# Mathematical Justification of a Compressible Bi-Fluid System with Different Pressure Laws: A Semi-Discrete approach and Numerical illustrations

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# Mathematical Justification of a Compressible Bi-Fluid System with Different Pressure Laws: A Semi-Discrete approach and Numerical illustrations

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**Abstract**

In this paper, we justify mathematically a compressible bifluid system with two different pressure state laws starting from an ODE system which somehow mimics the physical situation at the mesoscopic scale using the Lagrangian coordinates to fix the interfaces. In the last part, we show how the derivation of the macroscopic system may be helpful from a numerical point of view to simulate a mixture at the mesoscopic scale. This paper is a semi-discrete version of [5] where the first justification of such a mixture model is proposed. In the present justification, based on a semi-discrete scheme, the purpose and the frame might be more clear to the reader, and the numerical illustrations given in the last part (with a totally discrete scheme that is asymptotic preserving) give some strong hints on the phenomena that are involved.

**Résumé**

Dans ce papier, nous justifions mathématiquement un système bifluide compressible avec deux lois de pression pouvant être différentes en partant d'un système d'équations différentielles ordinaires qui représentent la situation d'un mélange à l'échelle mésoscopique quand on utilise les coordonnées lagrangiennes pour fixer les interfaces. Cet article est une version semi-discrète de [5] où la première justification rigoureuse de tels modèles multi-fluides est proposée. La présente démarche, complémentaire de la précédente et basée sur un schéma semi-discret, les buts et le cadre de travail sont sans doute plus aisés à comprendre, et les illustrations numériques de la dernière partie (obtenue avec un schéma totallement discret préservant l'asymptotique) permettent de saisir mieux les phénomènes en jeu dans cette modélisation.

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**Keywords.** Compressible Flows, Bi-Fluid System, semi-discrete approach, Hoff Solution, Homogenization, Numerical Schemes.

## 1 Introduction

The mathematical derivation of bifluid systems with the same pressure law for the two components starting from a continuous isentropic compressible Navier-Stokes system with highly oscillating–concentrated initial density has been firstly studied in the one-dimension in space case by W.E. [16], D. Serre [34] in parallel with A.A. Amosov and A.A. Zlotinkov [3] for instance. Recently, P. Plotnikov

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and I. Sokolowski [32] on the one hand and D. Bresch and M. Hillairet [6]- [7] on the other hand have investigated the multi-dimension in space case. More precisely, the first authors consider compressible Navier-Stokes equations with constant viscosities with rapidly oscillating initial data. Working on global weak-solution in the spirit of Leray, using Young measures theory, it is possible to derive kinetic equations (in the spirit of Lions, Perthame, Tadmor) which encode the mixing dynamic. However, as explained by D. Bresch, M. Hillairet and X. Huang [19], [6], [8], [7] and [20] multifluid systems are interpreted as reduced systems satisfied by particular Young measure (namely convex combinations of a finite number of Dirac masses) solutions of the homogenized compressible Navier-Stokes equation. Proving propagation of the number of Dirac masses in Young measure solutions to this homogenized equation is then the key point to derive the multifluid system with new relaxation terms. This requires to works on solutions with intermediate regularity in the spirit of D. Hoff [23] and B. Desjardins [12] namely with initial density in $L^\infty(\Omega)$ and initial velocity in $H^1(\Omega)$. However starting with the compressible Navier-Stokes equations with the monotone law $p(\rho) = a\rho^\gamma$ (with $a > 0$ and $\gamma > 1$) and choosing appropriate oscillating initial density provides bi-fluid system at the macroscopic scale with the same pressure law for the two components constituting the mixture. In order to obtain physical interesting systems from an application view-point, it is important to be able to consider different pressure laws depending on the components. More precisely it is this open problem, that we want to address in this paper, to mathematically justify the following system governing $(\alpha_\pm, \rho_\pm, u)$ (an equation on $g_\pm$ means two equations: one on $g_+$ and the other on $g_-$) with periodic boundary conditions on $(0,1)$ and corresponding initial data:

$$\begin{cases} \partial_t \alpha_\pm + u\,\partial_x \alpha_\pm = \dfrac{\alpha_+\alpha_-}{\alpha_+\mu_- + \alpha_-\mu_+}(\sigma_\pm - \sigma_\mp), \\ \partial_t(\alpha_\pm\rho_\pm) + \partial_x(\alpha_\pm\rho_\pm u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x(\mu_{\text{eff}}\partial_x u) + \partial_x p_{\text{eff}} = 0, \\ \alpha_+ + \alpha_- = 1 \text{ with } 0 \le \alpha_\pm \le 1, \\ \rho = \alpha_+\rho_+ + \alpha_-\rho_- \text{ with } 0 < \rho_\pm < \infty \end{cases} \tag{1.1}$$

where

$$\mu_{\text{eff}} = \frac{\mu_+\mu_-}{\alpha_+\mu_- + \alpha_-\mu_+}, \qquad p_{\text{eff}} = \frac{\alpha_+ p_+(\rho_+)\mu_- + \alpha_- p_-(\rho_-)\mu_+}{\alpha_+\mu_- + \alpha_-\mu_+} \tag{1.2}$$

with $s \mapsto p_+(s)$ and $s \mapsto p_-(s)$ two given monotone pressure laws satisfying

$$p_\pm \in \mathcal{C}^1([0,+\infty)) \quad \text{such that} \quad p_\pm(0) = 0 \quad \text{and} \quad a_\pm s^{\gamma_\pm - 1} - b_\pm \le p_\pm'(s) \le \frac{1}{a_\pm} s^{\gamma_\pm - 1} + b_\pm \tag{1.3}$$

for some constants $\gamma_\pm > 1$ and $a_\pm > 0$, $b_\pm \ge 0$ and $\mu_\pm$ two positive given constant viscosities that may be different for each component and where $\sigma_+$ and $\sigma_-$ are given through the formula

$$\sigma_\pm = -\mu_\pm \partial_x u + p_\pm(\rho_\pm). \tag{1.4}$$

Remark that the form of the expressions $\mu_{\text{eff}}$ and $p_{\text{eff}}$ is similar from what we could obtain in homogenization for elliptic equations in one dimension. We explain formally in the appendix how equation $(1.1)_1$ may be derived with a discrete approach of the mixture. This provides an equation on the volumic fractions $\alpha_\pm$ for each component with a relaxation term depending on the two viscosities $\mu_+$ and $\mu_-$ and two pressure state laws $p_+$ and $p_-$ that may be different.

In all the following, the term *meso* will denote what concerns the scale at which the two fluids are separated, while the term *macro* concerns the macroscopic mixture model in which the fluid are not separated.

In a first part, we mathematically justify that the system (1.1)–(1.4) can be obtained by homogenization of a system of ODEs which contains an order parameter $c$ and which describes the physics of the mixture at the mesoscale : this is the main result of the paper (see Theorem 1). The originality of the present paper is that the the mesocopic model is a system of ODEs rather than a PDE model as it is ussualy assumed see [19], [6], [8], [7], [20], [5]. For more recent applications of this method see M. Hillairet, H. Mathis and N. Seguin [21, 22]. We refer the reader to the recent work ok V. Perrier and E. Gutiérrez [31] where the mesoscopic model is of Euler type and stochastic homogenisation is performed in order to obtain an averaged model. See also the discrete equation method of R. Abgrall and R. Saurel in [1].

Then in a second part, we show how the derivation of (1.1)–(1.4) may be helpful from a numerical point of view to simulate mixture at the meso-scale. In some sense we revisit the seminal works by [13], [10], [11] and [2]. We present asymptotic preserving scheme using the macroscopic model to choose an appropriate flux quantity. In the appendix, for reader's convenience, we present a formal derivation of $(1.1)_1$ starting from the description of the physical situation at the mesoscale for readers who are not familiar with mathematical justifications.

*Important notation.* In all the paper long, we denote by $D_t$ the Lagrangian time derivative defined as follows when applied on a quantity $g$: $D_t g = \partial_t g + u\partial_x g$. This derivative will also be denoted $\dot{g}$ in the rest of the paper.

