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A function on an algebra is congruence preserving if, for any congruence, it maps pairs of congruent elements onto pairs of congruent elements. We show that on the algebra of binary trees whose leaves are labeled by letters of an alphabet containing at least three letters, a function is congruence preserving if and only if it is a polynomial function, thus exhibiting the first example of a non commutative and non associative affine complete algebra.

Keywords: algebras, trees, congruences

Merci à Maurice pour nombre de discussions algébriques passionnantes

1 Introduction

A function on an algebra is congruence preserving if, for any congruence, it maps pairs of congruent elements onto pairs of congruent elements.

A polynomial function on an algebra is a function defined by a term of the algebra using variables, constants and the operations of the algebra. Obviously, every polynomial function is congruence preserving.

Algebras where all congruence preserving functions are polynomial functions are called *affine complete* in the terminology introduced by Werner (1971). They are extensively studied in the book by Kaarli and Pixley (2001).

In the commutative case, many algebras have been shown to be affine complete: Boolean algebras (Grätzer , 1962), p-rings with unit (Iskander , 1972). For distributive lattices, Ploščica and Haviar (2008) described congruence preserving functions, and Grätzer (1964) determined which distributive lattices are affine complete. Affine completenes is an intrinsic property of an algebra, which fails to hold even for very simple algebras: e.g., in $\mathcal{A} = \langle \mathbb{Z}, + \rangle$, the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(x) = \texttt{if } x \geq 0 \texttt{ then } \frac{\Gamma(1/2)}{2 \times 4^x \times x!} \int_1^\infty e^{-t/2}(t^2-1)^x dt \texttt{ else } -f(-x).$$

has been proved to be congruence preserving (Cégielski et al. , 2015), but it is *not a polynomial function* because its power series is infinite. Hence $\mathcal{A} = \langle \mathbb{Z}, + \rangle$ is not affine complete.

In the non commutative case, very little is known about affine complete algebras. We proved in Arnold et al. (2020) that the free monoid Σ^* is an associative non commutative affine complete algebra if Σ has at least three letters, and we proved in Arnold et al. (2020) a partial result concerning a non commutative and non associative algebra: every *unary congruence preserving* function $f\colon T(\Sigma) \to T(\Sigma)$ is a polynomial function, where $T(\Sigma)$ is the algebra of *full* binary trees with leaves labelled by letters of an alphabet Σ having at least three letters. We here generalize this result proving that a congruence preserving function $f\colon \mathcal{T}(\Sigma)^n \to \mathcal{T}(\Sigma)$ of any arity n is a polynomial function, where $\mathcal{T}(\Sigma)$ is the algebra of arbitrary (possibly non full) binary trees with labelled leaves. This generalization is twofold: (1) non full binary trees are allowed in $\mathcal{T}(\Sigma)$, and (2) congruence preserving functions of arbitrary arity are allowed. This exhibits an example of a non commutative and non associative affine complete algebra. Non commutative and non associative algebras are of constant use in Computer Science, and congruences are also very often used, whence the potential usefulness of our result.

We first define binary trees and their congruences, we then study conditions which will enable us to prove that every congruence preserving function is a polynomial function, and to finally prove the affine completeness of $T(\Sigma)$.

2 The algebra of binary trees

2.1 Trees, congruences

For an algebra $\mathcal{A}$ with domain A, a *congruence* $\sim$ on $\mathcal{A}$ is an equivalence relation on A which is compatible with the operations of $\mathcal{A}$. We state the characterization of congruences by kernels of homomorphisms.

Lemma 2.1. *Let $\mathcal{A} = \langle A\,, \star\rangle$ be an algebra with a binary operation $\star$. An equivalence $\sim$ on A is a congruence iff there exists an algebra $\mathcal{B} = \langle B\,, *\rangle$ with a binary operation $*$ and there exists $\theta\colon A \to B$ a homomorphism such that $\sim$ coincides with the kernel congruence $\ker(\theta)$ of θ, defined by $x \sim_\theta y$ iff $\theta(x) = \theta(y)$.*

Let Σ be an alphabet not containing $\{0, 1\}$. We shall represent the algebra of binary trees over Σ, i.e., trees with leaves labeled by letters of Σ, as a set of words $\mathcal{T}(\Sigma)$ on the alphabet $\Sigma \cup \{0, 1\}$, together with the binary product operation $\star$.

