal{A}}(\ell|_T \cdot s).$$

This ℓ is uniquely determined up to addition by an element of $\mathcal{A}\!\ell t_{\mathcal{A},\sigma}(S)$.

Proof. Since S is affine, by Lemma 3.8 we know $\mathcal{S}\!ym_{\mathcal{A},\sigma}$ is a direct summand of $\mathcal{A}$. Therefore, for any quadratic triple $(\mathcal{A}, \sigma, f)$, the function f may be extended to $\mathcal{A}$. The result then follows from Lemma 4.4. □

By considering an affine open cover, Corollary 4.5 recovers [CF, 4.2.0.12], which states that Zariski locally a quadratic triple is of the form in Example 4.3(b).

The existence of quadratic pairs is related to line bundle-valued quadratic forms. We describe this in the following Lemma 4.6 for the neutral Azumaya algebra $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$, generalizing [KMRT, 5.11] for $S = \operatorname{Spec}(R)$ where R is a field. Part (i) of 4.6 is stated in [CF, 2.7.0.31] for quadratic forms with values in $\mathcal{O}$.

4.6. **Proposition.** *Consider the neutral Azumaya $\mathcal{O}$–algebra $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$ associated to a locally free $\mathcal{O}$–module of finite positive rank $\mathcal{M}$. Assume $\mathcal{A}$ has an orthogonal involution σ adjoint to a regular symmetric bilinear form $b\colon \mathcal{M} \times \mathcal{M} \to \mathcal{L}$ for an $\mathcal{O}$–line bundle $\mathcal{L}$. Let $\sum_{i=1}^n \ell_i^* \otimes \ell_i$ be the preimage of $1_{\mathcal{O}} \in \mathcal{O}(S)$ under the canonical isomorphism $\mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{L} \cong \mathcal{O}$. By Lemma 3.6, we have an isomorphism of $\mathcal{O}$–modules*

$$\begin{aligned}\varphi_b\colon \mathcal{M} \otimes_{\mathcal{O}} \mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^* &\xrightarrow{\sim} \mathcal{E}nd_{\mathcal{O}}(\mathcal{M}) \\ m_1 \otimes m_2 \otimes \ell^* &\longmapsto \ell^*(b(m_1, \underline{})) \cdot m_2.\end{aligned}$$

(i) *Assume $b = b_q$ for a regular quadratic form $q\colon \mathcal{M} \to \mathcal{L}$. Then there exists a unique linear form $f_q\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ such that*

$$((f_q \otimes 1) \circ (\varphi_b \otimes 1))(m \otimes m \otimes (\sum_{i=1}^n \ell_i^* \otimes \ell_i)) = q(m) \tag{4.6.1}$$

holds for all sections $m \in \mathcal{M}$ where we identify $\mathcal{O} \otimes_{\mathcal{O}} \mathcal{L} \cong \mathcal{L}$ canonically. The pair (σ, f) is a quadratic pair on $\mathcal{A}$.
(ii) *Assume (σ, f) is a quadratic pair on $\mathcal{A}$ for some f. Then, there exists a unique quadratic form $q_f\colon \mathcal{M} \to \mathcal{L}$, defined on sections $m \in \mathcal{M}$ by*

$$q_f(m) = ((f \otimes 1) \circ (\varphi_b \otimes 1))(m \otimes m \otimes (\sum_{i=1}^n \ell_i^* \otimes \ell_i)) \tag{4.6.2}$$

whose polar form is b.
(iii) *The involution σ is part of an orthogonal pair on $A = \mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$ if and only if b is the polar form of a regular quadratic form.*

Proof. Since in each statement a global function is given, it suffices to check the claims locally. It is therefore no harm to assume that $S = \operatorname{Spec}(R)$, assume that $\mathcal{M}$ and $\mathcal{L}$ are free, and then prove the lemma in the setting of a finite rank free R–module M with a regular symmetric bilinear form $b\colon M \times M \to R$ and isomorphism of R–modules with involution

$$\begin{aligned}\varphi_b\colon (M \otimes_R M, \tau) &\xrightarrow{\sim} (\operatorname{End}_R(M), \eta_b) \\ m_1 \otimes m_2 &\mapsto b(m_1, \underline{})m_2.\end{aligned}$$

where τ is the switch map. However, since M is free, we can use the arguments in the proof of [KMRT, 5.11] establishing the existence of f over fields with only minor changes.
(i): We assume that $b = b_q$ for a regular quadratic form $q\colon M \to R$ and need to prove that there exists a unique linear form $f_q\colon \operatorname{Sym}(\operatorname{End}_R(M), \eta_b) \to R$

satisfying $(f_q \circ \varphi_b)(m \otimes m) = q(m)$. Letting $\{m_1, \dots, m_n\}$ be a free basis of M, we have

$$\begin{aligned}&\mathrm{Sym}(\mathrm{End}_R(M), \eta_b)\\ =&\,\mathrm{Span}_R\{\varphi_b(m_i \otimes m_i), \varphi_b(m_i \otimes m_j + m_j \otimes m_i) \mid 1 \le i \ne j \le n\}.\end{aligned}$$

Then the linear map $f_q \colon \mathrm{Sym}(\mathrm{End}_R(M), \eta_b) \to R$ defined by

$$\begin{aligned}f_q(\varphi_b(m_i \otimes m_i)) &= q(m_i)\\ f_q(\varphi_b(m_i \otimes m_j + m_j \otimes m_i)) &= b(m_i, m_j)\end{aligned}$$

is the desired function.
(ii): We assume that (η_b, f) is a quadratic pair for the Azumaya R–algebra $\mathrm{End}_R(M)$ and define $q_f \colon M \to R$ by

$$q_f(m) = (f \circ \varphi_b)(m \otimes m).$$

The calculation which shows q has polar form is b carries over from [KMRT, 5.11] without any change.
(iii): This is a summary of (i) and (ii). □

4.7. **Remark.** Since ultimately the choice of dual basis in Proposition 4.6 does not matter, we note here that alternate forms of equations (4.6.1) and (4.6.2) respectively which avoid such a choice are

$$\begin{aligned}f_q \circ \varphi_b(m \otimes m \otimes \ell^*) &= \ell^*(q(m)), \text{ and}\\ \ell^*(q_f(m)) &= f \circ \varphi_b(m \otimes m \otimes \ell^*).\end{aligned}$$

4.8. **Examples.** We give some further examples. In particular we outline the *split examples* in all degrees.

(a) In the setting of Proposition 4.6, a quadratic pair (η_b, f) forces $b = b_q$ for a regular quadratic form q. Hence, if $\mathcal{M}$ has odd rank, then necessarily $2 \in \mathcal{O}^\times$.

(b) Let $(\mathcal{M}_0, q_0) = (\mathcal{O}^{2n}, q_0)$ where q_0 is the split hyperbolic form on $\mathcal{O}^{2n}$ defined on sections by

$$q_0(x_1, \dots, x_{2n}) = \sum_{i=1}^{n} x_{2i-1}x_{2i}.$$

It is regular. We let η_0 be the associated orthogonal involution of $\mathcal{A}_0 = \mathcal{E}nd_{\mathcal{O}}(\mathcal{M}_0) = \mathrm{M}_{2n}(\mathcal{O})$, and denote by f_0 the unique linear form on $\mathcal{S}ym_{\mathcal{A}_0,\eta_0}$ satisfying (4.6.1). It is worked out in the example after [CF, 2.7.0.31]. We will refer to $(\mathcal{A}_0, \eta_0, f_0)$ as the *split example* (in degree $2n$). We point out that there may be many different linear forms f making $(\mathcal{A}_0, \eta_0, f)$ a quadratic triple.

(c) Assume $2 \in \mathcal{O}^\times$ and let $(\mathcal{M}_0, q_0) = (\mathcal{O}^{2n+1}, q_0)$ where q_0 is the split hyperbolic form on $\mathcal{O}^{2n}$ defined on sections by

$$q_0(x_1, \dots, x_{2n+1}) = \left(\sum_{i=1}^{n} x_{2i-1}x_{2i}\right) + x_{2n+1}^2$$

It is regular since 2 is invertible. We let η_0 be the associated orthogonal involution of $\mathcal{A}_0 = \mathcal{E}nd_{\mathcal{O}}(\mathcal{M}_0) = \mathrm{M}_{2n+1}(\mathcal{O})$. It has a unique linear form on $\mathcal{S}ym_{\mathcal{A}_0,\eta_0}$, namely $f_0 = \frac{1}{2}\operatorname{Trd}_{\mathcal{A}_0}$ by Example 4.3(a), such that $(\mathcal{A}_0, \sigma_0, f_0)$ is a quadratic triple. We also refer to this as the *split example* (in degree $2n+1$).
(d) Other examples are given in [KMRT, 5.12 and 5.13].

4.9. **Corollary.** [CF, 2.7.0.32] *Every quadratic triple $(\mathcal{A}, \sigma, f)$ is split étale-locally.*

Proof. By first considering the cover by connected components, we may assume that $\mathcal{A}$ is of constant degree n. If n is odd, then by Example 4.8(a) above and Example 4.3(a), we must have $2 \in \mathcal{O}^\times$ and $f = \frac{1}{2}\operatorname{Trd}_{\mathcal{A},\sigma}$. Then there exists an étale cover over which σ and η_0 are isomorphic, which is sufficient since those isomorphisms will identify f and f_0 due to uniqueness.

Now assume n is even. We have already started with [CF, 2.7.0.25]: every quadratic triple is étale-locally of the form $(\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}), \eta_b)$ considered in Lemma 4.6. By Lemma 4.6(ii), the regular bilinear form of that lemma is the polar form of a regular quadratic form $q\colon \mathcal{M} \to \mathcal{O}$, given by (4.6.2). By [CF, 2.6.1.13], after a second étale extension, we can then assume that $(\mathcal{M}, q) = (\mathcal{M}_0, q_0)$ as in Example 4.8(b), and that the linear form is the one of Lemma 4.6(i), i.e., that the quadratic triple is split. □

4.10. **Corollary.** *Let $(\mathcal{A}, \sigma, f)$ be a quadratic triple. Assume $\mathcal{A}$ is of constant rank $n^2 \in \mathbb{N}$. Then, we have*

$$f(1_{\mathcal{A}}) = \frac{n}{2} \in \mathcal{O}(S).$$

Proof. Since $\mathcal{A}$ is of constant rank, there will exist a cover $\{T_i \to S\}_{i\in I}$ such that $\mathcal{A}|_{T_i} \cong \mathrm{M}_n(\mathcal{O}|_{T_i})$ for all $i \in I$. Evaluating locally, where $1_{\mathcal{A}}|_{T_i}$ is the identity matrix, we have that $2f|_{T_i}(1_{\mathcal{A}}|_{T_i}) = \operatorname{Trd}_{\mathrm{M}_n(\mathcal{O}|_{T_i})}(1_{\mathcal{A}}|_{T_i}) = n$, and hence

$$2f(1_{\mathcal{A}}) = n \in \mathcal{O}(S)$$

globally as well. If n is odd then $2 \in \mathcal{O}^\times$ and so $f(1_{\mathcal{A}}) = \frac{n}{2}$ makes sense. If $n = 2m$ is even, we need to argue that $f(1_{\mathcal{A}}) = m$. We may assume by Corollary 4.9 that in the above argument, the cover was chosen such that it splits the quadratic triple, and so our local evaluation may be done on the split example. In the calculations of f_0 following [CF, 2.7.0.31] the authors show that $f_0(E_{ii} + E_{i+1,i+1}) = 1$, and therefore it follows immediately that $f_0(1) = m$. □

4.11. **Example.** In contrast to Example 4.3(a), if $2 \notin \mathcal{O}^\times$, then not every orthogonal σ is part of a quadratic pair. For example, this is so for $S = \operatorname{Spec}(R)$ with $2R = 0 \neq R$, and the Azumaya $\mathcal{O}$–algebra $\mathcal{A} = \mathrm{M}_n(\mathcal{O})$ equipped with $\tau =$ the transpose involution (note that $a + \sigma(a)$ has 0–diagonal for any $a \in \mathrm{M}_n(\mathcal{O}(T))$ and $T \in \mathfrak{Sch}_S$). But there exists orthogonal involutions σ of $\mathrm{M}_n(\mathcal{O})$ which are part of an orthogonal pair (σ, f), see

[KMRT, 5.9]. We will characterize orthogonal involutions belonging to quadratic pairs in section 6.