## 2 Statement of the main result

Let us first describe the mesoscopic system under consideration on $[0,1]$. It corresponds to a physical description of a two-components system governed by ODEs on each cell (number of cells $J \in \mathbb{N}\backslash\{0\}$ in $[0,1)$ and location of cell interfaces position $x_j$). Letting the number of cells go to infinity with appropriate assumptions on the data, we are able to mathematically justify the derivation of the bifluid system (1.1)–(1.4). More precisely, let $J \in \mathbb{N}\setminus\{0\}$ be the number of cells in $[0,1)$. Let $(x_{j-1/2}(t))_{j=1}^J$ be the collection of cell interface positions at time $t$. One assumes $0 \le x_{j-1/2} < x_{j+1/2} < 1$ for any $j \in \{1,\dots,J-1\}$ (this set will also be denoted $\overline{1,J-1}$. In order to take into account the fact that the problem under consideration is posed on $\mathbb{T}$ in a simple manner, i.e. without taking care of the cells and quantities on the boundary, we extend all the data over $\mathbb{R}$ and $\mathbb{Z}$ by periodicity. The cells themselves are denoted by $[x_{j-1/2}, x_{j+1/2})$ for $j \in \mathbb{Z}$. The maximum length of these cells is intended to be small (and to tend to 0 as $J$ tends to $\infty$ to reach convergence).

We first consider the following mesoscopic system of ODEs, which is inspired by the structure of the system (1.1) in Lagrangian coordinates. As the fluids have to remain pure (not mixed) in every cell (this is encoded by the constraint $c_j(1-c_j) = 0$ in the following), we consider a *Lagrangian*, or *pseudo-Lagrangian*[1] mesoscale approach in which the cells follow the fluid in its transport, namely in which the edges of every cell moves at the fluid velocity namely

$$\dot{x}_{j+\frac{1}{2}} = u_{j+\frac{1}{2}}, \tag{2.1}$$

and the physical model in each cell

$$\left\{\begin{aligned} &\dot{c}_j = 0,\\ &\tfrac{d}{dt}\left(\rho_j \Delta x_j\right) = 0,\\ &\rho_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}\dot{u}_{j+\frac{1}{2}} + p_{j+1} - p_j\\ &\qquad = \left\{\mu\left(c_{j+1}\right)\frac{u_{j+\frac{3}{2}} - u_{j+\frac{1}{2}}}{\Delta x_{j+1}} - \mu\left(c_j\right)\frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x_j}\right\}, \end{aligned}\right. \tag{2.2}$$

[1] It can be called pseudo-Lagrangian because, although the solution is actually expressed in the classical Euler variable, the scheme strongly uses the Langrange formulation of the system.

where

$$p_j = p(\rho_j, c_j) \text{ with } p(\rho, c) = cp_+(\rho) + (1-c)p_-(\rho), \qquad \mu(c_j) = c_j\mu_+ + (1-c_j)\mu_-$$

with $s \mapsto p_-(s)$ and $s \mapsto p_+(s)$ are two increasing functions satisfying (1.3), and where

$$\begin{cases} \Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \\ \Delta x_{j+\frac{1}{2}} = \dfrac{\Delta x_j + \Delta x_{j+1}}{2}, \\ \rho_{j+\frac{1}{2}} = \dfrac{\rho_j \Delta x_j + \rho_{j+1}\Delta x_{j+1}}{\Delta x_j + \Delta x_{j+1}}. \end{cases} \tag{2.3}$$

for all $j \in \overline{0, J-1}$ with the periodic condition

$$\begin{cases} c_0 = c_J, \\ \rho_0 = \rho_J, \\ u_{\frac{1}{2}} = u_{J+\frac{1}{2}}, \qquad u_{-\frac{1}{2}} = u_{J-\frac{1}{2}}. \end{cases} \tag{2.4}$$

System (2.1) has to be completed with initial condition

$$c_j|_{t=0} = c_j^0, \quad \rho_j|_{t=0} = \rho_j^0, \quad x_{j+1/2}|_{t=0} = x_{j+1/2}^0, \quad u_{j+1/2}|_{t=0} = u_{j+1/2}^0, \tag{2.5}$$

satisfying the following constraints:

$$\begin{cases} x^0_{-\frac{1}{2}} < x^0_{\frac{1}{2}} < x^0_{\frac{3}{2}} < \cdots < x^0_{J-\frac{1}{2}}, \\ c_j^0 \in \{0,1\}, \\ \text{there exist } \underline{\rho^0} \text{ and } \underline{\rho^0} \text{ such that } 0 < \underline{\rho^0} \le \rho_j^0 \le \overline{\rho^0} < \infty, \\ \text{there exists } A \text{ such that } \left\| \left( u^0_{j+\frac{1}{2}} \right)_{j \in \overline{0,J-1}} \right\|^2_{\hat{H}^1_J} = \displaystyle\sum_{j=0}^{J-1} \left| u^0_{j+\frac{1}{2}} \right|^2 \Delta x_j^0 + \sum_{j=0}^{J-1} \left| \frac{u^0_{j+\frac{1}{2}} - u^0_{j-\frac{1}{2}}}{\Delta x_j^0} \right|^2 \Delta x_j^0 \le A < \infty. \end{cases} \tag{2.6}$$

**Remark:** From the previous system of equations we also deduce that

$$\begin{cases} (\dot{\Delta x_j}) = u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}, \\ \frac{d}{dt}\left( \rho_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}} \right) = 0. \end{cases} \tag{2.7}$$

For any $J \in \mathbb{N}^*$, having constructed the functions $\left( c_j, \rho_j, u_{j+\frac{1}{2}} \right)_{j \in \overline{0,J-1}}$ as above we construct

$$(\hat{c}_J, \hat{\rho}_J, \hat{u}_J, \hat{\sigma}_J) : \mathbb{R} \times \mathbb{T}^1 \to \{0,1\} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$$

defined by

$$\hat{c}_J(t,x) = c_j(t) \text{ if } x \in [x_{j-\frac{1}{2}}(t), x_{j+\frac{1}{2}}(t)), \tag{2.8}$$

$$\hat{\rho}_J(t,x) = \rho_j(t) \text{ if } x \in [x_{j-\frac{1}{2}}(t), x_{j+\frac{1}{2}}(t)), \tag{2.9}$$

$$\hat{u}_J(t,x) = \frac{x - x_{j-\frac{1}{2}}}{\Delta x_j} u_{j+\frac{1}{2}}(t) + \frac{x_{j+\frac{1}{2}} - x}{\Delta x_j} u_{j-\frac{1}{2}}(t) \text{ if } x \in [x_{j-\frac{1}{2}}(t), x_{j+\frac{1}{2}}(t)), \tag{2.10}$$

$$\hat{\sigma}_J(t,x) = \frac{x - x_j}{\Delta x_{j+\frac{1}{2}}} \sigma_{j+1} + \frac{x_{j+1} - x}{\Delta x_{j+\frac{1}{2}}} \sigma_j \text{ for } x \in [x_j(t), x_{j+1}(t)), \tag{2.11}$$

denoting

$$x_j = (x_{j-1/2} + x_{j+1/2})/2 \tag{2.12}$$

and

$$\sigma_j = \mu(c_j)\frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x_j} - p_j \tag{2.13}$$

with

$$\mu(c_j) = c_j\mu_+ + (1-c_j)\mu_-, \qquad p_j = c_j p_+(\rho_j) + (1-c_j)p_-(\rho_j).$$