Definition 2.2. The algebra $\mathcal{B} = \langle \mathcal{T}(\Sigma), \star\rangle$ of binary trees over Σ is defined as follows.

- A binary tree over Σ is a finite set of words $t \subseteq \{0, 1\}^*\Sigma$ such that: For any $ua, vb \in t$, if $ua \neq vb$ then u is not a prefix of v and v is not a prefix of u. The carrier set $\mathcal{T}(\Sigma)$ is the set of all binary trees. The empty set $\emptyset$ is a binary tree denoted by $\mathbf{0}$.
- The binary product operation $\star$ is defined by: for $t,\ t' \in \mathcal{T}(\Sigma)$, $t \star t' = 0.t \cup 1.t'$. In particular, $\mathbf{0} \star \mathbf{0} = \mathbf{0}$.

When the alphabet Σ is clear, we will denote by $\mathcal{T}$ the set of all binary trees. Trees are generated by $\{\mathbf{0}\} \cup \Sigma$ and the operation $\star$.

An essential property of this algebra $\mathcal{B}$ is that its elements are uniquely decomposable.

Lemma 2.3 (Unicity of decomposition)**.** *If t is a tree not in $\{\mathbf{0}\} \cup \Sigma$ then there exists a unique ordered pair $\langle t_1, t_2\rangle \neq \langle \mathbf{0}, \mathbf{0}\rangle$ in $\mathcal{T}^2$ such that $t = t_1 \star t_2$.*

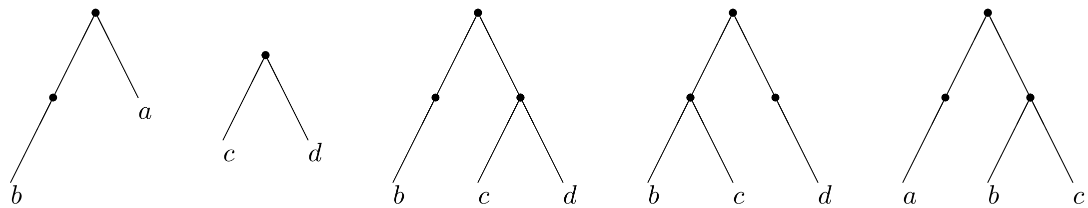

Fig. 1: From left to right, $t = \{00b, 1a\}$, $\tau = \{0c, 1d\}$, $t_1 = \gamma_{a\to\tau}(t) = \{00b, 01c, 11d\}$, $t_2 = \{00b, 01c, 11d\}$, $t_3 = \{00a, 10b, 11c\}$. Trees $t_1,\ t_2,\ t_3$ have the same size 6, trees t_1 and t_3 are similar (have the same skeleton.)

This property allows us to associate with each $t \in \mathcal{T}$ its *size* $|t|$ (number of nodes)
– $|\mathbf{0}| = 0$, and for all $a \in \Sigma$, $|a| = 1$,
– if $t \notin \{\mathbf{0}\} \cup \Sigma$ then $t = t_1 \star t_2$, and $|t| = |t_1| + |t_2| + 1$.

If $|t| > 1$ then there exist t_1, t_2 with $|t_i| < |t|$ such that $t = t_1 \star t_2$. Trees $t \star\ t'$, $\mathbf{0} \star t'$, $t \star \mathbf{0}$ are trees whose root has two sons, a single right son, a single left son, respectively. See Figure 1.

2.2 Homomorphisms, graftings

Lemma 2.4. *Let $\mathcal{B} = \langle B\,,\, *\rangle$ be an algebra with a binary operation $*$. Every mapping $h\colon \Sigma \to B$ can be uniquely extended to a homomorphism $h\colon \mathcal{T} \to B$.*

Remark 2.5. 1) Because of the universal property of Lemma 2.4, homomorphisms are (uniquely) defined by giving their values on Σ.

2) For every endomorphism, $h(\mathbf{0}) = \mathbf{0}$. Otherwise, as $\mathbf{0} = \mathbf{0} \star \mathbf{0}$, $h(\mathbf{0}) = h(\mathbf{0}) \star h(\mathbf{0})$; if $h(\mathbf{0}) = t$ with $|t| \geq 1$ then $t = t \star t$ implies $|t| = 2|t| + 1$, a contradiction.