5. Quadratic Pairs on Tensor Products

Given two Azumaya $\mathcal{O}$–algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, their tensor product $\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2$ is again an Azumaya $\mathcal{O}$–algebra. If in addition these algebras are equipped with involutions σ_1 and σ_2 respectively, then $(\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1 \otimes \sigma_2)$ is an algebra with involution. Throughout this section we will use the notation

$$(\mathcal{A}, \sigma) := (\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1 \otimes \sigma_2).$$

Let $\varepsilon_1, \varepsilon_2 \in \boldsymbol{\mu}_2(S)$ be the types of σ_1 and σ_2. Since the type of an involution can be determined locally, we may apply [Knu2, 8.1.3(1)] which applies over affine schemes, to see that the type of $\sigma_1 \otimes \sigma_2$ will be the product $\varepsilon_1\varepsilon_2$. For our purposes we want $\sigma_1 \otimes \sigma_2$ to be orthogonal, so we will focus on two cases: when σ_i are both orthogonal ($\varepsilon_i = 1$), and when they are both symplectic (in which case $\varepsilon_i = -1$). In preparation, we generalize [KMRT, 5.17] to our setting. We use the notation $\mathcal{F}^\sharp$ to denote the fppf sheafification of a presheaf $\mathcal{F}$ as in [St, Tag 03NQ].

5.1. **Lemma.** *Let $(\mathcal{A}_1, \sigma_1)$ and $(\mathcal{A}_2, \sigma_2)$ be two Azumaya $\mathcal{O}$–algebras with involution. If σ_i are both orthogonal, then*

(i) $\mathcal{S}ym_{\mathcal{A},\sigma} = \big(\mathcal{S}ymd_{\mathcal{A},\sigma} + (\mathcal{S}ym_{\mathcal{A}_1,\sigma_1} \otimes_{\mathcal{O}} \mathcal{S}ym_{\mathcal{A}_2,\sigma_2})\big)^\sharp$,

If instead σ_i are both weakly symplectic, then

(ii) $\mathcal{S}ym_{\mathcal{A},\sigma} = \big(\mathcal{S}ymd_{\mathcal{A},\sigma} + (\mathcal{S}kew_{\mathcal{A}_1,\sigma_1} \otimes_{\mathcal{O}} \mathcal{S}kew_{\mathcal{A}_2,\sigma_2})\big)^\sharp$.

Proof. (i): If there exists an fppf cover $\{T_i \to S\}_{i\in I}$ such that the restrictions of each sheaf are equal, then the sheaves are equal globally. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are Azumaya $\mathcal{O}$–algebras, there exists a sufficiently fine affine cover $\{U_i \to S\}_{i\in I}$ such that $(\mathcal{A}_1, \sigma_1)|_{U_i} \cong (\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{M}_1, \tau)$ and $(\mathcal{A}_2, \sigma_2)|_{U_i} \cong (\mathcal{M}_2 \otimes_{\mathcal{O}} \mathcal{M}_2, \tau)$ for free $\mathcal{O}|_{U_i}$–modules of finite rank $\mathcal{M}_1$ and $\mathcal{M}_2$ with the switch involution τ. Since the component sheaves are quasi-coherent and the U_i are affine, we have

$$\begin{aligned}
&\big(\mathcal{S}ymd_{\mathcal{A},\sigma} + (\mathcal{S}ym_{\mathcal{A}_1,\sigma_1} \otimes_{\mathcal{O}} \mathcal{S}ym_{\mathcal{A}_2,\sigma_2})\big)^\sharp|_{U_i}\\
=&\mathcal{S}ymd_{\mathcal{A},\sigma}|_{U_i} + (\mathcal{S}ym_{\mathcal{A}_1,\sigma_1} \otimes_{\mathcal{O}} \mathcal{S}ym_{\mathcal{A}_2,\sigma_2})|_{U_i}\\
=&\mathcal{S}ymd_{\mathcal{B}|_{U_i},\sigma'|_{U_i}} + (\mathcal{S}ym_{\mathcal{A}_1|_{U_i},\sigma_1|_{U_i}} \otimes_{\mathcal{O}|_{U_i}} \mathcal{S}ym_{\mathcal{A}_2|_{U_i},\sigma_2|_{U_i}})
\end{aligned}$$

Quasi-coherence also means that this sheaf is determined by the $\mathcal{O}(U_i)$–module

$$\begin{aligned}
&\mathcal{S}ymd_{\mathcal{A},\sigma}(U_i) + (\mathcal{S}ym_{\mathcal{A}_1,\sigma_1}(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{S}ym_{\mathcal{A}_2,\sigma_2}(U_i))\\
&= \mathrm{Symd}\big((\mathcal{M}_1(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{M}_1(U_i)) \otimes_{\mathcal{O}(U_i)} (\mathcal{M}_2(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{M}_2(U_i)), \tau \otimes \tau\big)\\
&\quad + \mathrm{Sym}(\mathcal{M}_1(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{M}_1(U_i), \tau) + \mathrm{Sym}(\mathcal{M}_2(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{M}_2(U_i), \tau).
\end{aligned}$$

and $\mathcal{S}ym_{\mathcal{A},\sigma}|_{U_i}$ is determined by the $\mathcal{O}(U_i)$–module

$$\begin{aligned}&\operatorname{Sym}(\mathcal{A}(U_i),\sigma(U_i))\\ &=\operatorname{Sym}\big((\mathcal{M}_1(U_i)\otimes_{\mathcal{O}(U_i)}\mathcal{M}_1(U_i))\otimes_{\mathcal{O}(U_i)}(\mathcal{M}_2(U_i)\otimes_{\mathcal{O}(U_i)}\mathcal{M}_2(U_i)),\tau\otimes\tau\big).\end{aligned}$$

Therefore it is sufficient to consider the case of M_1 and M_2 being free modules of finite rank over a ring R, and prove that

$$\begin{aligned}&\operatorname{Sym}\big((M_1\otimes_R M_1)\otimes_R(M_2\otimes_R M_2),\tau\otimes\tau\big)\\ &=\operatorname{Symd}\big((M_1\otimes_R M_1)\otimes_R(M_2\otimes_R M_2),\tau\otimes\tau\big)\\ &\quad+\operatorname{Sym}(M_1\otimes_R M_1,\tau)\otimes_R\operatorname{Sym}(M_2\otimes_R M_2,\tau)\end{aligned}$$

Clearly the second module is included in the first. Now, let $\{m_1,\dots,m_a\}$ and $\{n_1,\dots,n_b\}$ be free bases of M_1 and M_2 respectively. Then $\operatorname{Sym}\big((M_1\otimes_R M_1)\otimes_R(M_2\otimes_R M_2),\tau\otimes\tau\big)$ has a basis consisting of the elements

$$\begin{gathered}(m_i\otimes m_i)\otimes(n_k\otimes n_k),\\ (m_i\otimes m_j)\otimes(n_k\otimes n_l)+(m_j\otimes m_i)\otimes(n_l\otimes n_k)\end{gathered}$$

for $i,j\in\{1,\dots,a\}$, $k,l\in\{1,\dots,b\}$ with either $i\neq j$ or $k\neq l$. The first type of basis element is contained in $\operatorname{Sym}(M_1\otimes_R M_1,\tau)\otimes_R\operatorname{Sym}(M_2\otimes_R M_2,\tau)$, and the second type is contained in $\operatorname{Symd}\big((M_1\otimes_R M_1)\otimes_R(M_2\otimes_R M_2),\tau\otimes\tau\big)$, so therefore the converse inclusion holds as well, and claim (i) is proved.
(ii): Since σ_i are now weakly symplectic, i.e., of type -1, we will have that $(\mathcal{A}_j,\sigma_j)|_{U_i}\cong(\mathcal{M}_j\otimes_{\mathcal{O}}\mathcal{M}_j,-\tau)$ for $j=1,2$. Accounting for the fact that $(-\tau)\otimes(-\tau)=\tau\otimes\tau$,

$$\begin{gathered}\operatorname{Skew}(M_j\otimes_R M_j,-\tau)=\operatorname{Sym}(M_j\otimes_R M_j,\tau),\text{ and}\\ \operatorname{Alt}(M_j\otimes_R M_j,-\tau)=\operatorname{Symd}(M_j\otimes_R M_j,\tau),\end{gathered}$$

all other details are the same as in (i). □

5.2. **Remark.** With the same methods as above one can also generalize [KMRT, 5.16]: If σ_i are both orthogonal, then

(i) $\mathcal{S}ymd_{\mathcal{A},\sigma}\cap(\mathcal{S}ym_{\mathcal{A}_1,\sigma_1}\otimes_{\mathcal{O}}\mathcal{S}ym_{\mathcal{A}_2,\sigma_2})$

$$=\Big((\mathcal{S}ymd_{\mathcal{A}_1,\sigma_1}\otimes_{\mathcal{O}}\mathcal{S}ym_{\mathcal{A}_2,\sigma_2})+(\mathcal{S}ym_{\mathcal{A}_1,\sigma_1}\otimes_{\mathcal{O}}\mathcal{S}ymd_{\mathcal{A}_2,\sigma_2})\Big)^{\sharp}.$$

and if σ_i are both weakly symplectic, then

(ii) $\mathcal{S}ymd_{\mathcal{A},\sigma}\cap(\mathcal{S}kew_{\mathcal{A}_1,\sigma_1}\otimes_{\mathcal{O}}\mathcal{S}kew_{\mathcal{A}_2,\sigma_2})$

$$=\big((\mathcal{A}lt_{\mathcal{A}_1,\sigma_1}\otimes_{\mathcal{O}}\mathcal{S}kew_{\mathcal{A}_2,\sigma_2})+(\mathcal{S}kew_{\mathcal{A}_1,\sigma_1}\otimes_{\mathcal{O}}\mathcal{A}lt_{\mathcal{A}_2,\sigma_2})\big)^{\sharp}.$$

We warn that the naive generalization of [KMRT, 5.15], which states that for central simple algebras (A_1,σ_1) and (A_2,σ_2) with involutions of the first kind over a field $\mathbb{F}$ of characteristic 2 we have

$$\begin{aligned}&\operatorname{Symd}(A_1,\sigma_1)\otimes_{\mathbb{F}}\operatorname{Symd}(A_2,\sigma_2)\\ =&(\operatorname{Symd}(A_1,\sigma_1)\otimes_{\mathbb{F}}\operatorname{Sym}(A_2,\sigma_2))\cap(\operatorname{Sym}(A_1,\sigma_1)\otimes_{\mathbb{F}}\operatorname{Symd}(A_2,\sigma_2)),\end{aligned}$$

does not hold in our generality. It fails when 2 is neither invertible nor zero. For example, take $S = \mathrm{Spec}(\mathbb{Z})$ and consider the Azumaya algebra $\mathcal{A} = \mathrm{M}_2(\mathcal{O})$ with the split orthogonal involution η_0 in degree 2 from Example 4.8(b). In particular,

$$\eta_0\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$

Since all relevant $\mathcal{O}$ modules are quasi-coherent and S is affine, we may deal with the $\mathbb{Z}$–module with involution $(A, \eta_0) = (\mathrm{M}_2(\mathbb{Z}), \eta_0)$. Then

$$\mathrm{Sym}(A,\eta_0) = \left\{\begin{bmatrix} a & b \\ c & a \end{bmatrix}\right\} \text{ and } \mathrm{Symd}(A,\eta_0) = \left\{\begin{bmatrix} a & 2b \\ 2c & a \end{bmatrix}\right\}.$$

Therefore,

$$\mathrm{Symd}(A,\eta_0) \otimes_Z \mathrm{Symd}(A,\eta_0) = \left\{\begin{bmatrix} a & 2b & 2c & 4d \\ 2e & a & 4f & 2c \\ 2g & 4h & a & 2b \\ 4i & 2g & 2e & a \end{bmatrix}\right\}.$$

while

$$\begin{aligned} &(\mathrm{Symd}(A,\eta_0) \otimes_{\mathbb{Z}} \mathrm{Sym}(A,\eta_0)) \cap (\mathrm{Sym}(A,\eta_0) \otimes_{\mathbb{Z}} \mathrm{Symd}(A,\eta_0)) \\ &= \left\{\begin{bmatrix} a & 2b & 2c & 2d \\ 2e & a & 2f & 2c \\ 2g & 2h & a & 2b \\ 2i & 2g & 2e & a \end{bmatrix}\right\} \end{aligned}$$

and so these modules are not equal.

Lemma 5.1 will be used similarly to how 5.16 and 5.17 are used in [KMRT], to prove uniqueness claims about quadratic pairs on tensor products. If the algebra $(\mathcal{A},\sigma)$ extends to a quadratic triple with some linear form f, then the behaviour of f is prescribed on $\mathcal{S}ymd_{\mathcal{A},\sigma}$. Therefore, by Lemma 5.1 such f will be uniquely determined by its behaviour on $\mathcal{S}ym_{\mathcal{A}_1,\sigma_1} \otimes_{\mathcal{O}} \mathcal{S}ym_{\mathcal{A}_2,\sigma_2}$. We use this in the next two propositions, generalizing [KMRT, 5.18, 5.20].