In Proposition 3.5, we will show the following bounds on $(\hat{c}_J, \hat{\rho}_J, \hat{u}_J, \hat{\sigma}_J)$:

$$\begin{cases} \hat{c}_J(t,x) \in \{0,1\}, \\ 0 < \dfrac{1}{C} \le \hat{\rho}_J(t,x) \le C, \\ \displaystyle\int_0^1 \hat{\rho}_J |\hat{u}_J|^2(t,x)\,dx + \int_0^1 H(\hat{\rho}_J(t,x), \hat{c}_J(t,x))\,dx \\ \qquad\qquad\qquad\qquad \displaystyle + \int_0^t\int_0^1 \mu(\hat{c}_J(\tau,x))|\partial_x \hat{u}_J(\tau,x)|^2\,dx d\tau \le C, \\ \|\partial_x \hat{u}_J\|_{L^2_{t,x}} + \|\partial_t \hat{u}_J\|_{L^2_{t,x}} + \min\{1,t\}\|\partial_t \hat{u}_J(t)\|_{L^2_x} \le C, \\ \displaystyle\int_0^t\int_0^1 |\partial_x \hat{\sigma}_J(\tau,x)|^2\,dx d\tau + \min\{1,t\}\int_0^1 |\partial_x \hat{\sigma}_J(t,x)|^2\,dx \\ \qquad\qquad\qquad\qquad \displaystyle + \int_0^t (\sup_{x\in[0,1]}|\partial_x \hat{\sigma}_J(\tau,x)|)^{\frac{4}{3}-}\,d\tau \le C \\ \displaystyle\int_0^t\int_0^1 \min\{1,\tau\}|\partial_t \hat{\sigma}_J(\tau,x)|^2\,dx d\tau \le C \end{cases} \tag{2.14}$$

where $C$ a positive constant depending only on the initial data and the time variable $t$, and where

$$H(\rho_j, c_j) = c_j H_+(\rho_j) + (1-c_j)H_-(\rho_j) \text{ with } H_\pm(s) = \rho\int_0^\rho p_\pm(\tau)/\tau^2\,d\tau. \tag{2.15}$$

With such uniform estimates, we will be able to formulate the main theorem, namely the convergence of the mesoscopic system (2.1)– (2.6) through definitions (2.8)–(2.13) to the macroscopic system (1.1)–(1.4): this result will be obtained using Propositions 3.2–3.5 and the classical uniqueness results for transport equations with measure initial data.

**Theorem 1.** *Consider $p_+, p_-$ two given monotone pressure laws satisfying (1.3) and assume the initial sequence of data satisfies (2.6). Then, there exists a unique global solution $\{(c_j, \rho_j, x_{j+\frac{1}{2}}, u_{j+\frac{1}{2}})\}_{j=0,\ldots,J}$ of the mesoscopic system of odes (2.1)–(2.5). Then $(\hat{c}_J, \hat{\rho}_J, \hat{u}_J, \hat{\sigma}_J)$ defined by (2.8)–(2.13), satisfy the uniform estimates (2.14) on $[0,T]$ for any $T \in (0,\infty]$. Let $\Theta_J^0$ be defined by*

$$\left\langle \Theta_J^0, b\right\rangle \stackrel{def.}{:=} \int_{\mathbb{T}^1} b\left(x, \hat{\rho}_J^0(x), \hat{c}_J^0(x)\right)dx, \qquad \forall\, b \in C_c\left(\mathbb{T}_x^1 \times \mathbb{R}_\xi \times [0,1]\right). \tag{2.16}$$

*Assume there exists $\alpha_+^0, \alpha_-^0 \in L^1(\mathbb{T}^1, [0,1])$ and $\rho_+^0, \rho_-^0 \in L^\infty(\mathbb{T}^1, \mathbb{R}_+)$ such that*

$$\left\langle \Theta_J^0, b\right\rangle \longrightarrow_{J\to+\infty} \langle \Theta^0, b\rangle = \int_{\mathbb{T}^1} (\alpha_+^0(x) b\left(x, \rho_+^0(x), 1\right) + \alpha_-^0(x) b\left(x, \rho_-^0(x), 0\right))dx \tag{2.17}$$

*for all $b \in \mathcal{C}_c(\mathbb{T}_x^1 \times \mathbb{R}_\xi \times [0,1])$, and $u^0 \in H^1(\mathbb{T}^1)$ such that $\hat{u}_J^0 \rightharpoonup u^0$ in $H^1(\mathbb{T}^1)$. Then there exists $\alpha_+, \alpha_- \in L^1((0,T)\times\mathbb{T}^1, [0,1])$, $\rho_+, \rho_- \in L^\infty((0,T)\times\mathbb{T}^1)$ and $u \in H^1((0,T)\times\mathbb{T}^1)$ such that, for all $b \in \mathcal{C}_c(\mathbb{T}_x^1 \times \mathbb{R}_\xi \times \{0,1\})$:*

- *$\hat{u}_J \rightharpoonup u$ in $H^1((0,T)\times\mathbb{T}^1)$,*

- $$\begin{aligned}\langle \Theta_J, b\rangle &\stackrel{def.}{:=} \int_{\mathbb{T}^1} b\left(x, \hat{\rho}_J(t,x), \hat{c}_J(t,x)\right) dx \\ \longrightarrow_{J\to+\infty} \langle \Theta, b\rangle &= \int_{\mathbb{T}^1} (\alpha_+(t,x) b\left(x, \rho_+(t,x), 1\right) + \alpha_-(t,x) b\left(x, \rho_-(t,x), 0\right)) dx, \end{aligned} \tag{2.18}$$

- *$(\alpha_+, \alpha_-, \rho_+, \rho_-, u)$ satisfy (1.1)–(1.4) with the initial conditions*

$$\alpha_\pm|_{t=0} = \alpha^0_\pm, \qquad \rho_\pm|_{t=0} = \rho^0_\pm, \qquad u|_{t=0} = u^0.$$

*In particular, one has that*

$$\hat{\rho}_J \rightharpoonup \alpha_+\rho_+ + \alpha_-\rho_- \ \textit{weakly} - \star \ \textit{in} \ L^\infty((0,T)\times\mathbb{T}^1),$$

$$p(\hat{\rho}_J, \hat{c}_J) \rightharpoonup \alpha_+ p_+(\rho_+) + \alpha_- p_-(\rho_-) \ \textit{weakly} - \star \ \textit{in} \ L^\infty((0,T)\times\mathbb{T}^1).$$

In the numerical part, we will consider a time discretization of the semi-discrete physical description that has been proposed to determine the limit macroscopic system. Note that knowing theoretical properties will help to define appropriate quantities at the numerical level. We will present some illustrations, both with equal viscosities and with different viscosities.

## 3 Main steps

The proof of Theorem 1 will be divided in several steps:

- In a first stage, we prove global existence and uniqueness of solutions for the system of ordinary differential equations (2.1)–(2.6), see Proposition 3.1.
- Next, we study the functions introduced in (2.8)-(2.11). An important feature in our development is that $\hat{c}_J$ and $\hat{\rho}_J$ verify transport equations with velocity $\hat{u}_J$ (see Proposition 3.2). Morever we establish the uniform estimates (2.14) for $(\hat{c}_J, \hat{\rho}_J, \hat{u}_J, \hat{\sigma}_J)$ , see Proposition 3.3.
- Finally, introducing the measure $\Theta_J$ defined by 3.10 and passing to the limit $J \to +\infty$ we obtain a kinetic equation for $\Theta = \lim_{J\to+\infty} \Theta_J$, (see Proposition 3.5). From this kinetic equation and with the appropriate assumptions (2.17) for the initial data, we are able to characterize the measure $\Theta$ and thus to prove the main Theorem 1 obtained finally from Proposition 3.6. More precisely, we show that if $\Theta$ is initially a convex combination of two Dirac masses then, this structure is preserved for all later times.

The following quantities (mass, energy, Hoff energy functionals) will play a crucial role to show uniform estimates which will help us to pass to the limit as $J \to +\infty$ and get the macroscopic model:

– The total mass:

$$M(t) = \sum_{j=0}^{J-1} \rho_j(t)\, \Delta x_j(t) \tag{3.1}$$

– The basic energy functional:

$$E(t) = \frac{1}{2}\sum_{j=0}^{J-1} \rho_{j+\frac{1}{2}}(t) \left|u_{j+\frac{1}{2}}(t)\right|^2 \Delta x_{j+\frac{1}{2}}(t) + \sum_{j=0}^{J-1} H(\rho_j(t), c_j(t)) \Delta x_j(t)$$

$$+\int_0^t \sum_{j=0}^{J-1} \mu\left(c_j(\tau)\right)\left|\frac{u_{j+\frac{1}{2}}(\tau)-u_{j-\frac{1}{2}}(\tau)}{\Delta x_j(\tau)}\right|^2 \Delta x_j(\tau)\, d\tau \tag{3.2}$$

where $H$ is defined in (2.15).