Definition 2.6. For a given $a \in \Sigma$, let ν_a be the endomorphism sending Σ onto a. If for some $a \in \Sigma$, $\nu_a(t) = \nu_a(t')$, trees t and t' are said to be *similar*, which is denoted by $t \sim_s t'$.

Note that the congruence $\sim_s$ does not depend on the choice of the letter $a \in \Sigma$ since $\nu_b(t) = \nu_b(\nu_a(t))$. From an intuitive viewpoint, $t \sim_s t'$ means that t and t' have the same skeleton, i.e., they are identical except for the leaf labels. See Figure 1.

Other congruences fundamental for our proof are the kernels of the grafting endomorphisms, defined below.

Definition 2.7 (Grafting)**.** Let $a \in \Sigma$ and $\tau \in \mathcal{T}$. Then the grafting $\gamma_{a\to\tau}\colon \mathcal{T} \to \mathcal{T}$ is the endomorphism defined by its restriction on Σ

$$\gamma_{a\to\tau}(b) = \begin{cases} \tau & \text{if } b = a, \\ b & \text{if } b \neq a. \end{cases}$$

In other words, for any $a \in \Sigma$ and any $\tau \in \mathcal{T}$, $\gamma_{a\to\tau}$ is the endomorphism sending the letter a on τ and each other letter on itself.

An endomorphism h of $\langle \mathcal{T}(\Sigma), \star\rangle$ is *idempotent* if for every $t \in \mathcal{T}$, $h(h(t)) = h(t)$. By Lemma 2.4, h is idempotent iff for every $a \in \Sigma$, $h(h(a)) = h(a)$. For instance if a does not occur in τ then $\gamma_{a\to\tau}$ is idempotent.

Proposition 2.8. *Let $\tau \in \mathcal{T}$, let $t,\ t' \in \mathcal{T}$, and let $a_1 \neq a_2$ be two letters in Σ. If $\gamma_{a_i \to \tau}(t) = \gamma_{a_i \to \tau}(t')$ for $i = 1, 2$, then $t = t'$.*

Proof: By induction on $\min(|t|, |t'|)$.

Basis Case 0: If $\min(|t|, |t'|) = 0$ then one of t, t' is $\mathbf{0}$, say $t = \mathbf{0}$. If $t' \neq \mathbf{0}$ then t' contains at least one occurrence of some letter b. As $\gamma_{a_i \to \tau}(t') = \gamma_{a_i \to \tau}(t) = \gamma_{a_i \to \tau}(\mathbf{0}) = \mathbf{0}$, we have $\gamma_{a_i \to \tau}(t') = \mathbf{0}$, which implies (because $t' \neq \mathbf{0}$ was supposed) that $\tau = \mathbf{0}$. Then $\gamma_{a_i \to \tau}(t') = \mathbf{0}$ implies that all leaves of t' are equal to both a_1 and a_2, a contradiction. Hence $t' = \mathbf{0}$ and $t = t'$.

Basis Case 1: If $\min(|t|, |t'|) = 1$ then t or t' is a letter, say $t = b$, and there is one i, say $i = 1$, such that $a_1 \neq b$, thus $b = \gamma_{a_1 \to \tau}(t) = \gamma_{a_1 \to \tau}(t')$.

- If t' is a letter $c \neq b$, then $\gamma_{a_1 \to \tau}(c) = b$. If $c = a_1$ then $b = \gamma_{a_1 \to \tau}(c) = \tau$. Since $\gamma_{a_2 \to \tau}(c) = c = \gamma_{a_2 \to \tau}(b) \in \{\tau, b\} = \{b\}$, we have that $c = b$, a contradiction. If $c \neq a_1$ and $\gamma_{a_1 \to \tau}(c) = c \neq b = \gamma_{a_1 \to \tau}(c)$, a contradiction. Hence $t' = t = b$.

- If $|t'| > 1$ then $t' = t'_1 \star t'_2$, and $\gamma_{a_1 \to \tau}(t') = \gamma_{a_1 \to \tau}(t'_1) \star \gamma_{a_1 \to \tau}(t'_2)$ which can be only of size 0 or ≥ 2, contradicting $\gamma_{a_1 \to \tau}(t') = b$. this case is excluded.