5.3. **Proposition.** *Let $(\mathcal{A}_1,\sigma_1,f_1)$ be a quadratic triple and let $(\mathcal{A}_2,\sigma_2)$ be an Azumaya $\mathcal{O}$–algebra with orthogonal involution. Then there exists a unique quadratic triple $(\mathcal{A},\sigma,f_{1*})$ such that, for sections $s_i \in \mathcal{S}ym_{\mathcal{A}_i,\sigma_i}$ we have*

$$f_{1*}(s_1 \otimes s_2) = f_1(s_1)\,\mathrm{Trd}_{\mathcal{A}_2}(s_2). \tag{5.3.1}$$

Proof. Consider an affine cover $\{U_i \to S\}_{i\in I}$ of S. We know by Lemma 4.5 that each $f|_{U_i}$ is described by an $\ell_i \in \mathcal{A}_1(U_i)$ with $\ell_i + \sigma_1(\ell_i) = 1$. Then

$$(\ell_i \otimes 1) + (\sigma_1 \otimes \sigma_2)(\ell_i \otimes 1) = (\ell_i + \sigma_1(\ell_i)) \otimes 1 = 1 \otimes 1 = 1$$

and since $\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}} \in \mathcal{A}\ell t_{\mathcal{A}_1,\sigma_1}(U_{ij})$, we also have that

$$(\ell_i \otimes 1)|_{U_{ij}} - (\ell_j \otimes 1)|_{U_{ij}} = (\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}}) \otimes 1 \in \mathcal{A}\ell t_{\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1\otimes\sigma_2}(U_{ij}).$$

Therefore, we may use the construction of Example 4.3(c) with the elements $\ell_i \otimes 1 \in \mathcal{A}(U_i)$ to define a global f_{1*}. It remains to show that this f_{1*} behaves as claimed in (5.3.1) on sections $s_i \in \mathcal{S}ym_{\mathcal{A}_i,\sigma_i}(T)$ for some $T \in \mathfrak{Sch}_S$. Consider such sections s_1 and s_2. With respect to the cover $\{T_i := U_i \times_S T \to T\}_{i\in I}$, our new linear form f_{1*} will be locally described by the elements $(\ell_i \otimes 1)|_{T_i}$. So, locally we have

$$\begin{aligned} f_{1*,i}((s_1 \otimes s_2)|_{T_i}) &= \operatorname{Trd}_{\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2}\left((\ell_i \otimes 1)|_{T_i}(s_1 \otimes s_2)|_{T_i}\right) \\ &= \operatorname{Trd}_{\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2}\left((\ell_i|_{T_i} \cdot s_1|_{T_i}) \otimes s_2|_{T_i}\right) \\ &= \operatorname{Trd}_{\mathcal{A}_1}\left(\ell_i|_{T_i} \cdot s_1|_{T_i}\right) \operatorname{Trd}_{\mathcal{A}_2}\left(s_2|_{T_i}\right) \\ &= f_1|_{T_i}(s_1|_{T_i}) \operatorname{Trd}_{\mathcal{A}_2}\left(s_2|_{T_i}\right) \end{aligned}$$

and hence gluing yields $f_{1*}(s_1 \otimes s_2) = f_1(s_1) \operatorname{Trd}_{\mathcal{A}_2}(s_2)$ as desired. □

5.4. **Example.** Here we generalize [KMRT, Example 5.19]. Let $(\mathcal{M}_1, q_1)$ be a regular quadratic form and let $(\operatorname{End}_{\mathcal{O}}(\mathcal{M}_1), \sigma_1, f_{q_1})$ be the quadratic pair defined in Proposition 4.6(i). On the other hand, let $(\mathcal{M}_2, b_2)$ be a regular symmetric bilinear form on a locally free $\mathcal{O}$–module of finite positive rank $\mathcal{M}_2$. We consider the tensor product quadratic form q on $\mathcal{M} = \mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{M}_2$ as defined by Sah [Sah, Thm. 1] in the ring case. This q is regular, and so we consider the attached quadratic pair $(\operatorname{End}_{\mathcal{O}}(\mathcal{M}, \sigma, f_q)$. The proof of [KMRT, Example 5.19] shows that there is an isomorphism

$$(\operatorname{End}_{\mathcal{O}_S}(\mathcal{M}_1), \sigma_1, f_{q_1}) \otimes (\operatorname{End}_{\mathcal{O}_S}(\mathcal{M}_2), \sigma_{b_2}) \xrightarrow{\sim} (\operatorname{End}_{\mathcal{O}_S}(\mathcal{M}), \sigma, f_q)$$

where the tensor product stands for the construction of Proposition 5.3.

Turning our attention to the weakly symplectic case, it turns out that two weakly symplectic involutions may not have a quadratic triple structure on their tensor product, see Example 5.7 below. To obtain a result analogous to [KMRT, 5.20] we will assume we have two symplectic involutions, i.e., where the types are $\varepsilon_i = -1$ and the local bilinear forms are alternating. This is due in part to the following lemma.

5.5. **Lemma.** *Let $(\mathcal{A}, \sigma)$ be an Azumaya $\mathcal{O}$–algebra with a weakly symplectic involution. Then, the following are equivalent*

(i) *σ is symplectic,*
(ii) *$\operatorname{Trd}_{\mathcal{A}}(s) = 0$ for all sections $s \in \mathcal{S}kew_{\mathcal{A},\sigma}$,*
(iii) *$1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$.*

Proof. Since $(\mathcal{A}, \sigma)$ is weakly symplectic, there exists an fppf cover $\{T_i \to S\}_{i\in I}$ over which

$$(\mathcal{M}_i \otimes_{\mathcal{O}|_{T_i}} \mathcal{M}_i, -\tau) \xrightarrow{\varphi_b} (\mathcal{E}nd_{\mathcal{O}|_{T_i}}(\mathcal{M}_i), \eta_b) \cong (\mathcal{A}, \sigma)|_{T_i}$$

where τ is the switch involution and φ_b is the isomorphism in Remark 3.7 for a regular skew-symmetric bilinear form b.
(i)⇔(ii): For any $T \in \mathfrak{Sch}_S$ and a section $s \in \mathcal{S}kew_{\mathcal{A},\sigma}(T)$, we may compute locally with respect to the cover $\{T_i \times_S T \to T\}_{i\in I}$ where the section will

be a linear combination of elements of the form

$$\varphi_b(m\otimes m)$$
$$\varphi_b(m\otimes n)-\eta_b(\varphi_b(m\otimes n))$$

for $m,n\in\mathcal{M}_i(T_i\times_S T)$. Since we have

$$\mathrm{Trd}_{\mathcal{E}nd_{\mathcal{O}|_{T_i}}(\mathcal{M}_i)}(\varphi_b(m\otimes n))=b(m,n),$$

the trace of the second type of element is clearly 0, while the trace vanishes for all elements of the first type if and only if b is alternating. Hence, $\mathrm{Trd}_{\mathcal{A}}(s)=0$ for all sections $s\in\mathcal{S}kew_{\mathcal{A},\sigma}$ if and only if the underlying bilinear forms are alternating, i.e., σ is symplectic.
(ii)⇔(iii): By Lemma 3.11(iv) we know $\mathcal{S}kew_{\mathcal{A},\sigma}^{\perp}=\mathcal{S}ymd_{\mathcal{A},\sigma}$ and $\mathcal{S}ymd_{\mathcal{A},\sigma}^{\perp}=\mathcal{S}kew_{\mathcal{A},\sigma}$, from which the equivalence is immediate. □

Similar to the orthogonal case, having $1_{\mathcal{A}}\in\mathcal{S}ymd_{\mathcal{A},\sigma}$ will grant us local ℓ_i with $\ell_i+\sigma(\ell_i)=1$ from which we can build a quadratic triple. The following result generalizes [KMRT, 5.20].

5.6. **Proposition.** *Let $(\mathcal{A}_1,\sigma_1)$ and $(\mathcal{A}_2,\sigma_2)$ be two Azumaya $\mathcal{O}$–algebras with symplectic involutions. Then there exists a unique quadratic triple $(\mathcal{A}_1\otimes_{\mathcal{O}}\mathcal{A}_2,\sigma_1\otimes\sigma_2,f_\otimes)$ such that, for sections $s_i\in\mathcal{S}kew_{\mathcal{A}_i,\sigma_i}$ we have*

$$f_\otimes(s_1\otimes s_2)=0.$$

Proof. We first establish uniqueness. Let $f,f':\mathcal{S}ym_{\mathcal{A},\sigma}\to\mathcal{O}_S$ be two solutions. Then $f-f'$ vanishes on $\mathcal{S}ymd_{\mathcal{A},\sigma}$ and on $\mathcal{S}kew_{\mathcal{A}_1,\sigma_1}\otimes\mathcal{S}kew_{\mathcal{A}_2,\sigma_2}$. According to Lemma 5.1(ii), $\mathcal{S}ymd_{\mathcal{A},\sigma}$ and $\mathcal{S}kew_{\mathcal{A}_1,\sigma_1}\otimes\mathcal{S}kew_{\mathcal{A}_1,\sigma_1}$ generate $\mathcal{S}ymd_{\mathcal{A},\sigma}$ so that $f'=f$.

We now establish existence. Since $(\mathcal{A}_1,\sigma_1)$ is symplectic we have $1_{\mathcal{A}_1}\in\mathcal{S}ymd_{\mathcal{A}_1,\sigma_1}$ by Lemma 5.5, and so there is an fppf cover $\{T_i\to S\}_{i\in I}$ such that for each $i\in I$ there exists $\ell_i\in\mathcal{A}_1(T_i)$ with $\ell_i+\sigma_1(\ell_i)=1$. Then we will have

$$(\ell_i\otimes 1)+(\sigma_1\otimes\sigma_2)(\ell_i\otimes 1)=1,$$

so to use the construction of Examples 4.3(c) we need to check that $(\ell_i\otimes 1)|_{T_{ij}}-(\ell_j\otimes 1)|_{T_{ij}}\in\mathcal{A}lt_{\mathcal{A},\sigma}(T_{ij})$. By Lemma 3.11(ii) and Lemma 5.1(i), it is sufficient to show that these elements are orthogonal to sections of the form $a+\sigma(a)$ for any $a\in\mathcal{A}|_{T_{ij}}$, and $s_1\otimes s_2$ for sections $s_i\in\mathcal{S}kew(\mathcal{A}_i,\sigma_i)|_{T_ij}$. For the first type, we have

$$\mathrm{Trd}_{\mathcal{A}}((\ell_i\otimes 1)|_{T_{ij}}(a+\sigma(a)))=\mathrm{Trd}_{\mathcal{A}_1\otimes_{\mathcal{O}}\mathcal{A}_2}(a)=\mathrm{Trd}_{\mathcal{A}}((\ell_j\otimes 1)|_{T_{ij}}(a+\sigma(a)))$$

and so

$$\mathrm{Trd}_{\mathcal{A}}((\ell_i\otimes 1|_{T_{ij}}-\ell_j\otimes 1|_{T_{ij}})(a+\sigma(a)))=0.$$

For the second type, we have

$$\mathrm{Trd}_{\mathcal{A}}\left((\ell_i\otimes 1)|_{T_{ij}}(s_1\otimes s_2)\right)=\mathrm{Trd}_{\mathcal{A}_1}(\ell_i|_{T_{ij}}s_1)\,\mathrm{Trd}_{\mathcal{A}_2}(s_2)=0$$

since $\mathrm{Trd}_{\mathcal{A}_2}(s_2)=0$ by Lemma 5.5(ii). Similarly $\mathrm{Trd}_{\mathcal{A}}\left((\ell_j\otimes 1)|_{T_{ij}}(s_1\otimes s_2)\right)=0$ and so $(\ell_i\otimes 1)|_{T_{ij}}-(\ell_j\otimes 1)|_{T_{ij}}$ is orthogonal to $s_1\otimes s_2$ as well. Thus there exists a global $f_\otimes\colon \mathcal{Sym}_{\mathcal{A},\sigma} \to \mathcal{O}$ constructed as in Example 4.3(c). The above local calculations also imply that $f_\otimes(s_1 \otimes s_2) = 0$ for sections $s_1 \in \mathcal{Skew}_{\mathcal{A}_1,\sigma_1}$ and $s_2 \in \mathcal{Skew}_{\mathcal{A}_2,\sigma_2}$, and so we have constructed the unique quadratic triple we desired. □

Note that both $(\mathcal{A}_1, \sigma_1)$ and $(\mathcal{A}_2, \sigma_2)$ were required to be symplectic in the above proposition. We needed $(\mathcal{A}_1, \sigma_1)$ to be symplectic to have $1_{\mathcal{A}_1} \in \mathcal{Symd}_{\mathcal{A}_1,\sigma_1}$, and $(\mathcal{A}_2, \sigma_2)$ to be symplectic to have $\mathrm{Trd}_{\mathcal{A}_2}(s_2) = 0$. If $(\mathcal{A}_2, \sigma_2)$ were only weakly symplectic, then we could construct the local maps $f_{\otimes,i}$ just the same, but it is not clear if the elements $(\ell_i \otimes 1)|_{T_{ij}} - (\ell_j \otimes 1)|_{T_{ij}}$ belong to $\mathcal{Alt}_{\mathcal{A},\sigma}(T_{ij})$, and so there may be examples with no quadratic triple on the tensor product.