– The first Hoff energy functional:

$$E_{H_1}(t)=\frac{1}{2}\sum_{j=0}^{J-1} \mu\left(c_j(t)\right)\left|\frac{u_{j+\frac{1}{2}}(t)-u_{j-\frac{1}{2}}(t)}{\Delta x_j(t)}\right|^2 \Delta x_j(t) + \int_0^t \sum_{j=0}^{J-1} \rho_{j+\frac{1}{2}}(\tau)\left|\dot{u}_{j+\frac{1}{2}}(\tau)\right|^2 \Delta x_{j+\frac{1}{2}}(\tau)\, d\tau, \tag{3.3}$$

– The second Hoff energy functional:

$$E_{H_2}(t)=\frac{1}{2}\min\{1,t\}\sum_{j=0}^{J-1} \rho_{j+\frac{1}{2}}(t)\left|\dot{u}_{j+\frac{1}{2}}(t)\right|^2 \Delta x_{j+\frac{1}{2}}(t) + \int_0^t \min\{1,\tau\}\sum_{j=0}^{J-1} \mu\left(c_j(\tau)\right)\left|\frac{\dot{u}_{j+\frac{1}{2}}(\tau)-\dot{u}_{j-\frac{1}{2}}(\tau)}{\Delta x_j(t)}\right|^2 \Delta x_j(\tau)\, d\tau. \tag{3.4}$$

We formalize our first result in the following proposition.

**Proposition 3.1.** *System (2.1) with initial data satisfying (2.6) admits a unique global solution. Moreover, it satisfies the following uniform estimates with respect to $J$:*

$$\begin{cases} \dfrac{1}{C^1_{ini}(t)} \leq \rho_j(t) \leq C^1_{ini}(t), \text{ for all } j \in \overline{0, J-1}, \\ \dfrac{1}{C^2_{ini}(t)}\Delta x^0_j \leq \Delta x_j(t) \leq \Delta x^0_j C^2_{ini}(t), \text{ for all } j \in \overline{0, J-1}, \\ c_j \in \{0,1\} \end{cases}$$

*and the bounds*

$$\begin{cases} M(t)=M_0, \qquad E(t)=E_0 \\ E_{H_1}(t)+E_{H_2}(t) \leq C^3_{ini}(t). \end{cases}$$

*where $C^1_{ini}(\cdot)$, $C^2_{ini}(\cdot)$ and $C^3_{ini}(\cdot)$ are strictly positive increasing continuous functions that depend only on $M_0=M(0)$, $E_0=E(0)$, $\overline{\rho^0}$, $\underline{\rho^0}$, and $\left\|\left(u^0_{j+\frac{1}{2}}\right)_{j\in\overline{0,J-1}}\right\|_{\dot{H}^1_J}$.*

The proof of this proposition is the purpose of Section (3.1) hereafter.

Once we will have obtained the qualitative information stated in Proposition 3.1 for the system of ordinary differential equations, we will translate this into information for $(\hat{c}_J, \hat{\rho}_J, \hat{u}_J, \hat{\sigma}_J)$ defined by (2.8)-(2.11). First, we observe the following remarkable equations which will be crucial to derive System (1.1)–(1.4).

**Proposition 3.2.** *The functions $(\hat{c}_J, \hat{\rho}_J, \hat{u}_J)$ verify the following transport equations*

$$\begin{cases} \partial_t \hat{c}_J + \hat{u}_J \partial_x \hat{c}_J = 0, \\ \partial_t \hat{\rho}_J + \partial_x\left(\hat{\rho}_J \hat{u}_J\right) = 0, \end{cases} \tag{3.5}$$

*with initial data*

$$\begin{cases} \hat{\rho}_{J|t=0} = \hat{\rho}_J^0, \\ \hat{c}_{J|t=0} = \hat{c}_J^0, \end{cases}$$

*in the sense of distributions.*

Of course, the estimates announced in Proposition 3.1 can be used in order to estimate various norms of the functions $(\hat{c}_J, \hat{\rho}_J, \hat{u}_J, \hat{\sigma}_J)$. More precisely, we have the following result.

**Proposition 3.3.** *Consider discrete initial data verifying the hypothesis (2.6), and the corresponding globally defined solution of the system of ODEs (2.1)–(2.4). Consider the functions $(\hat{c}_J, \hat{\rho}_J, \hat{u}_J, \hat{\sigma}_J)$ given by (2.8)– (2.11). Then $(\hat{c}_J, \hat{\rho}_J, \hat{u}_J, \hat{\sigma}_J)$ satisfies (2.14). Up to a subsequence, we have*

$$\begin{cases} \hat{\rho}_J \rightharpoonup^\star \rho, \qquad p(\rho_J, c_J) \rightharpoonup^\star p_{eff} \text{ in } L^\infty((0,T)\times\mathbb{T}^1), \\ \hat{u}_J \rightharpoonup u \text{ in } H^1((0,T)\times\mathbb{T}^1) \text{ and strongly in } \mathcal{C}([0,T]; L^2(\mathbb{T}^1)), \\ \hat{\sigma}_J \rightharpoonup \sigma = \mu_{eff}\partial_x u - p_{eff} \text{ in } L^2(0,T; H^1(\mathbb{T}^1)) \end{cases} \tag{3.6}$$

*with*

$$\mu_{eff} = \frac{1}{\left\langle \dfrac{1}{\mu(\hat{c}_J)} \right\rangle}, \qquad p_{eff} = \frac{1}{\left\langle \dfrac{1}{\mu(\hat{c}_J)} \right\rangle} \left\langle \hat{c}_J \frac{p_+(\hat{\rho}_J)}{\mu_+} + (1-\hat{c}_J)\frac{p_-(\hat{\rho}_J)}{\mu_-} \right\rangle, \tag{3.7}$$

*where $\langle\cdot\rangle$ denotes the weak limit. Moreover, we have that*

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x(\mu_{eff}\partial_x u) + \partial_x p_{eff} = 0. \end{cases} \tag{3.8}$$

Since nonlinear functions are involved, we cannot link *a priori* $\mu_{eff}, p_{eff}$ with the weak limits of the sequences $\hat{\rho}_J, \hat{c}_J$. Moreover, as explained previously, the density $\hat{\rho}_J$ and the parameter $\hat{c}_J$ are expected to widely oscillated w.r.t. the space variable. For this reason, one cannot hope to obtain strong convergence in Lebegue spaces for these sequences.

In order to identify $\hat{\sigma}_J$ with the weak limits of the other unknowns, we follow the approach from [6] where the authors noticed that this identification is similar to the classical homogenisation problem for the elliptic equation

$$-\partial_x(a_\varepsilon \partial_x u_\varepsilon) = F$$

where $a_\varepsilon(x) = a(x/\varepsilon)$ where $a$ is 1-periodic. It is well know from works by F. Murat and L. Tartar that the system verified by the weak limit of $\bar{u}$ is

$$-\partial_x(\bar{a}\partial_x\bar{u}) = F$$

where

$$\bar{a} = \frac{1}{\left\langle \dfrac{1}{a} \right\rangle}$$

(this can be proved using the uniform bounds of $a_\varepsilon \partial_x u_\varepsilon$ in $H^1$ under the assumption $a_\varepsilon \geq c > 0$). In our setting, some form of compactness is known to hold for $\hat{\sigma}_J$ although we have to take care of the fact that $\hat{c}_J p_+(\hat{\rho}_J) + (1-\hat{c}_J)p_-(\hat{\rho}_J)$ appears in the definition of $\sigma$. This kind of information is deduced through the uniform estimates from the two Hoff functionals.

In order to finish the proof of our main result, we still have to obtain an equation for the volume fraction and to identify the limits $\mu_{eff}$ and $p_{eff}$ through a closed system. To this end, we associate the sequence $(\hat{\rho}_J, \hat{c}_J)_{n\in\mathbb{N}}$ with a sequence of measures on the space $\mathbb{T}^1_x \times \mathbb{R}_\xi \times [0,1]$ (here $\mathbb{R}_\xi$ must be understood as the range of the $\hat{\rho}_J$ while $[0,1]$ is the interval where $\hat{c}_J$ belong to). Namely, given $n \geq 0$ and $t \geq 0$, we consider the measure on $\mathbb{T}^1_x \times \mathbb{R}_\xi \times [0,1]$ as defined by

$$\langle \Theta_J(t), b\rangle \stackrel{def.}{:=} \int_{\mathbb{T}^1} b\left(x, \hat{\rho}_J(t,x), \hat{c}_J(t,x)\right) dx, \qquad \text{for } b \in \mathcal{C}_c\left(\mathbb{T}^1_x \times \mathbb{R}_\xi \times [0,1]\right). \tag{3.9}$$

We have the following proposition.