Induction: If $\min(|t|, |t'|) > 1$ then $t = t_1 \star t_2$ and $t' = t'_1 \star t'_2$ with $\min(|t_i|, |t'_i|) < \min(|t|, |t'|)$, for $i = 1, 2$. By Lemma 2.3, $\gamma_{a_j \to \tau}(t_1) \star \gamma_{a_j \to \tau}(t_2) = \gamma_{a_j \to \tau}(t'_1) \star \gamma_{a_j \to \tau}(t'_2)$ implies $\gamma_{a_j \to \tau}(t_i) = \gamma_{a_j \to \tau}(t'_i)$, for $j = 1, 2$. By the induction hypothesis $t_i = t'_i$, hence $t = t'$. □

Proposition 2.9. *Let us fix $a \in \Sigma$, with $|\Sigma| \geq 3$, $t,\ t' \in \mathcal{T}$ such that $t \sim_s t'$.*
(1) If, for some $\tau \in \mathcal{T}$ of size $|\tau| \neq 1$, $\gamma_{a \to \tau}(t) = \gamma_{a \to \tau}(t')$, then $t = t'$.
(2) If, for all $b \neq a$, $b \in \Sigma$, $\gamma_{a \to b}(t) = \gamma_{a \to b}(t')$, then $t = t'$.

Proof: Both (1) and (2) are proved by induction on $|t| = |t'|$, and in both cases, the result obviously holds if $t = t' = \mathbf{0}$.

Basis: If $|t| = |t'| = 1$.
(1) We assume that $t = b \neq c = t'$.
(i) If $a \notin \{b, c\}$ then $\gamma_{a \to \tau}(t) = b \neq c = \gamma_{a \to \tau}(t')$, a contradiction.
(ii) Otherwise, $a \in \{b, c\}$, e.g., $a = b = t$, then $\gamma_{a \to \tau}(t) = \gamma_{a \to \tau}(a) = \tau$ and $\gamma_{a \to \tau}(t') = \gamma_{a \to \tau}(c) = c$, hence $\tau = c$, which contradicts $|\tau| \neq 1$.
(2) We assume that $t = b \neq c = t'$.
(i) The case $a \notin \{b, c\}$ yields a contradiction as in case (1).
(ii) Otherwise, e.g., $a = b$, there exists $d \notin \{a, c\}$, and we get $\gamma_{a \to d}(t) = \gamma_{a \to d}(a) = d$ and $\gamma_{a \to d}(t') = \gamma_{a \to d}(c) = c$, contradicting $\gamma_{a \to d}(t) = \gamma_{a \to d}(t')$.

Induction: As in Proposition 2.8 in both cases: since t and t' are similar, $t = t_1 \star t_2$ and $t' = t'_1 \star t'_2$ with t_i similar to t'_i and $|t_i| < |t'_i|$. □

2.3 Congruence preserving functions on trees

Definition 2.10. A function $f\colon \mathcal{T}^n \to \mathcal{T}$ is *congruence preserving* (abbreviated into CP) if for all congruences $\sim$ on $\mathcal{T}$, for all $t_1,\ldots,t_n,\ t'_1,\ldots,t'_n$ in $\mathcal{T}$, $t_i \sim t'_i$ for all $i=1,\ldots,n$, implies $f(t_1,\ldots,t_n) \sim f(t'_1,\ldots,t'_n)$.

Remark 2.11. (1) It follows from Lemma 2.1 that CP functions are characterized by the fact that for all homomorphisms h from $\langle \mathcal{T},\star\rangle$ to any algebra $\langle A,\star_A\rangle$, $h(t_i)=h(t'_i)$ for all $i=1,\ldots,n$, implies $h(f(t_1,\ldots,t_n)) = h(f(t'_1,\ldots,t'_n))$.

(2) If f is CP and endomorphism h is idempotent then $h(f(t_1,\ldots,t_n)) = h(f(h(t_1),\ldots,h(t_n)))$. Indeed, let $\sim_h$ be the congruence associated with h, for $i=1,\ldots,n$, we have $h(t_i)=h(h(t_i))$, hence $t_i \sim_h h(t_i)$, whence the result.

We will show that congruence preserving functions on the algebra $\langle \mathcal{T}(\Sigma),\star\rangle$ are polynomial functions. Let us first formally define polynomials on trees.

Definition 2.12. Let $x_1,\ldots,x_n \notin \Sigma$ be called *variables*. A *polynomial* $P(x_1,\ldots,x_n)$ is a tree on the alphabet $\Sigma \cup \{x_1,\ldots,x_n\}$.