If both involutions are only weakly symplectic there is almost no hope, even over affine schemes as in the following example.

5.7. **Example.** Let $S = \mathrm{Spec}(\mathbb{Z}/6\mathbb{Z})$. Since we are over an affine scheme we may simply work with Azumaya algebras over $R = \mathbb{Z}/6\mathbb{Z}$. Consider the Azumaya algebra $\mathrm{M}_2(R)$ with involution $\sigma(B) = u^{-1}B^T u$ for

$$u = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}.$$

This is weakly symplectic since $u^T = -u$, but not symplectic (for example by Lemma 5.5). Note that

$$s = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \in \mathrm{Skew}(\mathrm{M}_2(R), \sigma)$$

is a skew-symmetric element. If there were a quadratic triple $(\mathrm{M}_2(R) \otimes_R \mathrm{M}_2(R), \sigma \otimes \sigma, f)$, then necessarily

$$2f(s \otimes s) = \mathrm{Trd}_{\mathrm{M}_2(R)\otimes_R \mathrm{M}_2(R)}(s \otimes s) = \mathrm{Trd}_{\mathrm{M}_2(R)}(s)^2 = 3^2 = 3,$$

however there is no $x \in \mathbb{Z}/6\mathbb{Z}$ such that $2x = 3$. Therefore no such f exists. Since we are over an affine scheme this means that $\sigma \otimes \sigma$ fails to even be locally quadratic as defined below in Definition 6.3.

6. Obstructions to Quadratic Pairs

In light of Example 4.11 above, we seek to characterize which orthogonal involutions can participate in a quadratic pair. In this section we introduce cohomological obstructions which provide this characterization.

6.1. **Lemma.** *Let $(\mathcal{A}, \sigma, f)$ be a quadratic triple. Then, $1_{\mathcal{A}} \in \mathcal{Symd}_{\mathcal{A},\sigma}(S)$.*

Proof. We know by Corollary 4.5 that for an affine cover $\{U_i \to S\}_{i\in I}$ of S we have that

$$f|_{U_i} = \mathrm{Trd}_{\mathcal{A}}(\ell_i \_\_)$$

for some $\ell_i \in \mathcal{A}(U_i)$ with $\ell_i + \sigma(\ell_i) = 1_{\mathcal{A}(U_i)}$. In particular, $1_{\mathcal{A}(U_i)} \in \mathcal{Symd}_{\mathcal{A},\sigma}(U_i)$. Since the elements $1_{\mathcal{A}(U_i)}$ clearly agree on overlaps of the cover $\{U_i \to S\}_{i\in I}$ and $\mathcal{Symd}_{\mathcal{A},\sigma}$ is a sheaf, we obtain that $1_{\mathcal{A}} \in \mathcal{Symd}_{\mathcal{A},\sigma}(S)$. □

6.2. **Lemma.** *Let $(\mathcal{A},\sigma)$ be an Azumaya $\mathcal{O}$–algebra with orthogonal involution. Then the following are equivalent.*

(i) $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$
(ii) *For all affine $U \in \mathfrak{Sch}_S$, there exists $f_U \colon \mathcal{S}ym_{\mathcal{A}|_U,\sigma|_U} \to \mathcal{O}|_U$ such that $(\mathcal{A}|_U,\sigma|_U,f_U)$ is a quadratic triple.*
(iii) *For all affine open subschemes $U \subseteq S$, there exists $f_U \colon \mathcal{S}ym_{\mathcal{A}|_U,\sigma|_U} \to \mathcal{O}|_U$ such that $(\mathcal{A}|_U,\sigma|_U,f_U)$ is a quadratic triple.*
(iv) *There exists an fppf cover $\{T_i \to S\}_{i\in I}$ such that for each $i \in I$, there exists $f_i \colon \mathcal{S}ym_{\mathcal{A}|_{T_i},\sigma|_{T_i}} \to \mathcal{O}|_{T_i}$ such that $(\mathcal{A}|_{T_i},\sigma|_{T_i},f_i)$ is a quadratic triple.*
(v) *There exists an fppf cover $\{T_i \to S\}_{i\in I}$ and regular quadratic forms $(\mathcal{M}_i,q_i,\mathcal{L}_i)$ over $\mathcal{O}|_{T_i}$ such that for each $i \in I$,*

$$(\mathcal{A}|_{T_i},\sigma|_{T_i}) \cong (\mathcal{E}nd_{\mathcal{O}|_{T_i}}(\mathcal{M}_i),\eta_q)$$

where η_q is the adjoint involution of the polar bilinear form of q.

Proof. We argue that (i)⇒(ii)⇒(iii)⇒(iv)⇒(i), and then that (iv)⇔(v).
(i)⇒(ii): Assume $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$. Then, for an affine $U \in \mathfrak{Sch}_S$, by restriction we have that $1_{\mathcal{A}(U)} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(U)$. Now consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{S}kew_{\mathcal{A},\sigma} \longrightarrow \mathcal{A} \longrightarrow \mathcal{S}ymd_{\mathcal{A},\sigma} \longrightarrow 0$$

and its associated cohomology sequence

$$\dots \longrightarrow \mathcal{A}(U) \longrightarrow \mathcal{S}ymd_{\mathcal{A},\sigma}(U) \longrightarrow H^1_{\mathrm{fppf}}(U,\mathcal{S}kew_{\mathcal{A},\sigma}) \longrightarrow \dots$$

According to [St, Tag 03P2], $H^1_{\mathrm{fppf}}(U,\mathcal{S}kew_{\mathcal{A},\sigma})$ is the Zariski H^1 of the quasi-coherent $\mathcal{O}_S$–module underlying $\mathcal{S}kew_{\mathcal{A},\sigma}$, and so vanishes by [St, Tag 01XB] since U is affine. It follows that $1+\sigma\colon \mathcal{A}(U) \to \mathcal{S}ymd_{\mathcal{A},\sigma}(U)$ is surjective and thus there exists $\ell \in \mathcal{A}(U)$ such that $\ell + \sigma(\ell) = 1_{\mathcal{A}(U)}$. We may then construct our desired quadratic triple as in Example 4.3(b).
(ii)⇒(iii) is obvious, and (iii)⇒(iv) by taking an affine open cover of S.
(iv)⇒(i): Let such a cover be given. By Lemma 6.1 we have $1_{\mathcal{A}(T_i)} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(T_i)$ for all $i \in I$. These agree on overlaps and so $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$ by the sheaf property.
(iv)⇒(v): Since $(\mathcal{A},\sigma)$ is locally isomorphic to a neutral algebra, we may assume by refining the given cover that $(\mathcal{A}|_{T_i},\sigma|_{T_i},f_i) = (\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}_i),\sigma|_{T_i},f_i)$ for some $\mathcal{O}|_{T_i}$–modules $\mathcal{M}_i$. There are then quadratic forms $q_i\colon \mathcal{M}_i \to \mathcal{L}_i$ whose polar bilinear form has adjoint involution $\sigma|_{T_i}$ by Lemma 4.6(ii).
(v)⇒(iv): This is Lemma 4.6(i). □

6.3. **Definition** (Locally Quadratic Involution)**.** We call an orthogonal involution on an Azumaya $\mathcal{O}$–algebra *locally quadratic* if it satisfies the equivalent conditions of Lemma 6.2.

Let $(\mathcal{A}, \sigma)$ be an Azumaya $\mathcal{O}$–algebra with a locally quadratic involution. Consider the exact sequence of $\mathcal{O}$–modules

$$0 \to \mathcal{S}\!kew_{\mathcal{A},\sigma} \to \mathcal{A} \to \mathcal{S}\!ymd_{\mathcal{A},\sigma} \to 0.$$

where the second map is inclusion, and the third map is $a \mapsto a + \sigma(a)$. A portion of the associated long exact cohomology sequence over S is

$$\ldots \to \mathcal{A}(S) \to \mathcal{S}\!ymd_{\mathcal{A},\sigma}(S) \xrightarrow{\delta} H^1_{\text{fppf}}(S, \mathcal{S}\!kew_{\mathcal{A},\sigma}) \to \ldots$$

where we denote the connecting morphism by δ. In addition, since $\mathcal{A}\ell t_{\mathcal{A},\sigma} \subseteq \mathcal{S}\!kew_{\mathcal{A},\sigma}$, there is a map of sheaves $\mathcal{S}\!kew_{\mathcal{A},\sigma} \to \mathcal{S}\!kew_{\mathcal{A},\sigma}/\mathcal{A}\ell t_{\mathcal{A},\sigma}$ whose induced map on cohomology we call π. Therefore we have,

$$\begin{array}{ccccc} \mathcal{S}\!ymd_{\mathcal{A},\sigma}(S) & \xrightarrow{\delta} & H^1_{\text{fppf}}(S, \mathcal{S}\!kew_{\mathcal{A},\sigma}) & \xrightarrow{\pi} & H^1_{\text{fppf}}(S, \mathcal{S}\!kew_{\mathcal{A},\sigma}/\mathcal{A}\ell t_{\mathcal{A},\sigma}) \\ 1_{\mathcal{A}} & \longmapsto & \Omega(\mathcal{A},\sigma) & \longmapsto & \omega(\mathcal{A},\sigma), \end{array}$$

i.e., $\Omega(\mathcal{A}, \sigma) = \delta(1_{\mathcal{A}})$ and $\omega(\mathcal{A}, \sigma) = \pi(\Omega(\mathcal{A}, \sigma))$.

6.4. **Definition** (Strong and Weak Obstructions)**.** We call $\Omega(\mathcal{A}, \sigma)$ the *strong obstruction* and call $\omega(\mathcal{A}, \sigma)$ the *weak obstruction*.

We will make use of the isomorphism between first fppf cohomology and first Čech cohomology in order to perform computations with $\Omega(A, \sigma)$. Recall from [M, III.2] (or [St, Section 03OK]) that for an abelian sheaf $\mathcal{F} \colon \mathfrak{Sch}_S \to \mathfrak{Ab}$ and $T \in \mathfrak{Sch}_S$, the first Čech cohomology is

$$\check{H}^1(T, \mathcal{F}) = \operatorname*{colim}_{\mathcal{U}=\{T_i \to T\}_{i\in I}} \check{H}^1(\mathcal{U}, \mathcal{F})$$

where the colimit is over fppf covers of T. The Čech cohomology relative a cover is $\check{H}^1(\mathcal{U}, \mathcal{F}) = \operatorname{Ker}(d^1)/\operatorname{Im}(d^0)$ where d^0 and d^1 are the natural maps in the sequence

$$0 \to \mathcal{F}(T) \to \prod_{i\in I} \mathcal{F}(T_i) \xrightarrow{d^0} \prod_{i,j\in I} \mathcal{F}(T_{ij}) \xrightarrow{d^1} \prod_{i,j,k\in I} \mathcal{F}(T_{ijk}) \xrightarrow{d^2} \ldots$$

That is,

$$\begin{aligned} d^0((f_i)_{i\in I}) &= (f_i|_{T_{ij}} - f_j|_{T_{ij}})_{i,j\in I}, \text{ and} \\ d^1((f_{ij})_{i,j\in I}) &= (f_{ij}|_{T_{ijk}} - f_{ik}|_{T_{ijk}} + f_{jk}|_{T_{ijk}})_{i,j,k\in I}. \end{aligned}$$

By [M, III, 2.10], there is an isomorphism $H^1_{\text{fppf}}(T, \mathcal{F}) \cong \check{H}^1(T, \mathcal{F})$, which we will use freely going forward. The following lemma is a general cohomological fact.

6.5. **Lemma.** *Let $0 \to \mathcal{F}' \to \mathcal{F} \xrightarrow{\pi} \mathcal{F}'' \to 0$ be an exact sequence of abelian fppf sheaves on $\mathfrak{Sch}_S$. Then, the first connecting morphism in the long exact cohomology sequence over S behaves as*

$$\begin{aligned} \delta \colon \mathcal{F}''(S) &\to \check{H}^1(S, \mathcal{F}') \\ x &\mapsto \Big[(f_i|_{T_{ij}} - f_j|_{T_{ij}})_{i,j\in I}\Big] \end{aligned}$$

where $\{T_i \to S\}_{i\in I}$ *is any fppf cover and* $f_i \in \mathcal{F}(T_i)$ *are any elements such that* $\pi(f_i) = x|_{T_i}$*.*

We are interested in a special case of the above fact since the strong obstruction is an image under such a connecting morphism.