**Proposition 3.4.** *For fixed $J \in \mathbb{N}$ one has*

$$\Theta_J \in C_w([0,\infty); \mathcal{M}_+(\mathbb{T}^1_x \times \mathbb{R}_\xi \times [0,1])) \tag{3.10}$$

*with*

$$Supp(\Theta_J(t)) \subset \mathbb{T}^1_x \times [C^{-1}, C] \times [0,1] \quad \langle \Theta_J, 1\rangle = 1. \quad \forall t \geq 0, \tag{3.11}$$

*where $C$ is a strictly positive real number depending only on the data and the existence time $T$.*

*Proof.* The second identity (3.11) being obvious we only discuss (3.10). First, we note that, by definition $\Theta_J$ is continuous in $b$ for the topology of $L^1(\mathbb{T}^1_x; \mathcal{C}(\mathbb{R}_\xi \times [0,1]))$. Consequently, a standard density argument entail that we only need to prove that $t \mapsto \langle \Theta_J(t), b\rangle$ is continuous when $b \in C^1_c\left(\mathbb{T}^1_x \times \mathbb{R}_\xi\right)$. For this, we write that

$$|\langle \Theta_J(t), b\rangle - \langle \Theta_J(s), b\rangle| \leq \|\partial_2 b\|_{L^\infty} \int_{\mathbb{T}^1} |\hat{\rho}_J(t,x) - \hat{\rho}_J(s,x)|\, dx, \quad \forall (t,s) \in [0,\infty)^2,$$

and the fact that $\hat{\rho}_J \in C\left([0,\infty), L^1\left(\mathbb{T}^1\right)\right)$ allows to conclude. □

Once these measures are constructed, the rigorous justification of system (1.1)–(1.4) (namely the main result of the paper) reduces to the following two propositions:

**Proposition 3.5.** *Up to the extraction of a subsequence, we have $\Theta_J \rightharpoonup \Theta$ in $C_w([0,\infty); \mathcal{M}_+(\mathbb{T}^1_x \times \mathbb{R}_\xi \times [0,1]))$ where $\Theta$ satisfies*

$$\partial_t \Theta + \partial_x(u\Theta) - \partial_\xi\left(\frac{1}{\mu(\eta)}(\sigma\xi + \xi p(\eta,\xi))\Theta\right) - \frac{1}{\mu(\eta)}(\sigma + p(\eta,\xi))\Theta = 0 \tag{3.12}$$

*with*

$$\mu(\eta) = \eta\mu_+ + (1-\eta)\mu_-, \qquad p(\eta,\xi) = \eta p_+(\xi) + (1-\eta)p_-(\xi)$$

*and $(u,\sigma)$ defined in (3.6)–(3.7).*

The measure $\Theta$ encodes information regarding weak limits of (nonlinear) functions of $(\hat{\rho}_J, \hat{c}_J)$. More precisely, observe that for all $\psi \in C^\infty_{per}(\mathbb{R})$ we have that

$$\begin{aligned}\int_{\mathbb{T}^1} \psi(x) \langle b(\rho,c)\rangle(t,x) &= \lim_{J\to\infty} \int_{\mathbb{T}^1} \psi(x)\, b\left(\hat{\rho}_J(t,x), \hat{c}_J(t,x)\right) = \lim_{J\to\infty} \langle \Theta_J(t), \psi b\rangle \\ &= \langle \theta(t), \psi b\rangle.\end{aligned} \tag{3.13}$$

The next proposition states that if initially we have some extra information on the structure of the measure $\Theta$ then, this structure is conserved for later times.

**Proposition 3.6.** *Assume there exists $(\alpha^0_+, \alpha^0_-, \rho^0_+, \rho^0_-) \in L^\infty(\mathbb{T}^1)$ such (2.17) holds:*

$$\langle \Theta^0_J, b\rangle \to \langle \Theta^0, b\rangle = \int_{\mathbb{T}^1} (\alpha^0_+(x) b\left(x, \rho^0_+(x), 1\right) + \alpha^0_-(x) b\left(x, \rho^0_-(x), 0\right)) dx$$

*for all* $b \in \mathcal{C}(\mathbb{T}^1_x \times \mathbb{R}_\xi \times [0,1])$ *then there exists* $(\alpha_+, \alpha_-, \rho_+, \rho_-) \in L^\infty((0,T)\times\mathbb{T}^1)$ *such that, for all* $b \in \mathcal{C}(\mathbb{T}^1_x \times \mathbb{R}_\xi \times [0,1])$*:*

$$\begin{aligned} \langle \Theta_J, b\rangle &:\overset{def.}{=} \int_{\mathbb{T}^1} b\left(x, \hat{\rho}_J(t,x), \hat{c}_J(t,x)\right) dx \\ \to \langle \Theta, b\rangle &= \int_{\mathbb{T}^1} (\alpha_+(t,x) b\left(x, \rho_+(t,x), 1\right) + \alpha_-(t,x) b\left(x, \rho_-(t,x), 0\right)) dx. \end{aligned} \tag{3.14}$$

*In particular, this helps to conclude that together with* $u$ *such that*

$$\hat{\rho}_J \rightharpoonup \alpha_+\rho_+ + \alpha_-\rho_- \ \ weakly - \star \ in \ L^\infty((0,T)\times\mathbb{T}^1),$$

$$p(\hat{\rho}_J, \hat{c}_J) \rightharpoonup \alpha_+ p_+(\rho_+) + \alpha_- p_-(\rho_-) \ \ weakly - \star \ in \ L^\infty((0,T)\times\mathbb{T}^1),$$

*along with*

$$\hat{u}_J \rightharpoonup u \ in \ H^1((0,T)\times\mathbb{T}^1)$$

*and that* $(\alpha_+, \alpha_-, \rho_+, \rho_-)$ *satisfy* (1.1)–(1.4) *with the initial conditions*

$$\alpha_\pm|_{t=0} = \alpha^0_\pm, \qquad \rho_\pm|_{t=0} = \rho^0_\pm, \qquad u|_{t=0} = u^0.$$

Let us briefly explain why the above proposition is sufficient to conclude the proof of Theorem 1. Using (3.14) along with the identity (3.13) we obtain that for all $\psi \in C^\infty_{per}(\mathbb{R})$

$$\begin{aligned} \lim_{J\to+\infty} \langle \Theta_J, \psi b\rangle = \int_{\mathbb{T}^1} \psi(x) \langle b(\rho, c)\rangle (t,x) = \langle \Theta(t), \psi b\rangle \\ = \int_{\mathbb{T}^1} (\alpha_+(t,x)\, b(\rho_+(t,x), 1) + \alpha_-(t,x)\, b(\rho_-(t,x), 0))\, \psi(x) \end{aligned}$$

which allows us to make the identification

$$\langle b(\rho, c)\rangle (t,x) = \alpha_+(t,x)\, b(\rho_+(t,x), 1) + \alpha_-(t,x)\, b(\rho_-(t,x), 0).$$

Then, recovering the equations verified by $\alpha_\pm$ is achieved via equation (3.12) written for functions $\psi\eta$ and $\psi(1-\eta)$, and the above formula. We refer to [7, 19] for complete details regarding this procedure.

# 4 Proof of Proposition 3.1

The objective of this section is to prove the results announced in Proposition 3.1. First we prove a local existence result, then we show uniform estimates and an upper and lower bound for the density to conclude on the global existence. Then we prove some high-order estimates with appropriate weights in time.