With every polynomial $P(x_1,\ldots,x_n)$ we will associate a *polynomial function* $\tilde{P}\colon \mathcal{T}^n \to \mathcal{T}$ defined by: for any $\vec{u} = \langle t_1,\ldots,t_i,\ldots,t_n\rangle \in \mathcal{T}^n$,

$$\tilde{P}(\vec{u}) = \begin{cases} P & \text{if } P = \mathbf{0} \text{ or } P \in \Sigma \\ t_i & \text{if } P = x_i \\ \widetilde{P_1}(\vec{u}) \star \widetilde{P_2}(\vec{u}) & \text{if } P = P_1 \star P_2 \end{cases}$$

Obviously, every polynomial function is CP. Our goal is to prove the converse, namely

Theorem 2.13. *Let $|\Sigma| \geq 3$. If $g\colon \mathcal{T}^n \to \mathcal{T}$ is CP then there exists a polynomial P_g such that $g = \widetilde{P_g}$.*

3 Equality of CP functions

Notation 3.1. For any $f\colon \mathcal{T}^n \to \mathcal{T}$, we denote by $f|_{\Sigma^n}$ its restriction to Σ^n.

In this section we prove that if f and g are two CP functions, then $f|_{\Sigma^n} = g|_{\Sigma^n}$ implies $f=g$, *provided that Σ contains at least three letters.*

Lemma 3.2. *Suppose Σ has at least three letters. If f and g are unary CP functions on $\mathcal{T}$ such that for all $a\in\Sigma$, $f(a)=g(a)$ then for all $t\in\mathcal{T}$, $f(t)$ and $g(t)$ are similar.*

Proof: We have to show that $\nu_a(f(t)) = \nu_a(g(t))$ for some $a\in\Sigma$ and for all t. As ν_a is idempotent and f is CP, by Remark 2.11 (2), $\nu_a(f(t)) = \nu_a(f(\nu_a(t)))$, and similarly for g. Hence it suffices to prove $f(\nu_a(t)) = g(\nu_a(t))$. Let $b_1,\ b_2 \in \Sigma$ such that $a,\ b_1,\ b_2$ are pairwise distinct. As $\gamma_{b_i\to\nu_a(t)}$ is idempotent, by Remark 2.11 (2), we have $\gamma_{b_i\to\nu_a(t)}(f(b_i)) = \gamma_{b_i\to\nu_a(t)}(f(\nu_a(t)))$. The same holds for g, i.e., $\gamma_{b_i\to\nu_a(t)}(g(b_i)) = \gamma_{b_i\to\nu_a(t)}(g(\nu_a(t)))$. From $f(b_i)=g(b_i)$, we deduce that $\gamma_{b_i\to\nu_a(t)}(f(\nu_a(t))) = \gamma_{b_i\to\nu_a(t)}(g(\nu_a(t)))$. This equality holds for $i=1,2$, thus Proposition 2.8 implies that $f(\nu_a(t)) = g(\nu_a(t))$. □

The following proposition shows that a unary CP function f is completely determined by its values on Σ.

Proposition 3.3. *Suppose Σ has at least three letters. If f and g are unary CP functions on $\mathcal{T}$ such that for all $a\in\Sigma$, $f(a)=g(a)$ then for all $t\in\mathcal{T}$, $f(t)=g(t)$.*

Proof: Let a be a letter that occurs in t. For any other letter b, the endomorphisms $\gamma_{a\to b}$ and $\gamma_{a\to t_b}$ are idempotent, where $t_b = \gamma_{a\to b}(t)$. Thus by Remark 2.11 (2), $\gamma_{a\to t_b}(f(a)) = \gamma_{a\to t_b}(f(t_b))$, and $\gamma_{a\to t_b}(g(a)) = \gamma_{a\to t_b}(g(t_b))$. As $f(a) = g(a)$ we have $\gamma_{a\to t_b}(f(t_b)) = \gamma_{a\to t_b}(g(t_b))$. By Lemma 3.2, $f(t_b)$ and $g(t_b)$ are similar, and by Proposition 2.9 (1) $f(t_b) = g(t_b)$.

On the other hand, as f and g are CP and $t \sim_{\gamma_{a\to b}} t_b$, we get $\gamma_{a\to b}(f(t)) = \gamma_{a\to b}(f(t_b))$ and $\gamma_{a\to b}(g(t)) = \gamma_{a\to b}(g(t_b))$, hence $\gamma_{a\to b}(f(t)) = \gamma_{a\to b}(g(t))$. As this is true for all $b \neq a$, we have by Proposition 2.9 (2), $f(t) = g(t)$. □

Proposition 3.3 now can be generalized.