6.6. **Lemma.** *Let* $(\mathcal{A}, \sigma)$ *be an Azumaya* $\mathcal{O}$*–algebra with locally quadratic involution. Using the isomorphism* $H^1_{\mathrm{fppf}}(S, \mathcal{S}kew_{\mathcal{A},\sigma}) \cong \check{H}^1(S, \mathcal{S}kew_{\mathcal{A},\sigma})$*, the strong obstruction takes the form*

$$\Omega(\mathcal{A}, \sigma) = \Big[(\ell_i|_{T_{ij}} - \ell_j|_{T_{ij}})_{i,j\in I}\Big]$$

for any cover $\{T_i \to S\}_{i\in I}$ *and elements* $\ell_i \in \mathcal{A}(T_i)$ *such that* $\ell_i + \sigma(\ell_i) = 1$*.*

6.7. **Remarks.** (a) By Lemma 6.2, if we wish we may represent $\Omega(\mathcal{A}, \sigma)$ as in Lemma 6.6 with an affine open cover $\{U_i \to S\}_{i\in I}$.
(b) Since the weak obstruction $\omega(\mathcal{A}, \sigma)$ is simply an image of the strong obstruction, it will be represented similarly by the class

$$\Big[\overline{(\ell_i|_{T_{ij}} - \ell_j|_{T_{ij}})}_{i,j\in I}\Big] \in \check{H}^1(S, \mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma})$$

for a cover $\{T_i \to S\}_{i\in I}$ and elements $\ell_i \in \mathcal{A}(T_i)$ with $\ell_i + \sigma(\ell_i) = 1$, where the overline denotes the image in $(\mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma})(T_{ij})$. This cover can of course also be taken to be an affine open cover if desired.

The strong and weak obstructions prevent the existence of a quadratic triple involving σ in the following way.

6.8. **Theorem.** *Let* $(\mathcal{A}, \sigma)$ *be an Azumaya* $\mathcal{O}$*–algebra with a locally quadratic involution. Then,*

(i) *There exists a linear map* $f\colon \mathcal{A} \to \mathcal{O}$ *such that* $(\mathcal{A}, \sigma, f|_{\mathcal{S}ym_{\mathcal{A},\sigma}})$ *is a quadratic triple if and only if* $\Omega(\mathcal{A}, \sigma) = 0$*. In this case* $f = \mathrm{Trd}_{\mathcal{A}}(\ell\_\_)$ *for an element* $\ell \in \mathcal{A}(S)$ *with* $\ell + \sigma(\ell) = 1$*.*
(ii) *There exists a linear map* $f\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ *such that* $(\mathcal{A}, \sigma, f)$ *is a quadratic triple if and only if* $\omega(\mathcal{A}, \sigma) = 0$*.*

Proof. (i): Assume we have $f\colon \mathcal{A} \to \mathcal{O}$ such that $(\mathcal{A}, \sigma, f|_{\mathcal{S}ym_{\mathcal{A},\sigma}})$ is a quadratic triple. Then trivially $f|_{\mathcal{S}ym_{\mathcal{A},\sigma}}$ can be extended to $\mathcal{A}$, and so by Lemma 4.4 there exists an $\ell \in \mathcal{A}(S)$ such that $\ell + \sigma(\ell) = 1_{\mathcal{A}}$. Thus $1_{\mathcal{A}}$ is in the image of $\mathcal{A}(S) \to \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$ in the long exact cohomology sequence, and so $\Omega(\mathcal{A}, \sigma) = 0$. Conversely, if $\Omega(\mathcal{A}, \sigma) = 0$ then $1_{\mathcal{A}}$ is in the same image and so we obtain such an ℓ. Using Example 4.3(b) we can construct $f\colon \mathcal{A} \to \mathcal{O}$ with the desired property.
(ii): Assume $(\mathcal{A}, \sigma, f)$ is a quadratic triple. Over an affine cover $\{U_i \to S\}_{i\in I}$ we have that $f_i := f|_{U_i}$ are of the form $\mathrm{Trd}_{\mathcal{A}}(\ell_i\_\_)$ for some $\ell_i \in \mathcal{A}(U_i)$ with $\ell_i + \sigma(\ell_i) = 1$. Then by Lemma 6.6, $\Omega(\mathcal{A}, \sigma)$ is represented by the 1-cocycle $(\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}})_{i,j\in I}$ in $\mathcal{S}kew_{\mathcal{A},\sigma}$, and so $\omega(\mathcal{A}, \sigma)$ takes the same form but in $\mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma}$. However, because the f_i come from a global f, they agree on overlaps, i.e, $\mathrm{Trd}_{\mathcal{A}}((\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}})s) = 0$ for all $s \in \mathcal{S}ym_{\mathcal{A},\sigma}(U_{ij})$. By the

orthogonality of $\mathcal{S}ym_{\mathcal{A},\sigma}$ and $\mathcal{A}lt_{\mathcal{A},\sigma}$ via the trace form, this means that each $\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}}$ is in $\mathcal{A}lt_{\mathcal{A},\sigma}(U_{ij})$. Hence, $\omega(\mathcal{A},\sigma) = 0$.

Conversely, assume $\omega(\mathcal{A},\sigma) = 0$. We may take an affine cover $\{U_i \to S\}_{i\in I}$ with respect to which $\Omega(\mathcal{A},\sigma)$ will be represented by $\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}}$ for some $\ell_i \in \mathcal{A}(U_i)$. Now, as in Example 4.3(b), defining $f_i\colon \mathcal{S}ym_{\mathcal{A}|_{U_i},\sigma|_{U_i}} \to \mathcal{O}|_{U_i}$ produces quadratic triples $(\mathcal{A}|_{U_i}, \sigma|_{U_i}, f_i)$. Due to the assumption that the weak obstruction vanishes, each $\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}}$ lies in $\mathcal{A}lt_{\mathcal{A},\sigma}(U_{ij})$ and so $f_i|_{U_{ij}} = f_j|_{U_{ij}}$. Hence we can glue to produce $f\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ which makes $(\mathcal{A},\sigma,f)$ a quadratic triple. □

For examples of locally quadratic involutions with non-trivial strong or weak obstructions, see section 7.

6.9. **Remark.** Since the sequence $0 \to \mathcal{S}kew_{\mathcal{A},\sigma} \to \mathcal{A} \to \mathcal{S}ymd_{\mathcal{A},\sigma} \to 0$ is killed by 2, it follows that $2\Omega(\mathcal{A},\sigma) = 0$ so that $2\omega(\mathcal{A},\sigma) = 0$. In particular, if 2 is invertible over S we recover the fact that $(\mathcal{A},\sigma)$ extends in a quadratic triple.

6.10. **Alternate Obstructions to Extend a Form.** Let $(\mathcal{A},\sigma,f)$ be a quadratic triple. Lemma 4.4 above states that f extends from $\mathcal{S}ym_{\mathcal{A},\sigma}$ to $\mathcal{A}$ if and only if f arises from some $\ell \in \mathcal{A}(S)$ satisfying $\ell + \sigma(\ell) = 1$. The cohomology theory of quasi-coherent sheaves provides an obstruction to such an extension, in particular an obstruction $c(f) \in H^1_{\text{fppf}}(S, \mathcal{A}lt_{\mathcal{A},\sigma})$ as follows. Consider an affine covering $\{U_i \to S\}_{i\in I}$ of S. By Corollary 4.5, each restriction $f_i := f|_{U_i}$ will be of the form $\mathrm{Trd}_{\mathcal{A}|_{U_i}}(\ell_i\_\_)$ for an $\ell_i \in \mathcal{A}(U_i)$ with $\ell_i + \sigma(\ell_i) = 1_{\mathcal{A}_{U_i}}$. The 1-cocycle $(\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}})_{i,j\in I}$ represents the cohomological obstruction to the existence of a global ℓ which would extend f. Denote this cohomology class by $c(f)$. It belongs to $H^1_{\text{fppf}}(S, \mathcal{A}lt_{\mathcal{A},\sigma})$ because $f_i|_{U_{ij}} = f_j|_{U_{ij}}$, which is equivalent to $\mathrm{Trd}_{\mathcal{A}|_{U_{ij}}}(\ell_i|_{U_{ij}} s) = \mathrm{Trd}_{\mathcal{A}|_{U_{ij}}}(\ell_j|_{U_{ij}} s)$ for all $s \in \mathcal{S}ym_{\mathcal{A},\sigma}(U_{ij})$, and so $\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}} \in \mathcal{A}lt_{\mathcal{A},\sigma}(U_{ij})$ by Lemma 3.11(iii).

However, we also obtain another obstruction in the following way. We have an exact sequence of sheaves

$$0 \to \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{A} \to \mathcal{A}lt_{\mathcal{A},\sigma} \to 0$$

which, since the sheaves are finite locally free, we may dualize to obtain another exact sequence,

$$0 \to \mathcal{A}lt^\vee_{\mathcal{A},\sigma} \to \mathcal{A}^\vee \to \mathcal{S}ym^\vee_{\mathcal{A},\sigma} \to 0.$$

A portion of the long exact cohomology sequence associated with this exact sequence is

$$\ldots \to \mathcal{A}^\vee \to \mathcal{S}ym^\vee_{\mathcal{A},\sigma} \to H^1_{\text{fppf}}(S, \mathcal{A}lt^\vee_{\mathcal{A},\sigma}) \to \ldots$$

Since $f \in \mathcal{S}ym^\vee_{\mathcal{A},\sigma}(S)$, we then get a class $c'(f) \in H^1(S, \mathcal{A}lt^\vee_{\mathcal{A},\sigma})$ which is the obstruction to extend f to $\mathcal{A}$. Not surprisingly we can compare the two obstructions.

6.11. **Lemma.** *Let $(\mathcal{A},\sigma,f)$ be a quadratic triple, and let $\widehat{b_-} : \mathcal{A}lt_{A,\sigma} \xrightarrow{\sim} \mathcal{A}lt^{\vee}_{A,\sigma}$ be the isomorphism associated to the regular bilinear form b_- provided by Lemma 3.12(ii). Then,*

(i) $c'(f) = \widehat{b_-}(c(f))$.
(ii) *The image of $c(f)$ under the map $H^1_{\text{fppf}}(S,\mathcal{A}lt_{\mathcal{A},\sigma}) \to H^1_{\text{fppf}}(S,\mathcal{S}kew_{\mathcal{A},\sigma})$ induced by the inclusion $\mathcal{A}lt_{\mathcal{A},\sigma} \to \mathcal{S}kew_{\mathcal{A},\sigma}$, is $\Omega(\mathcal{A},\sigma)$.*

Proof. (i): We use Čech cohomology by taking an affine open covering $\{U_i \to S\}_{i\in I}$ of S. Then $f|_{U_i}(s) = \mathrm{Trd}_{\mathcal{A}}(\ell_i s)$ for some $\ell_i \in \mathcal{A}(U_i)$ satisfying $\ell_i + \sigma(\ell_i) = 1$ by Lemma 4.5. By definition $c(f)$ is the class of the 1–cocycle $c_{ij} = \ell_i|_{U_{ij}} - \ell_j|_{U_{ij}}$ with values in $\mathcal{A}lt_{\mathcal{A},\sigma}$. We define $\widetilde{f}_i(a) = \mathrm{Trd}_{\mathcal{A}}(\ell_i a)$ on $\mathcal{A}|_{U_i}$ for $i \in I$.

On the other hand, $c'(f)$ is the class of the 1–cocycle $c'_{ij} = (\widetilde{f}_i)|_{U_{ij}} - (\widetilde{f}_j)|_{U_{ij}}$ with values in $\mathcal{A}lt^{\vee}_{\mathcal{A},\sigma}$.

We need to check that $c'_{ij} = \widehat{b_-}(c_{ij})$ for all $i,j \in I$ or equivalently that $c'_{ij} = b_-(c_{ij}, \_\_)$ over U_{ij}. Up to localization to an affine open subset $V \subseteq U_{ij}$, we can deal with elements of the form $x = x' - \sigma(x')$ for some $x' \in \mathcal{A}(V)$. It follows that

$$\begin{aligned} c'_{ij}(x) &= (\widetilde{f}_i)|_{U_{ij}}(x' - \sigma(x')) - (\widetilde{f}_j)|_{U_{ij}}(x' - \sigma(x')) \\ &= \mathrm{Trd}_{\mathcal{A}}(\ell_i|_{U_{ij}}(x' - \sigma(x')) - \mathrm{Trd}_{\mathcal{A}}(\ell_j|_{U_{ij}}(x' - \sigma(x')) \\ &= \mathrm{Trd}_{\mathcal{A}}((\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}})(x' - \sigma(x')) \\ &= b_-(c_{ij}, x) \end{aligned}$$

as desired.