## 4.1 Existence of a local solution for Cauchy problems associated with (2.1)

In order to obtain local existence of a unique solution, let us observe first from $(2.2)_1$ and $(2.2)_2$ that

$$c_j(t) = c_j(0), \qquad \rho_j(t) = \frac{\rho_j(0)\,\Delta x_j(0)}{x_{j+\frac12}(t) - x_{j-\frac12}(t)}, \qquad \rho_{j+\frac12}(t)\,\Delta x_{j+\frac12}(t) = \rho_{j+\frac12}(0)\,\Delta x_{j+\frac12}(0), \quad j = 0,\dots,J-1,$$

and therefore that System (2.1) is nothing but a system of ordinary differential equations of the form

$$\begin{cases} D_t X = U, \\ D_t U = F(X,U), \\ (X,U) = (X_0, U_0), \end{cases} \tag{4.1}$$

with $X = \left(x_{j+\frac{1}{2}}\right)_{j=0,J-1} \in \mathbb{R}^J$ and $U = \left(u_{j+\frac{1}{2}}\right)_{j=0,J-1} \in \mathbb{R}^J$ and $F : D \to \mathbb{R}^J$ is a nonlinear function where

$$D = \left\{(X,U) \in \mathbb{R}^J \times \mathbb{R}^J : \; x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{J-\frac{1}{2}}\right\}$$

which is an open set of $\mathbb{R}^J \times \mathbb{R}^J$. Owing to the fact that $F$ is $C^\infty$ on $D$, we obtain via the Cauchy-Lipschitz-Peano theorem that for any initial data $(X_0, U_0) \in D$, there exists a unique maximal solution for (4.1)

$$(X,U) : [0, T_{\max}) \to D$$

with

- either $T_{\max} = \infty$
- or $T_{\max} < \infty$ and
$$\lim_{t \to T_{\max}} (X(t), U(t)) \in \partial D.$$

This second case means that which means that

- either
$$\lim_{t \to T_{\max}} |X(t)| + |U(t)| = \infty \tag{4.2}$$
- or
$$\begin{array}{l} \sup\limits_{t \in [0,T_{\max})} \{|X(t)| + |U(t)|\} < \infty \\ \text{and there exists } j_0 \in \overline{1, J-1} \text{ such that,} \\ \text{for a sequence of times } (t_n)_{n \geq 0} \text{ such that} t_n \to T_{\max}, \\ \lim\limits_{n\to\infty} x_{j_0}(t_n) = \lim\limits_{n\to\infty} x_{j_0+1}(t_n) \end{array} \tag{4.3}$$

where $|\cdot|$ stands for the euclidean norm of $\mathbb{R}^J$. In the following sections we obtain estimates for the system of ODEs (2.1) that will prove that none of the scenarios (4.2) nor (4.3) happen. Note that these estimates mimic the one from the continuous case.

### 4.2 Mass conservation and basic energy estimates

First of all owing to the periodicity condition (2.4), we have that

$$\sum_{j=0}^{J-1} \Delta x_j(t) = 1.$$

Moreover, from $(2.2)_2$, we have that

$$\sum_{j=0}^{J-1} \rho_j \Delta x_j(t) = \sum_{j=0}^{J-1} \rho_j^0 \Delta x_j^0 \overset{\text{not.}}{=} M_0. \tag{4.4}$$

We start from the following three equations

$$\begin{cases} \Delta x_j \dfrac{d\rho_j}{dt} + \rho_j \left(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right) = 0, \\ \dot{c}_j = 0 \\ \dot{\Delta x_j} = u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}. \end{cases} \tag{4.5}$$

Consider $H$ given by (2.15). We multiply the first equation of (4.5) with $\partial_\rho H(\rho_j, c_j)$, the second one with $\partial_c H(\rho_j, c_j)$ respectively the third one with $H(\rho_j, c_j)$ in order to obtain

$$\frac{d}{dt}\left(H(\rho_j, c_j)\Delta x_j\right) + \left(\rho_j \partial_\rho H(\rho_j, c_j) - H(\rho_j, c_j)\right)\left(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right) = 0,$$

which gives, using the definition of $H$, the following equation:

$$\frac{d}{dt}\left(H(\rho_j, c_j)\Delta x_j\right) + p_j\left(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right) = 0. \tag{4.6}$$

Next, multiplying the momentum equation with $u_{j+\frac{1}{2}}$ we get that

$$\rho_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}\frac{1}{2}\frac{d}{dt}\left|u_{j+\frac{1}{2}}\right|^2 + (p_{j+1} - p_j)\, u_{j+\frac{1}{2}} = \left\{\mu(c_{j+1})\frac{u_{j+\frac{3}{2}} - u_{j+\frac{1}{2}}}{\Delta x_{j+1}} - \mu(c_j)\frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x_j}\right\} u_{j+\frac{1}{2}}.$$

Taking in account (2.7) and summing over $j \in \overline{0, J}$ we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{J-1}\rho_{j+\frac{1}{2}}\left|u_{j+\frac{1}{2}}\right|^2\Delta x_{j+\frac{1}{2}} + \sum_{j=0}^{J-1}(p_{j+1} - p_j)\, u_{j+\frac{1}{2}} \\
&= \sum_{j=0}^{J-1}\left\{\mu(c_{j+1})\frac{u_{j+\frac{3}{2}} - u_{j+\frac{1}{2}}}{\Delta x_{j+1}} - \mu(c_j)\frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x_j}\right\} u_{j+\frac{1}{2}}.
\end{aligned} \tag{4.7}$$

Observe that using the periodic boundary conditions (2.4) and equation (4.6), we get that

$$\sum_{j=0}^{J-1}(p_{j+1} - p_j)\, u_{j+\frac{1}{2}} = -\sum_{j=0}^{J-1} p_j\left(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right) = \frac{d}{dt}\sum_{j=0}^{J-1} H(\rho_j, c_j)\Delta x_j. \tag{4.8}$$

Also, we see that

$$\sum_{j=0}^{J-1}\left\{\mu(c_{j+1})\frac{u_{j+\frac{3}{2}} - u_{j+\frac{1}{2}}}{\Delta x_{j+1}} - \mu(c_j)\frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x_j}\right\} u_{j+\frac{1}{2}} = -\sum_{j=0}^{J-1}\mu(c_j)\frac{1}{\Delta x_j}\left(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right)^2. \tag{4.9}$$

Gathering (4.7), (4.8) and (4.9) we obtain that

$$\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{J-1}\rho_{j+\frac{1}{2}}\left|u_{j+\frac{1}{2}}\right|^2\Delta x_{j+\frac{1}{2}} + \frac{d}{dt}\sum_{j=0}^{J-1} H(\rho_j, c_j)\Delta x_j = -\sum_{j=0}^{J-1}\mu(c_j)\frac{1}{\Delta x_j}\left(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right)^2,$$

which in turn implies that

$$E(t) = E_0 \tag{4.10}$$

for all $t \geq 0$, where $E$ is the basic energy functional defined by relation (3.2).

### 4.3 Upper and lower bound for the density

Using $(2.2)_2$ and $(2.7)_1$, we get

$$\frac{d\rho_j}{dt}\Delta x_j + \rho_j\left(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right) = 0,$$

which we rewrite as

$$\frac{d\log\rho_j}{dt} + \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x_j} = 0. \tag{4.11}$$

Multiplying the last relation by $\mu(c_j)$ and using that $\dot{c}_i = 0$ we get that

$$\frac{d}{dt}\left[\mu(c_j)\log\rho_j\right] + \mu(c_j)\frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x_j} = 0. \tag{4.12}$$

Next, fix arbitrary $q, \ell \in \overline{0, J-1}$ and take the sum in the momentum equation from $j = \ell$ to $j = q-1$ in order to obtain

$$\frac{d}{dt}\sum_{j=\ell}^{q-1}\rho_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}} + p_q - p_\ell = \mu(c_q)\frac{u_{q+\frac{1}{2}} - u_{q-\frac{1}{2}}}{\Delta x_q} - \mu(c_\ell)\frac{u_{\ell+\frac{1}{2}} - u_{\ell-\frac{1}{2}}}{\Delta x_\ell}.$$

Using (4.12) we get that, for any $l \in \overline{0, J-1}$,

$$\begin{aligned}
&\frac{d}{dt}\left\{\sum_{j=0}^{q-1}\rho_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}} + \mu(c_q)\log\rho_q\right\} + p_q \\
&= \frac{d}{dt}\left\{\sum_{j=0}^{\ell-1}\rho_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}\right\} - \mu(c_\ell)\frac{u_{\ell+\frac{1}{2}} - u_{\ell-\frac{1}{2}}}{\Delta x_\ell} + p_\ell.
\end{aligned}$$