Notation 3.4. For any function $f\colon \mathcal{T}^{n+1} \to \mathcal{T}$, any $t \in \mathcal{T}$, and $\vec{u} = \langle t_1, \dots, t_n\rangle$, we define
(1) a n-ary function $f_{\cdots,t}$ obtained by "freezing" the (n+1)th argument to the value t, and defined by: for all $\vec{u} \in \mathcal{T}^n$, $f_{\cdots,t}(\vec{u}) = f(\vec{u}, t)$,
(2) a unary function $f_{\vec{u},\cdot}$ obtained by "freezing" the n first arguments to the value $\vec{u} = \langle t_1, \dots, t_n\rangle$, and defined by: for all $t \in \mathcal{T}$, $f_{\vec{u},\cdot}(t) = f(\vec{u}, t)$.

Proposition 3.5. *Let f and g be n-ary CP functions on $\mathcal{T}$ such that for all $a_1, \dots, a_n \in \Sigma$, $f(a_1, \dots, a_n) = g(a_1, \dots, a_n)$ then for all $t_1, \dots, t_n \in \mathcal{T}$, $f(t_1, \dots, t_n) = g(t_1, \dots, t_n)$.*

Proof: By induction on n. For $n = 1$ the result was proved in Proposition 3.3. Assume the result holds for n. By the hypothesis, for all $a_1, \dots, a_n, a \in \Sigma$, we have $f(a_1, \dots, a_n, a) = g(a_1, \dots, a_n, a)$, i.e., $f_{\cdots,a}(a_1, \dots, a_n) = g_{\cdots,a}(a_1, \dots, a_n)$. By the induction applied to $f_{\cdots,a}$, for all $\vec{u} \in \mathcal{T}^n$, $f_{\cdots,a}(\vec{u}) = g_{\cdots,a}(\vec{u})$, or equivalently $f_{\vec{u},\cdot}(a) = g_{\vec{u},\cdot}(a)$. As $f_{\vec{u},\cdot}(a) = g_{\vec{u},\cdot}(a)$, applying now Proposition 3.3 to $f_{\vec{u},\cdot}$ and $g_{\vec{u},\cdot}$ yields $f_{\vec{u},\cdot}(t) = g_{\vec{u},\cdot}(t)$ for all t, hence $f(\vec{u}, t) = g(\vec{u}, t)$. □

4 The algebra of binary trees is affine complete

To prove that any CP function is a polynomial function, as a consequence of Proposition 3.5 and of the fact that a polynomial function is CP, it is enough to show that the restriction $f|_{\Sigma^n}$ of $f\colon \mathcal{T}^n \to \mathcal{T}$ to Σ^n is equal to the restriction $\tilde{P}|_{\Sigma^n}$ of a n-ary polynomial function. For such restricted functions we introduce a weakened version WCP of the CP condition, namely,

Definition 4.1. Function $g\colon \mathcal{T}^n \to \mathcal{T}$ is said to be WCP iff for any idempotent mapping $h\colon \Sigma \to \Sigma$, $\forall \vec{u}, \vec{v} \in \Sigma^n$, $h(\vec{u}) = h(\vec{v}) \implies h(g(\vec{u})) = h(g(\vec{v}))$, where for $\vec{u} = \langle u_1, \dots, u_n\rangle$, $h(\vec{u})$ denotes $\langle h(u_1), \dots, h(u_n)\rangle$.

Every CP function is clearly WCP.

Lemma 4.2. *If g is WCP then for all $\vec{u}, \vec{v} \in \Sigma^n$, $g(\vec{u})$ and $g(\vec{v})$ are similar.*

Proof: As $\nu_a(\vec{u}) = \nu_a(\vec{v}) = \langle a, \dots, a\rangle$ for all $\vec{u}, \vec{v} \in \Sigma^n$ and g is WCP, $\nu_a(g(\vec{u})) = \nu_a(g(\vec{v}))$. □

We often use a different form of the condition WCP, which deals only with alphabetic graftings.