(ii): We have that $1_{\mathcal{A}(U_i)} = \ell_i + \sigma(\ell_i)$ for each i. By Lemma 6.6, the strong obstruction $\Omega(A,\sigma)$ is represented by the 1–cocycle $\ell_i|_{U_{i,j}} - \ell_j|_{U_{i,j}}$ with values in $\mathcal{S}kew_{\mathcal{A},\sigma}$. We conclude that the image of $c(f)$ by the map $H^1_{\text{fppf}}(S,\mathcal{A}lt_{\mathcal{A},\sigma}) \to H^1_{\text{fppf}}(S,\mathcal{S}kew_{\mathcal{A},\sigma})$ is $\Omega(A,\sigma)$. □

6.12. **Obstructions for Tensor Products.** Consider Azumaya algebras $(\mathcal{A}_1,\sigma_1)$ and $(\mathcal{A}_2,\sigma_2)$ either both with orthogonal or both with symplectic involutions. Their tensor product $(\mathcal{A},\sigma) = (\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1 \otimes \sigma_2)$ is an Azumaya algebra with orthogonal involution. There is a natural map

$$\begin{aligned} \mathcal{A}_1 \times \mathcal{A}_2 &\to \mathcal{A} \\ (a_1,a_2) &\mapsto a_1 \otimes a_2 \end{aligned}$$

which restricts to a morphism

$$\mathcal{S}kew_{\mathcal{A}_1,\sigma_1} \times \mathcal{S}ym_{\mathcal{A}_2,\sigma_2} \to \mathcal{S}kew_{\mathcal{A},\sigma}$$

which in turn induces a morphism

$$(\mathcal{S}kew_{\mathcal{A}_1,\sigma_1}/\mathcal{A}lt_{\mathcal{A}_1,\sigma_1}) \times \mathcal{S}ym_{\mathcal{A}_2,\sigma_2} \to \mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma}.$$

We will make use of these tensor morphisms throughout this section to investigate how the strong and weak obstructions for $(\mathcal{A},\sigma)$. When σ_i are both orthogonal we will assume σ_1 is locally quadratic and we will relate $\Omega(\mathcal{A},\sigma)$ to $\Omega(\mathcal{A}_1,\sigma_1)$. When σ_i are both symplectic we also have $1_{\mathcal{A}_1} \in \mathcal{S}\mathit{ymd}_{\mathcal{A}_1,\sigma_1}$ by Lemma 5.5 and a connecting morphism $\delta_1 \colon \mathcal{S}\mathit{ymd}_{\mathcal{A}_1,\sigma_1} \to H^1_{\mathrm{fppf}}(S, \mathcal{S}\mathit{ymd}_{\mathcal{A}_1,\sigma_1})$. In this case we will relate $\Omega(\mathcal{A},\sigma)$ to $\delta_1(1_{\mathcal{A}_1})$, which is only technically a strong obstruction when $2 = 0 \in \mathcal{O}$ and so σ_1 would be simultaneously locally quadratic as well as symplectic.

6.13. **Lemma.** *Let $(\mathcal{A}_1,\sigma_1)$ and $(\mathcal{A}_2,\sigma_2)$ be two Azumaya $\mathcal{O}$–algebra with orthogonal involution. Let $(\mathcal{A},\sigma)$ be their tensor product. Assume that $(\mathcal{A}_1,\sigma_1)$ is locally quadratic.*

(i) *$(\mathcal{A},\sigma)$ is locally quadratic.*

(ii) *We have $\Omega(\mathcal{A},\sigma) = \Omega(\mathcal{A}_1,\sigma_1) \cup 1_{\mathcal{A}_2} \in H^1_{\mathrm{fppf}}(S, \mathcal{S}\mathit{kew}_{\mathcal{A},\sigma})$ where $1_{\mathcal{A}_2} \in \mathcal{S}\mathit{ym}_{\mathcal{A}_2,\sigma_2}(S) = H^0(S, \mathcal{S}\mathit{ym}_{\mathcal{A}_2,\sigma_2})$ and the cup-product arises from the morphism*

$$\mathcal{S}\mathit{kew}_{\mathcal{A}_1,\sigma_1} \times \mathcal{S}\mathit{ym}_{\mathcal{A}_2,\sigma_2} \to \mathcal{S}\mathit{kew}_{\mathcal{A},\sigma}.$$

(iii) *We have $\omega(\mathcal{A},\sigma) = \omega(\mathcal{A}_1,\sigma_1) \cup 1_{\mathcal{A}_2} \in H^1_{\mathrm{fppf}}(S, \mathcal{S}\mathit{kew}_{\mathcal{A},\sigma}/\mathcal{A}\mathit{lt}_{\mathcal{A},\sigma})$ where the cup-product arises from the morphism*

$$(\mathcal{S}\mathit{kew}_{\mathcal{A}_1,\sigma_1}/\mathcal{A}\mathit{lt}_{\mathcal{A}_1,\sigma_1}) \times \mathcal{S}\mathit{ym}_{\mathcal{A}_2,\sigma_2} \to \mathcal{S}\mathit{kew}_{\mathcal{A},\sigma}/\mathcal{A}\mathit{lt}_{\mathcal{A},\sigma}.$$

Proof. (i): The statement is local and follows then from the construction of tensor products in Proposition 5.3.

(ii): The map $\otimes 1_{\mathcal{A}_2} \colon \mathcal{A}_1 \to \mathcal{A}$, $a_1 \mapsto a_1 \otimes 1_{\mathcal{A}_2}$, restricts to both $\mathcal{S}\mathit{kew}_{\mathcal{A}_1,\sigma_1} \to \mathcal{S}\mathit{kew}_{\mathcal{A},\sigma}$ and $\mathcal{S}\mathit{ymd}_{\mathcal{A}_1,\sigma_1} \to \mathcal{S}\mathit{ymd}_{\mathcal{A},\sigma}$. Therefore, we have a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S}\mathit{kew}_{\mathcal{A}_1,\sigma_1} & \longrightarrow & \mathcal{A}_1 & \xrightarrow{1+\sigma_1} & \mathcal{S}\mathit{ymd}_{\mathcal{A}_1,\sigma_1} & \longrightarrow & 0. \\
 & & \downarrow{\scriptstyle \otimes 1_{\mathcal{A}_2}} & & \downarrow{\scriptstyle \otimes 1_{\mathcal{A}_2}} & & \downarrow{\scriptstyle \otimes 1_{\mathcal{A}_2}} & & \\
0 & \longrightarrow & \mathcal{S}\mathit{kew}_{\mathcal{A},\sigma} & \longrightarrow & \mathcal{A} & \xrightarrow{1+\sigma} & \mathcal{S}\mathit{ymd}_{\mathcal{A},\sigma} & \longrightarrow & 0.
\end{array}$$

whose rows are exact sequences of $\mathcal{O}$–modules. We claim this diagram commutes. Commutativity of the left square is clear, so we check commutativity of the right square with a computation. Let $a \in \mathcal{A}_1$, then

$$\begin{aligned}
(1+\sigma) \circ (\otimes 1_{\mathcal{A}_2})(a) &= (1+\sigma)(a \otimes 1_{\mathcal{A}_2}) = a \otimes 1_{\mathcal{A}_2} + (\sigma_1 \otimes \sigma_2)(a \otimes 1_{\mathcal{A}_2}) \\
&= a \otimes 1_{\mathcal{A}_2} + \sigma_1(a) \otimes 1_{\mathcal{A}_2} = (a + \sigma_1(a)) \otimes 1_{\mathcal{A}_2} = (\otimes 1_{\mathcal{A}_2}) \circ (1+\sigma_1)(a).
\end{aligned}$$

Thus, we get another commutative diagram involving boundary maps,

$$\begin{array}{ccc}
\mathcal{S}\mathit{ymd}_{\mathcal{A},\sigma_1}(S) & \xrightarrow{\delta_1} & H^1_{\mathrm{fppf}}(S, \mathcal{S}\mathit{kew}_{\mathcal{A}_1,\sigma_1}) \\
\downarrow{\scriptstyle \otimes 1_{\mathcal{A}_2}} & & \downarrow{\scriptstyle \cup 1_{\mathcal{A}_2}} \\
\mathcal{S}\mathit{ymd}_{\mathcal{A},\sigma}(S) & \xrightarrow{\delta} & H^1_{\mathrm{fppf}}(S, \mathcal{S}\mathit{kew}_{\mathcal{A},\sigma}).
\end{array}$$

Since $1_{\mathcal{A}} = 1_{\mathcal{A}_1} \otimes 1_{\mathcal{A}_2}$, the commutativity of the diagram implies that $\delta(1_{\mathcal{A}}) = \delta_1(1_{\mathcal{A}_1}) \cup 1_{\mathcal{A}_2}$ whence the desired formula

$$\Omega(\mathcal{A}, \sigma) = \Omega(\mathcal{A}_1, \sigma_1) \cup [1_{\mathcal{A}_2}] \in H^1_{\mathrm{fppf}}(S, \mathcal{S}kew_{\mathcal{A},\sigma}).$$

(iii): This follows from (ii) and the commutativity of the square

$$\begin{array}{ccc} \mathcal{S}kew_{\mathcal{A}_1,\sigma_1} \times \mathcal{S}ym_{\mathcal{A}_2,\sigma_2} & \longrightarrow & \mathcal{S}ym_{\mathcal{A},\sigma} \\ \downarrow{\scriptstyle \pi_1 \times \mathrm{Id}} & & \downarrow{\scriptstyle \pi} \\ (\mathcal{S}kew_{\mathcal{A}_1,\sigma_1} / \mathcal{A}lt_{\mathcal{A}_1,\sigma_1}) \times \mathcal{S}ym_{\mathcal{A}_2,\sigma_2} & \longrightarrow & \mathcal{S}kew_{\mathcal{A},\sigma} / \mathcal{A}lt_{\mathcal{A},\sigma}. \end{array}$$

where the horizontal maps are the tensor morphisms. □

6.14. **Remark.** Lemma 6.13(iii) shows that if $(\mathcal{A}_1, \sigma_1)$ is extendable to a quadratic pair, then so is $(\mathcal{A}, \sigma)$. We actually already knew this from Proposition 5.3. We shall see later in Proposition 7.7 that the converse is false.

A consequence of Proposition 5.6 is the following.

6.15. **Corollary.** *Let $(\mathcal{A}_1, \sigma_1)$ and $(\mathcal{A}_2, \sigma_2)$ be two Azumaya $\mathcal{O}$–algebras with symplectic involutions. Let $(\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1 \otimes \sigma_2)$ be the corresponding Azumaya $\mathcal{O}$–algebra with orthogonal involution. Then $\omega(\mathcal{A}, \sigma) = 0$.*

6.16. **Remark.** In the setting of Corollary 6.15, Remark 7.3 provides an explicit example such that $\Omega(\mathcal{A}, \sigma) \neq 0$.

The case of the strong obstruction is more involved.

6.17. **Lemma.** *Let $(\mathcal{A}_1, \sigma_1)$ and $(\mathcal{A}_2, \sigma_2)$ be two Azumaya $\mathcal{O}$–algebras with symplectic involutions. Let $(\mathcal{A}, \sigma) = (\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1 \otimes \sigma_2)$ be the tensor Azumaya $\mathcal{O}$–algebra with orthogonal involution. According to Lemma 5.5, we have $1_{\mathcal{A}_i} \in \mathcal{S}ymd_{\mathcal{A}_i,\sigma_i}(S)$ for $i = 1, 2$. Consider the boundary map $\delta_1 : \mathcal{S}ymd_{\mathcal{A}_1,\sigma_1} \to H^1_{\mathrm{fppf}}(S, \mathcal{S}kew_{\mathcal{A}_1,\sigma_1})$ arising from the exact sequence of $\mathcal{O}$–modules $0 \to \mathcal{S}kew_{\mathcal{A}\,\sigma} \to \mathcal{A} \to \mathcal{S}ymd_{\mathcal{A},\sigma} \to 0$. Then we have*

$$\Omega(\mathcal{A}, \sigma) = \delta_1(1_{\mathcal{A}_1}) \cup 1_{\mathcal{A}_2}$$

where $1_{\mathcal{A}_2} \in \mathcal{S}ymd_{\mathcal{A}_2,\sigma_2}(S) = H^0(S, \mathcal{S}ym_{\mathcal{A}_2,\sigma_2})$ and the cup-product arises from the tensor morphism $\mathcal{S}kew_{\mathcal{A}_1,\sigma_1} \times \mathcal{S}ym_{\mathcal{A}_2,\sigma_2} \to \mathcal{S}kew_{\mathcal{A},\sigma}$.