Now, for all $\ell \in \overline{0, J-1}$, multiply the previous relation with $\Delta x_\ell$ and take the sum over $\ell \in \overline{0, J-1}$ in order to obtain that:

$$\begin{aligned}
&\frac{d}{dt}\left\{\sum_{j=0}^{q-1}\rho_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}} + \mu(c_q)\log\rho_q\right\} + p_q \\
&= \sum_{\ell=0}^{J-1}\Delta x_\ell\frac{d}{dt}\sum_{j=0}^{\ell-1}\rho_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}} + \sum_{\ell=0}^{J-1}\left[p_\ell\Delta x_\ell - \mu(c_\ell)\left(u_{\ell+\frac{1}{2}} - u_{\ell-\frac{1}{2}}\right)\right].
\end{aligned} \tag{4.13}$$

Observe that

$$\begin{aligned}
&\sum_{\ell=0}^{J-1}\Delta x_\ell\frac{d}{dt}\sum_{j=0}^{\ell-1}\rho_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}} \\
&= \frac{d}{dt}\left\{\sum_{\ell=0}^{J-1}\Delta x_\ell\sum_{j=0}^{\ell-1}\rho_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}\right\} - \sum_{\ell=0}^{J-1}\frac{d\Delta x_\ell}{dt}\sum_{j=0}^{\ell-1}\rho_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}} \\
&= \frac{d}{dt}\left\{\sum_{\ell=0}^{J-1}\Delta x_\ell\sum_{j=0}^{\ell-1}\rho_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}\right\} - \sum_{\ell=0}^{J-1}\left(u_{\ell+\frac{1}{2}} - u_{\ell-\frac{1}{2}}\right)\sum_{j=0}^{\ell-1}\rho_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}.
\end{aligned} \tag{4.14}$$

Integrating (4.13) in time and using (4.14), we get, for any $q$,

$$\begin{aligned}
&\mu(c_q(t))\log\rho_q(t) + \int_0^t p_q(\tau)\,d\tau = \mu(c_q(0))\log\rho_q(0) \\
&+\sum_{j=\ell}^{q-1}\rho_{j+\frac{1}{2}}(0)\,u_{j+\frac{1}{2}}(0)\,\Delta x_{j+\frac{1}{2}}(0) - \sum_{j=\ell}^{q-1}\rho_{j+\frac{1}{2}}(t)\,u_{j+\frac{1}{2}}(t)\,\Delta x_{j+\frac{1}{2}}(t) - \int_0^t\sum_{\ell=0}^{J-1}\mu(c_\ell(\tau))\left(u_{\ell+\frac{1}{2}}(\tau) - u_{\ell-\frac{1}{2}}(\tau)\right) \\
&+ \int_0^t\sum_{\ell=0}^{J-1}\left(u_{\ell+\frac{1}{2}}(\tau) - u_{\ell-\frac{1}{2}}(\tau)\right)\sum_{j=0}^{\ell-1}\rho_{j+\frac{1}{2}}(\tau)\,u_{j+\frac{1}{2}}(\tau)\,\Delta x_{j+\frac{1}{2}}(\tau) + \int_0^t\sum_{\ell=0}^{J-1}p_\ell(\tau)\,\Delta x_\ell(\tau)
\end{aligned}$$

$$+\left\{\sum_{\ell=0}^{J-1}\Delta x_\ell\sum_{j=0}^{\ell-1}\rho_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}\right\}(t)-\left\{\sum_{\ell=0}^{J-1}\Delta x_\ell\sum_{j=0}^{\ell-1}\rho_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}\right\}(0) \quad (4.15)$$

We will use the above equation to show that $\rho_j(t)$ is bounded for any $j$ and $t$.
Recalling that $E_0 = E(0)$ where $E$ is the energy functional defined by (3.2) and $M_0 = M(0)$ where $M$ is the discrete total mass defined by (3.1), we get (thanks to the Cauchy-Schwarz inequality) the following bounds:

$$\sum_{j=\ell}^{q}\rho_{j+\frac{1}{2}}(0)\,u_{j+\frac{1}{2}}(0)\,\Delta x_{j+\frac{1}{2}}(0)-\sum_{j=\ell}^{q}\rho_{j+\frac{1}{2}}(t)\,u_{j+\frac{1}{2}}(t)\,\Delta x_{j+\frac{1}{2}}(t)\le 2\sqrt{2}E_0^{\frac{1}{2}}M_0^{\frac{1}{2}}, \quad (4.16)$$

$$\int_0^t\sum_{\ell=0}^{J-1}\mu\left(c_\ell(\tau)\right)\left(u_{\ell+\frac{1}{2}}(\tau)-u_{\ell-\frac{1}{2}}(\tau)\right)\le\sqrt{t}\left(\sup_{\ell\in\overline{0,J-1}}\mu\left(c_\ell(0)\right)\right)^{1/2}E_0^{\frac{1}{2}}, \quad (4.17)$$

$$\int_0^t\sum_{\ell=0}^{J-1}\left(u_{\ell+\frac{1}{2}}(\tau)-u_{\ell-\frac{1}{2}}(\tau)\right)\sum_{j=0}^{\ell-1}\rho_{j+\frac{1}{2}}(\tau)\,u_{j+\frac{1}{2}}(\tau)\,\Delta x_{j+\frac{1}{2}}(\tau)$$
$$+\int_0^t\sum_{\ell=0}^{J-1}p_\ell(\tau)\,\Delta x_\ell(\tau)\le\left(\frac{2M_0t}{\inf\mu}\right)^{\frac{1}{2}}E_0+t\mathcal{C}(\gamma_+,\gamma_-)E_0, \quad (4.18)$$

with $\mathcal{C}(\gamma_+,\gamma_-$ is a constant coming from the properties (1.3) of the pressure state laws, the expression of $H(\rho,c)$ and its control through the energy. For the fourth line of (4.15), we have similar estimates than the first one because of the control of the total mass. We conclude that there exists a constant $C_0>0$ depending on $E_0$, $M_0$, $\sup_{\ell\in\overline{0,J-1}}\mu\left(c_\ell(0)\right)$, $\inf_{\ell\in\overline{0,J-1}}\mu\left(c_\ell(0)\right)$ and $c_0$ such that

$$\max_{j\in\overline{0,J-1}}\rho_j(t)\le\max_{j\in\overline{0,J-1}}\rho_j(0)\exp\left((1+t)\,C_0\right). \quad (4.19)$$

Using once more the estimates (4.16), (4.17), (4.18) and (4.19) we conclude that there exists a constant $C_1>0$ depending on $E_0$, $M_0$, $\sup_{\ell\in\overline{0,J-1}}\mu\left(c_\ell(0)\right)$, $\inf_{\ell\in\overline{0,J-1}}\mu\left(c_\ell(0)\right)$ and $c_0$ such that

$$\min_{j\in\overline{0,J-1}}\rho_j(0)\exp\left(-\left(1+C_1\right)\exp\left((1+t)\,C_1\right)\right)\le\min_{j\in\overline{0,J-1}}\rho_j(t) \quad (4.20)$$

We deduce that there exists an increasing continuous function $C_{ini}^1(t)$ depending on $E_0$, $M_0$, $\max_{\ell\in\overline{0,J-1}}\mu\left(c_\ell(0)\right)$, $\min_{\ell\in\overline{0,J-1}}\mu\left(c_\ell(0)\right)$, $\overline{\rho^0}$,$\underline{\rho^0}$ such that

$$\frac{1}{C_{ini}^1(t)}\le\rho_q(t)\le C_{ini}^1(t) \quad (4.21)$$

for any $q\in\overline{0,J-1}$.