Proposition 4.3. *A function g is WCP if and only if*
(GCP) (G for graftings) for all $a_1, a_2, \dots, a_n \in \Sigma$, $i \in \{1, \dots, n\}$ and $b_i \in \Sigma$, $\gamma_{a_i\to b_i}(g(a_1, \dots, a_n)) = \gamma_{a_i\to b_i}(g(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n))$.

Proof: Since $\gamma_{a_i \to b_i}(a_1, \ldots, a_n) = \gamma_{a_i \to b_i}(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n)$, clearly WCP implies GCP. The proof of the converse is by induction on n. It is obviously true for $n = 0$.

Otherwise, let h be a mapping $h\colon \Sigma \to \Sigma$ and let $\vec{u}, \vec{v} \in \Sigma^n$ such that $h(\vec{u}) = h(\vec{v})$, and let $a, b \in \Sigma$ such that $h(a) = h(b)$. By (GCP), we have $\gamma_{a \to b}(g(\vec{u}, a)) = \gamma_{a \to b}(g(\vec{u}, b))$, hence $h(\gamma_{a \to b}(g(\vec{u}, a))) = h(\gamma_{a \to b}(g(\vec{u}, b)))$.
But $h(\gamma_{a \to b}(c)) = \begin{cases} h(c) & \text{if } c \neq a \\ h(b) = h(a) & \text{if } c = a \end{cases}$ hence $h \circ \gamma_{a \to b} = h$. Therefore $h(g(\vec{u}, a)) = h(g(\vec{u}, b))$, and by the induction applied to $g_{\ldots, b}$, $h(g(\vec{u}, a)) = h(g(\vec{u}, b)) = h(g(\vec{v}, b))$. □

Let us first study unary WCP functions whose restriction to Σ takes its values in Σ.

Proposition 4.4. *Assume $|\Sigma| \geq 3$. Let $f\colon \mathcal{T} \to \mathcal{T}$ be WCP and such that $f(\Sigma) \subseteq \Sigma$. Then $f|_{\Sigma}$ is (1) either a constant function (2) or the identity.*

Proof: If f is not the identity there exist $a,\ b$, with $a \neq b$ and $f(a) = b$. As $\gamma_{a \to b}(f(b)) = \gamma_{a \to b}(f(a)) = \gamma_{a \to b}(b) = b$, we get $f(b) \in \{a, b\}$.

For $c \notin \{a, b\}$, $\gamma_{a \to c}(f(c)) = \gamma_{a \to c}(f(a)) = b$ implies $f(c) = b$. It remains to prove that $f(b) = b$. From $\gamma_{b \to c}(f(b)) = \gamma_{b \to c}(f(c)) = c$, we deduce that $f(b) \in \{c, b\}$, hence $f(b) \in \{a, b\} \cap \{c, b\} = \{b\}$, which concludes the proof. □

We now will generalize Proposition 4.4 by Proposition 4.5 (replacing a unary f by a n-ary g).

Proposition 4.5. *Assume $|\Sigma| \geq 3$. If $g\colon \mathcal{T}^n \to \mathcal{T}$ is WCP and such that $g(\Sigma^n) \subseteq \Sigma$, then $g|_{\Sigma^n}$ is (1) either a constant function (2) or a projection π_i^n.*

Proof: The proof is by induction on n. By Proposition 4.4 it is true for $n = 1$. If g is of arity $n + 1$ then, by induction hypothesis, for any $a \in \Sigma$, the function $g_{\cdots, a}$ of arity n is either a constant or a projection π_i^n. We first show that these functions are all equal to a given π_i^n, or all equal to a same constant, or every $g_{\cdots, a}$ is the constant function a.

Let us assume that $g_{\cdots, a} = \pi_i^n$. Let $\vec{u} = \langle a, \ldots, a, c, a, \ldots, a\rangle$ and $\vec{v} = \langle a, \ldots, a, d, a, \ldots, a\rangle$ with a, c, d pairwise disjoint, so that for any b, $\gamma_{a \to b}(g(\vec{u}, a)) = c$ and $\gamma_{a \to b}(g(\vec{v}, a)) = d$. It follows from the GCP condition that $\gamma_{a \to b}(g(\vec{u}, a)) = \gamma_{a \to b}(g(\vec{u}, b)) = c$ and $\gamma_{a \to b}(g(\vec{v}, a)) = \gamma_{a \to b}(g(\vec{v}, b)) = d$, which is impossible if $g_{\cdots, b}$ is either a constant or a projection π_j^n with $j \neq i$. Hence all $g_{\cdots, a}$ are equal to π_i^n, implying $g = \pi_i^{n+1}$.