Proof. There is an fppf cover $\{T_i \to S\}_{i \in I}$ such that for each $i \in I$ there exists $\ell_i \in \mathcal{A}_1(T_i)$ with $\ell_i + \sigma_1(\ell_i) = 1$. Then we will have

$$(\ell_i \otimes 1) + (\sigma_1 \otimes \sigma_2)(\ell_i \otimes 1) = 1,$$

According to Lemma 6.6, $\Omega(\mathcal{A}, \sigma)$ is represented by the 1–cocycle

$$(\ell_i \otimes 1)|_{T_{ij}} - (\ell_j \otimes 1)|_{T_{ij}} \in \mathcal{S}kew_{\mathcal{A},\sigma}(T_{ij})$$

which also represents the class $\Omega(\mathcal{A}_1, \sigma_1) \cup 1_{\mathcal{A}_2}$. □

7. Examples of Non-trivial Obstructions

We use here Brion's theory of homogeneous torsors over abelian varieties [Br1, Br2, Br3] which extends that of Mukai for vector bundles [Mu]. All concepts not defined below can be found in a standard textbook on elliptic curves such as [KM].

7.1. **Non-trivial Strong Obstruction.** Let k be an algebraically closed field of characteristic 2, and let $S = E$ be an ordinary elliptic curve over k. Since E is ordinary, the 2-torsion points are $E[2] \cong \boldsymbol{\mu}_2 \times_k \mathbb{Z}/2\mathbb{Z}$. We set $E' = E$ and consider the $\boldsymbol{\mu}_2 \times_k \mathbb{Z}/2\mathbb{Z}$–torsor $E' \to E$ arising from multiplication by 2, where $\boldsymbol{\mu}_2 \times_k \mathbb{Z}/2\mathbb{Z}$ acts on E' by addition. Writing $\boldsymbol{\mu}_2$ multiplicatively and $\mathbb{Z}/2\mathbb{Z}$ additively, we have an injection of group schemes $i\colon \boldsymbol{\mu}_2 \times_k \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathrm{PGL}_2$ given by

$$(\varepsilon, 0) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \text{ and } (1,1) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which permits us to define a PGL_2–torsor $P = E' \wedge^{\boldsymbol{\mu}_2 \times_k \mathbb{Z}/2\mathbb{Z}} \mathrm{PGL}_2$ over E. Note that the isomorphism class of P is the image of the isomorphism class of E' under the induced map on cohomology

$$i_*\colon H^1(E, \boldsymbol{\mu}_2 \times_k \mathbb{Z}/2\mathbb{Z}) \to H^1(E, \mathrm{PGL}_2).$$

The twist of $\mathrm{M}_2(\mathcal{O})$ by P is a quaternion $\mathcal{O}$–algebra $\mathcal{Q}$. We can describe this quaternion algebra explicitly using a Čech 1-cocycle in PGL_2 with respect to the cover $\{E' \to E\}$, which is an fppf cover by [KM, 2.3], i.e., an element $\phi \in \mathrm{PGL}_2(E' \times_E E')$ satisfying

$$p_{23}^*(\phi|_{E' \times_E E' \times_E E'}) \circ p_{12}^*(\phi|_{E' \times_E E' \times_E E'}) = p_{13}^*(\phi|_{E' \times_E E' \times_E E'})$$

where $p_{ij}\colon E' \times_E E' \times_E E' \to E' \times_E E'$ is the projection which keep the i^{th} and j^{th} factors. The cocycle condition above also means that such a ϕ will be a glueing datum as in [St, Tag 04TP]. To give this cocycle, we first find the global sections of E' and $E' \times_E E'$. We have that $\mathcal{O}(E') = k$, and since $E' \times_E E' \cong E' \times_k (\boldsymbol{\mu}_2 \times_k \mathbb{Z}/2\mathbb{Z})$ we also have

$$\mathcal{O}(E' \times_E E') \cong (k[x]/\langle x^2 - 1\rangle)^2$$

because $\boldsymbol{\mu}_2 \times_k \mathbb{Z}/2\mathbb{Z}$ is represented as an affine k–group scheme by $(k[x]/\langle x^2 - 1\rangle)^2$. Note that the two projections $p_i\colon E' \times_E E' \to E'$ have the same associated k–algebra morphisms

$$\begin{aligned} p_1^*(E') = p_2^*(E')\colon k &\to (k[x]/\langle x^2 - 1\rangle)^2 \\ c &\mapsto (c, c). \end{aligned}$$

The torsor $E' \to E$ corresponds to a cocycle in $\boldsymbol{\mu}_2 \times_k \mathbb{Z}/2\mathbb{Z}$, and one can compute that this cocycle is

$$\mathrm{Id} \in \mathrm{Hom}((k[x]/\langle x^2 - 1\rangle)^2, (k[x]/\langle x^2 - 1\rangle)^2) = (\boldsymbol{\mu}_2 \times_k \mathbb{Z}/2\mathbb{Z})(E' \times_E E').$$

By interpreting the image of x as the element in $\boldsymbol{\mu}_2$ and the image of $(1,0)$ as the element in $\mathbb{Z}/2\mathbb{Z}$, we view this as being the element

$$((x,x),(1,0)) \in (\boldsymbol{\mu}_2 \times_k \mathbb{Z}/2\mathbb{Z})((k[x]/\langle x^2-1\rangle)^2).$$

This then maps to the cocycle

$$\phi = \mathrm{Inn}\left(\begin{bmatrix} (0,1) & (1,0) \\ (x,0) & (0,x) \end{bmatrix}\right) \in \mathrm{PGL}_2(E' \times_E E').$$

which we use to twist $\mathrm{M}_2(\mathcal{O})$. The twisted algebra is then the result of gluing two copies of $\mathrm{M}_2(\mathcal{O})|_{E'}$ using ϕ, and so by [St, Tag 04TR] we obtain the quaternion algebra $\mathcal{Q}$. For $T \in \mathfrak{Sch}_E$, we have

$$\mathcal{Q}(T) = \{B \in \mathrm{M}_2(\mathcal{O}(T \times_E E')) \mid \phi(B|_{T\times_E E' \times_E \times E'}) = B|_{T\times_E E' \times_E \times E'}\}.$$

Since k has characteristic 2, the canonical quaternion involution on $\mathrm{M}_2(\mathcal{O})$ is the split orthogonal involution in degree 2 of Example 4.8(b)

$$\eta_0 \colon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$

The canonical involution on $\mathcal{Q}$, denoted by θ, is the descent of η_0 and hence $(\mathcal{Q},\theta)$ is a quaternion algebra with orthogonal involution.

Let n be a positive integer and now work over the abelian k–variety $S = E^n$. We define the Azumaya algebra

$$\mathcal{A} = p_1^*(\mathcal{Q}) \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} p_n^*(\mathcal{Q})$$

where $p_i \colon E^n \to E$ is the i^{th} projection and p_i^* the pullback of sheaves. It comes with the tensor product involution $\sigma = p_1^*(\theta) \otimes \cdots \otimes p_n^*(\theta)$, which is orthogonal. Alternatively, $\mathcal{A}$ may be viewed as the following twist of $\mathrm{M}_{2^n}(\mathcal{O})$. Set $S' = S$ and view it as a scheme over S with respect to the multiplication by 2 map. This makes $S' \to S$ a $(\boldsymbol{\mu}_2 \times_k \mathbb{Z}/2\mathbb{Z})^n$–torsor and $\{S' \to S\}$ an fppf cover. Since we are looking for a cocycle in $\mathrm{PGL}_{2^n}(S' \times_S S')$ we note that $\mathcal{O}(S) = k$, and since $S' \times_S S' \cong (E' \times_E E')^n$ we have

$$\mathcal{O}(S' \times_S S') = \left((k[x]/\langle x^2-1\rangle)^2\right)^{\otimes n}.$$

where the tensor product is over k. Using the decomposition

$$\mathrm{M}_{2^n}(S' \times_S S') \cong \mathrm{M}_2((k[x]/\langle x^2-1\rangle)^2) \otimes_k \ldots \otimes_k \mathrm{M}_2((k[x]/\langle x^2-1\rangle)^2)$$

we have $\phi \otimes \ldots \otimes \phi \in \mathrm{PGL}_{2^n}(S' \times_S S')$, and this is our desired cocycle. In this view, the tensor product involution on $\mathcal{A}$ is the descent of the involution $\eta' = \eta_0 \otimes \ldots \otimes \eta_0$ on $\mathrm{M}_{2^n}(k) \cong (\mathrm{M}_2(k))^{\otimes n}$.

7.2. **Lemma.** *Consider the Azumaya algebra with involution $(\mathcal{A},\sigma)$ defined above.*

(i) *$(\mathcal{A},\sigma)$ can be extended to a quadratic triple. In particular, $(\mathcal{A},\sigma)$ is locally quadratic with trivial weak obstruction.*
(ii) *$\mathcal{A}(S) \cong k$.*
(iii) *The strong obstruction $\Omega(\mathcal{A},\sigma)$ is non-trivial.*

Proof. (i): We begin by considering the linear form

$$f'\colon \mathcal{S}ym_{\mathrm{M}_{2^n}(\mathcal{O}),\sigma_{2^n}} \to \mathcal{O}$$
$$B \mapsto \mathrm{Trd}_{\mathrm{M}_{2^n}(\mathcal{O})}\left(\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{2^{n-1}} \right) B \right)$$

where $I_{2^{n-1}}$ denotes the $2^{n-1} \times 2^{n-1}$ identity matrix. This is an instance of the construction of Example 4.3(b), and so $(\mathrm{M}_{2^n}(\mathcal{O}), \sigma_{2^n}, f')$ is a quadratic triple. The form $f'|_{S'}$ will descend to a suitable linear form on $\mathcal{S}ym_{\mathcal{A},\sigma}$ if $f'|_{S'\times_S S'} \circ (\phi \otimes \ldots \otimes \phi) = f'|_{S'\times_S S'}$. We verify this with the following computation, which uses the fact that $\phi^2 = \mathrm{Id}$.

$$\begin{aligned}
&\mathrm{Trd}_{\mathrm{M}_{2^n}(\mathcal{O})}\left(\left(\begin{bmatrix} (1,1) & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{2^{n-1}} \right) \cdot (\phi \otimes \ldots \otimes \phi)(__) \right) \\
&= \mathrm{Trd}_{\mathrm{M}_{2^n}(\mathcal{O})}\left((\phi \otimes \ldots \otimes \phi)\left(\begin{bmatrix} (1,1) & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{2^{n-1}} \right) \cdot __ \right) \\
&= \mathrm{Trd}_{\mathrm{M}_{2^n}(\mathcal{O})}\left(\left(\begin{bmatrix} (0,1) & 0 \\ 0 & (1,0) \end{bmatrix} \otimes I_{2^{n-1}} \right) \cdot __ \right).
\end{aligned}$$

Then, because

$$\left(\begin{bmatrix} (1,1) & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{2^{n-1}} \right) - \left(\begin{bmatrix} (0,1) & 0 \\ 0 & (1,0) \end{bmatrix} \otimes I_{2^{n-1}} \right) = \left(\begin{bmatrix} (1,0) & 0 \\ 0 & (1,0) \end{bmatrix} \otimes I_{2^{n-1}} \right)$$

which is an element of $\mathcal{A}\ell t_{\mathrm{M}_{2^n}(\mathcal{O}),\eta'}(S' \times_S S')$, we know by 4.3(b) that

$$f'|_{S'\times_S S'} \circ (\phi \otimes \ldots \otimes \phi) = f'|_{S'\times_S S'}.$$

Therefore $f'|_{S'}$ descends, and there exists a linear form $f\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ such that $(\mathcal{A}, \sigma, f)$ is a quadratic triple.