Assume now all the $g_{\cdots, a}$ are constant. For every $\vec{u}, \vec{v}, a$, we have $g(\vec{u}, a) = g(\vec{v}, a)$. We choose an arbitrary $\vec{u} \in \Sigma^n$ which will be fixed. By the induction hypothesis $g_{\vec{u}, \cdot}$ is either (1) the identity, or (2) a constant c. In case (1), for all $\vec{v}, a$, $g(\vec{u}, a) = g(\vec{v}, a) = a$ and $g = \pi_{n+1}^{n+1}$. In case (2), for all $\vec{v}, a, b$, $g(\vec{u}, a) = g(\vec{v}, b) = c$ and g is a constant. □

As CP functions are WCP, for g a CP function such that for some $a_1, \ldots, a_n \in \Sigma$, $g(a_1, \ldots, a_n) \in \Sigma$, we have shown that there exists a polynomial P_g, which is either a constant $a \in \Sigma$ or an x_i, such that $g = \widetilde{P_g}$. We will now extend to the case when $g(a_1, \ldots, a_n) \notin \Sigma$.

Proposition 4.6. *Assume that $|\Sigma| \geq 3$. Let $g\colon \mathcal{T}^n \to \mathcal{T}$ be WCP. Then there exists a polynomial P_g such that $g|_{\Sigma^n} = \widetilde{P_g}|_{\Sigma^n}$.*

Proof: Let $\sigma(g)$ be the common size of all the $g(\vec{u}),\ \vec{u} \in \Sigma^n$. The proof is by induction on $\sigma(g)$.

Basis: If $\sigma(g) = 0$ then $g|_{\Sigma^n} = \tilde{P}|_{\Sigma^n} = \mathbf{0}$. If $\sigma(g) = 1$ then $g(a_1, \ldots, a_n) \in \Sigma$ and the result is proved in Proposition 4.5.

Induction: If $\sigma(g) > 1$ there exists two functions $g_i \colon \mathcal{T}^n \to \mathcal{T}$ for $i = 1, 2$ such that for all $\vec{u} \in \Sigma^n$, $g(\vec{u}) = g_1(\vec{u}) \star g_2(\vec{u})$, with $|\sigma(g_i)| < |\sigma(g)|$. It remains to show that both g_1 and g_2 are WCP. Let $\vec{u}, \vec{v} \in \Sigma^n$ be such that $h(\vec{u}) = h(\vec{v})$ for some mapping $h \colon \Sigma \to \Sigma$. Extend h as an endomorphism $\mathcal{T} \to \mathcal{T}$ by Lemma 2.4, then $h(g(\vec{u})) = h(g_1(\vec{u}) \star g_2(\vec{u})) = h(g_1(\vec{u})) \star h(g_2(\vec{u}))$. Similarly, $h(g(\vec{v})) = h(g_1(\vec{v})) \star h(g_2(\vec{v}))$. As g is WCP and $h(\vec{u}) = h(\vec{v})$, we have $h(g(\vec{u})) = h(g(\vec{v}))$. Then by Lemma 2.3 (unique decomposition) we get $h(g_i(\vec{u})) = h(g_i(\vec{v}))$ for $i = 1, 2$. This is true for any h, thus g_1 and g_2 are WCP. By the induction hypothesis there exists P_i such $\widetilde{P_i}|_{\Sigma^n} = g_i|_{\Sigma^n}$, hence $g|_{\Sigma^n} = \widetilde{P_1}|_{\Sigma^n} \star \widetilde{P_2}|_{\Sigma^n} = \widetilde{P_1 \star P_2}|_{\Sigma^n}$. □

Theorem 4.7. *If $f \colon \mathcal{T}^n \to \mathcal{T}$ is CP then there exists a polynomial P such that $f = \tilde{P}$.*

Proof: Since f is CP, f also is WCP. By the previous proposition, there exists P such that $f|_{\Sigma^n} = \tilde{P}|_{\Sigma^n}$, and by Proposition 3.5, $f = \tilde{P}$. □

5 Conclusion

We proved that, when Σ has at least three letters, the algebra of arbitrary binary trees with leaves labeled by letters of Σ is an affine complete algebra (non commutative and non associative).

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