(ii): By construction we have that

$$\mathcal{A}(S) = \{B \in \mathrm{M}_{2^n}(k)^2 \mid (\phi \otimes \ldots \otimes \phi)(B|_{S'\times_S S'}) = B|_{S'\times_S S'}\}.$$

Given $B \in \mathcal{A}(S)$, we may write B uniquely as

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes B_1 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes B_2 + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes B_3 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes B_4$$

with $B_i \in \mathrm{M}_{2^{n-1}}(k)$, and then setting $B|_{S'\times_S S'} = \overline{B}$ we have

$$\begin{aligned}
\overline{B} &= \begin{bmatrix} (1,1) & 0 \\ 0 & 0 \end{bmatrix} \otimes \overline{B_1} + \begin{bmatrix} 0 & (1,1) \\ 0 & 0 \end{bmatrix} \otimes \overline{B_2} \\
&+ \begin{bmatrix} 0 & 0 \\ (1,1) & 0 \end{bmatrix} \otimes \overline{B_3} + \begin{bmatrix} 0 & 0 \\ 0 & (1,1) \end{bmatrix} \otimes \overline{B_4}.
\end{aligned}$$

Now we apply $\phi \otimes \ldots \otimes \phi$, setting $\overline{\phi} = \phi^{\otimes n-1}$.

$$\begin{aligned}
(\phi \otimes \ldots \otimes \phi)(\overline{B}) &= \begin{bmatrix} (0,1) & 0 \\ 0 & (1,0) \end{bmatrix} \otimes \overline{\phi}(\overline{B_1}) + \begin{bmatrix} 0 & (0,x) \\ (x,0) & 0 \end{bmatrix} \otimes \overline{\phi}(\overline{B_2}) \\
&+ \begin{bmatrix} 0 & (x,0) \\ (0,x) & 0 \end{bmatrix} \otimes \overline{\phi}(\overline{B_3}) + \begin{bmatrix} (1,0) & 0 \\ 0 & (0,1) \end{bmatrix} \otimes \overline{\phi}(\overline{B_4}).
\end{aligned}$$

Since this is equal to $\overline{B}$, linear independence then requires that

$$\overline{B_1} = \overline{\phi}(\overline{B_4}) = \overline{B_4} = \overline{\phi}(\overline{B_1}), \text{ and}$$
$$\overline{B_2} = \overline{B_3} = \overline{\phi}(\overline{B_2}) = \overline{\phi}(\overline{B_3}) = 0$$

and so we can conclude that $\overline{B} = I_2 \otimes \overline{B'}$ for some $B' \in \mathrm{M}_{2^{n-1}}(k)$ such that $(\phi \otimes \ldots \otimes \phi)(\overline{B'}) = \overline{B'}$, now with only $n-1$ tensor factors. Hence, by induction we need only address the case of two 2×2 matrices. There we have that

$$\phi(\overline{B}) = \begin{bmatrix} (b_4, b_1) & (b_3 x, b_2 x) \\ (b_2 x, b_3 x) & (b_1, b_4) \end{bmatrix} = \begin{bmatrix} (b_1, b_1) & (b_2, b_2) \\ (b_3, b_3) & (b_4, b_4) \end{bmatrix} = \overline{B}.$$

Since $b_i \in k$, this can only happen when $B = aI_2$ for some $a \in k$. Therefore, overall

$$\mathcal{A}(S) = \{I_{2^{n-1}} \otimes aI_2 \mid a \in k\} \cong k.$$

(iii): The exact sequence

$$0 \to \mathcal{S}kew_{\mathcal{A},\sigma} \to \mathcal{A} \to \mathcal{S}ymd_{\mathcal{A},\sigma} \to 0$$

gives rise to the exact sequence

$$0 \to \mathcal{S}kew_{\mathcal{A},\sigma}(S) \to \mathcal{A}(S) \to \mathcal{S}ymd_{\mathcal{A},\sigma}(S) \xrightarrow{\delta} H^1_{\mathrm{fppf}}(S, \mathcal{S}kew_{\mathcal{A},\sigma}) \tag{7.2.1}$$

by taking global sections. Since $(\mathcal{A},\sigma)$ is locally quadratic, we have $1 \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$ and we recall that $\Omega(\mathcal{A},\sigma) = \delta(1)$ by Definition 6.4. Since $\mathcal{A}(S) = k$ and $\sigma(S)$ is k–linear, all elements of $\mathcal{A}(S)$ are symmetric, and since k is characteristic 2, all elements of $\mathcal{A}(S)$ are also skew-symmetric. Hence, $\mathcal{S}kew_{\mathcal{A},\sigma}(S) = k$ also. Therefore, the sequence (7.2.1) is of the form

$$0 \to k \to k \to \mathcal{S}ymd_{\mathcal{A},\sigma}(S) \xrightarrow{\delta} H^1_{\mathrm{fppf}}(S, \mathcal{S}kew_{\mathcal{A},\sigma})$$

and we obtain that $\Omega(\mathcal{A},\sigma) = \delta(1) \neq 0$. □

7.3. **Remark.** Note that for $n = 2$, $(\mathcal{A},\sigma)$ is a tensor product of two Azumaya algebras with symplectic involutions, and so Corollary 6.15 provides another proof that the weak obstruction is zero in Lemma 7.2(iii). However, the important point is that the strong obstruction does not vanish.

7.4. **Non-trivial Weak Obstruction.** We continue working over an algebraically closed base field k of characteristic 2. According to a result by Serre [Se, prop. 15], for each finite group Γ there exists a Galois Γ–cover $Y \to S$ as in [St, Tag 03SF] such that S and Y are connected smooth projective k–varieties. We use Serre's results with the group $\Gamma = \mathrm{PGL}_2(\mathbb{F}_4)$ to obtain a Γ–cover $\pi : Y \to S$ using Serre's result. We take S to be our base scheme. We denote by Γ_k the constant group scheme associated to Γ. It is affine and represented by $k^{|\Gamma|}$ with componentwise multiplication. We write this algebra as

$$k^{|\Gamma|} = \{(c_g)_{g\in\Gamma} \mid c_g \in k\}.$$

The map $Y \to S$ is then a Γ_k–torsor and $\{Y \to S\}$ is an fppf cover. Since Γ_k embeds in PGL_2, which we view as a group scheme over k, we can define the PGL_2–torsor $P = Y \wedge^{\Gamma_k} \mathrm{PGL}_2$ over S. The twist of $\mathrm{M}_2(\mathcal{O})$ by P is a quaternion $\mathcal{O}$–algebra $\mathcal{Q}$, which of course is not the same $\mathcal{Q}$ as in section 7.1. Here as well we may describe $\mathcal{Q}$ explicitly using cocycles. As before, we first describe the global sections of Y and $Y \times_S Y$. We know that $\mathcal{O}(Y) \cong k$, and since $Y \to S$ is a Galois extension with Galois group Γ_k, we have $Y \times_S Y \cong Y \times_k \Gamma_k$ and so

$$\mathcal{O}(Y \times_S Y) \cong k^{|\Gamma|}.$$

Therefore, we have that

$$\mathrm{M}_2(\mathcal{O}(Y \times_S Y)) = \mathrm{M}_2(k^{|\Gamma|}) \cong \mathrm{M}_2(k)^{|\Gamma|}.$$

Identifying Γ_k with its embedding in PGL_2, we have the element

$$(g)_{g\in\Gamma} \in \mathrm{PGL}_2(Y \times_S Y)$$

which we use as our cocycle. Thus, again by [St, Tag 04TR] the algebra $\mathcal{Q}$ is described over $T \in \mathfrak{Sch}_S$ by

$$\mathcal{Q}(T) = \{B \in \mathrm{M}_2(\mathcal{O}(T \times_S Y)) \mid g(B) = B,\ \forall\, g \in \Gamma\}.$$

Or, more concisely, $\mathcal{Q}(T) = \mathrm{M}_2(\mathcal{O}(T \times_S Y))^{\Gamma_k}$ are the fixed points. The canonical involution η_0 on $\mathrm{M}_2(\mathcal{O})$ of Example 4.8(b) is orthogonal and descends to an orthogonal involution θ on $\mathcal{Q}$, which is the canonical involution on $\mathcal{Q}$.

7.5. **Lemma.** *$(\mathcal{Q}, \theta)$ is locally quadratic, but cannot be extended to a quadratic triple. Thus, the weak obstruction is non-trivial, i.e., $\omega(\mathcal{Q}, \theta) \neq 0 \in H^1(S, \mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma})$.*

Proof. Since $(\mathcal{Q}, \theta)$ is a twisted form of $\mathrm{M}_2(\mathcal{O})$ which splits over $\{Y \to S\}$, we have that

$$(\mathcal{Q}, \theta)|_Y \cong (\mathrm{M}_2(\mathcal{O}), \eta_0)|_Y.$$

Since we have the split quadratic triple $(\mathrm{M}_2(\mathcal{O}), \eta_0, f_0)|_Y$ of Example 4.8(b) and $\{Y \to S\}$ is an fppf cover, we obtain that $(\mathcal{Q}, \theta)$ is locally quadratic by Lemma 6.2(iv).

Now, assume that we can extend $(\mathcal{Q}, \theta)$ to a quadratic triple $(\mathcal{Q}, \theta, f)$ over S. The linear map f then fits into the following commutative diagram with exact rows.

$$\begin{array}{ccccc}
\mathcal{S}ym_{\mathcal{Q},\theta}(S) & \longrightarrow & \mathcal{S}ym_{\mathrm{M}_2(\mathcal{O}),\eta_0}(Y)^2 & \xrightarrow{\pi_1} & \mathcal{S}ym_{\mathrm{M}_2(\mathcal{O}),\eta_0}(Y \times_S Y) \\
\downarrow{\scriptstyle f(S)} & & \downarrow{\scriptstyle f(Y)\times f(Y)} & & \downarrow{\scriptstyle f(Y\times_S Y)} \\
k & \longrightarrow & k \times k & \xrightarrow{\pi_2} & k^{|\Gamma|}
\end{array}$$

where $\pi_1(B_1, B_2) = (g(B_1) - B_2)_{g\in\Gamma}$ and $\pi_2(c_1, c_2) = (c_1 - c_2)_{g\in\Gamma}$. Commutativity of the diagram then enforces that

$$f(Y \times_S Y)\,((g(B_1) - B_2)_{g\in\Gamma}) = (f(Y)(B_1) - f(Y)(B_2))_{g\in\Gamma}$$

which is equivalent to

$$\Leftrightarrow (f(Y)(g(B_1)))_{g\in\Gamma} = (f(Y)(B_1))_{g\in\Gamma}$$

and hence $f(Y)$ must be Γ–equivariant. We now argue that this means $f(Y)$ must be zero. Certainly, $f(Y)$ is of the form

$$f\left(\begin{bmatrix} a & b \\ c & a \end{bmatrix}\right) = au + bv + cw$$

for some $u, v, w \in k = \mathcal{O}(Y)$. By Γ–equivariance we obtain

$$w = f\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = v,$$

as well as

$$w = f\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = u + v + w$$

which implies that $u+v = 0$, i.e., $u = v$. Therefore we must have $u = v = w$. Finally, consider $0, 1 \neq \lambda \in \mathbb{F}_4$. We also have

$$w = f\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}\right) = \lambda w$$

and so $u = v = w = 0$, meaning $f(Y) \equiv 0$. However, this contradicts Remark 4.2, and therefore no such f extending $(\mathcal{Q}, \theta)$ can exist. □

7.6. **Remark.** We also notice that the trace induces an isomorphism

$$H_0(\Gamma, \mathrm{M}_2(k)) \xrightarrow{\sim} k.$$

Since $\mathrm{Sym}(\mathrm{M}_2(k), \eta_0)$ has dimension 3, $H_0(\Gamma, \mathrm{M}_2(k))$ has dimension ≤ 1. The trace induces a surjective map $H_0(\Gamma, \mathrm{M}_2(k)) \to k$, it is an isomorphism by counting dimensions.

Enlarging the example of section 7.4 is quite surprising. Unlike in section 7.1, we cannot use the tensor product construction to produce Azumaya algebras with involution of larger degree which have non-trivial weak obstruction.

7.7. **Proposition.** *Let $n \geq 1$ and define on the k–variety S^n the following Azumaya algebra with orthogonal involution of degree 2^n*

$$(\mathcal{A}_n, \sigma_n) = p_1^*(\mathcal{Q}, \theta) \otimes_{\mathcal{O}_{S^n}} \cdots \otimes_{\mathcal{O}_{S^n}} p_n^*(\mathcal{Q}, \theta).$$

Then $(\mathcal{A}_n, \sigma_n)$ is locally quadratic for any $n \geq 1$, and $(\mathcal{A}_n, \sigma_n)$ can be extended to a quadratic triple if and only if $n \geq 2$. Equivalently, $\omega(\mathcal{A}_n, \sigma_n) \neq 0$ if and only if $n = 1$.

Proof. Lemma 7.5 is the case $n = 1$ and shows in particular that $(\mathcal{Q}, \theta)$ is locally quadratic. Since k is characteristic 2, this means by Lemma 5.5 that θ is also symplectic. Therefore $p_i^*(\mathcal{Q}, \theta)$ have symplectic involutions and so when $n = 2$, Proposition 5.6 shows that $p_1^*(\mathcal{Q}, \theta) \otimes_{\mathcal{O}} p_2^*(\mathcal{Q}, \theta)$ can be extended to a quadratic triple. Then Proposition 5.3 handles the cases of $n \geq 3$. □

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