### 4.4 Global existence for the Cauchy problem associated with (2.1)

The estimates obtained in the last two sections ensure that the solution for the system of ODEs (2.1) is globally defined. First of all, we see that owing to the energy conservation equation (4.10) for all $t\in[0,T_{\max})$ we have that

$$\left(\max_{j\in\overline{0,J-1}}\left|u_{j+\frac{1}{2}}(t)\right|\right)^2\le\sum_{j=0}^{J-1}\left|u_{j+\frac{1}{2}}(t)\right|^2\le\frac{1}{\min_{j\in\overline{0,J-1}}\left(\rho_{j+\frac{1}{2}}(0)\,\Delta x_{j+\frac{1}{2}}(0)\right)}\sum_{j=0}^{J-1}\rho_{j+\frac{1}{2}}(t)\left|u_{j+\frac{1}{2}}(t)\right|^2\Delta x_{j+\frac{1}{2}}(t)$$

$$\leq \frac{2E_0}{\min\limits_{j\in\overline{0,J-1}} \left(\rho_{j+\frac{1}{2}}(0)\,\Delta x_{j+\frac{1}{2}}(0)\right)}.$$

Next, owing to the fact that

$$\dot{x}_{j+\frac{1}{2}} = u_{j+\frac{1}{2}}$$

and to the last inequality, we obtain that

$$\left|x_{j+\frac{1}{2}}(t)\right| \leq \left|x_{j+\frac{1}{2}}(0)\right| + \left(\frac{2E_0}{\min\limits_{j\in\overline{0,J-1}} \left(\rho_{j+\frac{1}{2}}(0)\,\Delta x_{j+\frac{1}{2}}(0)\right)}\right)^{1/2} t.$$

Thus, the first blow-up scenario (4.2) cannot happen.

**Remark 4.1.** *The above estimates degenerate when $J \to +\infty$, however they are sufficient for the purpose of obtaining existence of global solutions for the ODE system (2.1) for a fixed value of $J \in \mathbb{N}^*$.*

Finally, recalling the relation (4.21) along with

$$\rho_q(t) = \frac{\rho_q(0)\,\Delta x_q(0)}{\Delta x_q(t)},$$

we infer that there exists an increasing continuous function $C_{ini}^2(t)$ depending on $E_0, M_0, \max\limits_{\ell\in\overline{0,J-1}} \mu(c_\ell(0))$, $\min\limits_{\ell\in\overline{0,J-1}} \mu(c_\ell(0)), \overline{\rho^0}, \underline{\rho^0}$ such that for all $q \in \overline{0,J-1}$ the following holds true:

$$\frac{\Delta x_q(0)}{C_{ini}^2(t)} \leq \Delta x_q(t) \leq C_{ini}^2(t)\,\Delta x_q(0).$$

Thus the second blow-up scenario (4.3) cannot happen: the unique solution is global.

### 4.5 Control of the first Hoff energy functional defined in (3.3)

Let us multiply the momentum equation with $\dot{u}_{j+\frac{1}{2}}$ and take the sum over all $j \in \overline{0,J-1}$ and observe that this yields

$$\sum_{j=0}^{J-1} \rho_{j+\frac{1}{2}} \left|\dot{u}_{j+\frac{1}{2}}\right|^2 \Delta x_{j+\frac{1}{2}} + \sum_{j=0}^{J-1} \mu(c_j)\,\frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x_j}\left(\dot{u}_{j+\frac{1}{2}} - \dot{u}_{j-\frac{1}{2}}\right) = \sum_{j=0}^{J-1} p_j(\dot{u}_{j+\frac{1}{2}} - \dot{u}_{j-\frac{1}{2}}). \tag{4.22}$$

Note that

$$\begin{aligned}
&\sum_{j=0}^{J-1} \mu(c_j)\,\frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x_j}\left(\dot{u}_{j+\frac{1}{2}} - \dot{u}_{j-\frac{1}{2}}\right)\\
&= \frac{1}{2}\sum_{j=0}^{J-1} \frac{1}{\Delta x_j}\frac{d}{dt}\left(\mu(c_j)\left[u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right]^2\right)\\
&= \frac{1}{2}\frac{d}{dt}\sum_{j=0}^{J-1} \frac{\mu(c_j)}{\Delta x_j}\left[u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right]^2 - \frac{1}{2}\sum_{j=0}^{J-1} \frac{d}{dt}\left(\frac{1}{\Delta x_j}\right)\mu(c_j)\left[u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right]^2\\
&= \frac{1}{2}\frac{d}{dt}\sum_{j=0}^{J-1} \frac{\mu(c_j)}{\Delta x_j}\left[u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right]^2 + \frac{1}{2}\sum_{j=0}^{J-1} \frac{\mu(c_j)}{(\Delta x_j)^2}\left[u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right]^3.
\end{aligned}$$

The above relation allows to put Equation (4.22) under the form

$$\begin{aligned}
&\sum_{j=0}^{J-1} \rho_{j+\frac{1}{2}} \left|\dot{u}_{j+\frac{1}{2}}\right|^2 \Delta x_{j+\frac{1}{2}} + \frac{1}{2}\frac{d}{dt}\sum_{j=0}^{J-1} \mu\left(c_j\right)\left[\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_j}\right]^2 \Delta x_j \\
&= \sum_{j=0}^{J-1} p_j(\dot{u}_{j+\frac{1}{2}}-\dot{u}_{j-\frac{1}{2}}) - \frac{1}{2}\sum_{j=0}^{J-1}\frac{1}{(\Delta x_j)^2}\mu\left(c_j\right)\left[u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}\right]^3 .
\end{aligned} \tag{4.23}$$

Next, observe that

$$\begin{aligned}
&\sum_{j=0}^{J-1} p_j(\dot{u}_{j+\frac{1}{2}}-\dot{u}_{j-\frac{1}{2}}) \\
&= \frac{d}{dt}\sum_{j=0}^{J-1} p_j(u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}) - \sum_{j=0}^{J-1} \dot{p}_j(u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}) \\
&= \frac{d}{dt}\sum_{j=0}^{J-1} p_j(u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}) + \sum_{j=0}^{J-1} \rho_j\partial_\rho p\left(\rho_j,c_j\right)\frac{1}{\Delta x_j}(u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}})^2.
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
&-\frac{1}{2}\sum_{j=0}^{J-1}\frac{1}{(\Delta x_j)^2}\mu\left(c_j\right)\left[u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}\right]^3 = -\frac{1}{2}\sum_{j=0}^{J-1}\frac{1}{\Delta x_j}\left[u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}\right]^2\left\{\mu\left(c_j\right)\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_j} - p\left(\rho_j,c_j\right)\right\} \\
&-\frac{1}{2}\sum_{j=0}^{J-1}\frac{p\left(\rho_j,c_j\right)}{\Delta x_j}\left[u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}\right]^2
\end{aligned}$$

Thus, we can put (4.23) under the form

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{J-1}\mu\left(c_j\right)\left[\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_j}\right]^2\Delta x_j + \sum_{j=0}^{J-1}\rho_{j+\frac{1}{2}}\left|\dot{u}_{j+\frac{1}{2}}\right|^2\Delta x_{j+\frac{1}{2}} \\
&= \frac{d}{dt}\sum_{j=0}^{J-1} p_j(u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}) + \sum_{j=0}^{J-1}\left(\rho_j\partial_\rho p\left(\rho_j,c_j\right) - \frac{p\left(\rho_j,c_j\right)}{2}\right)\frac{1}{\Delta x_j}(u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}})^2 \\
&-\frac{1}{2}\sum_{j=0}^{J-1}\frac{1}{\Delta x_j}\left[u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}\right]^2\left\{\mu\left(c_j\right)\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_j} - p\left(\rho_j,c_j\right)\right\}.
\end{aligned}$$

Thus, recalling the notation introduced in (3.3) we have that

$$\begin{aligned}
E_{H_1}\left(t\right) \le E_{H_1}\left(0\right) &+ \sum_{j=0}^{J-1} p_j\left(t\right)\left(u_{j+\frac{1}{2}}\left(t\right)-u_{j-\frac{1}{2}}\left(t\right)\right) - \sum_{j=0}^{J-1} p_j\left(0\right)\left(u_{j+\frac{1}{2}}\left(0\right)-u_{j-\frac{1}{2}}\left(0\right)\right) \\
&+ \int_0^t\sum_{j=0}^{J-1}\left(\rho_j\partial_\rho p\left(\rho_j,c_j\right) - \frac{p\left(\rho_j,c_j\right)}{2}\right)\frac{1}{\Delta x_j}(u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}})^2 \\
&- \frac{1}{2}\int_0^t\sum_{j=0}^{J-1}\frac{1}{\Delta x_j}\left[u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}\right]^2\left\{\mu\left(c_j\right)\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_j} - p\left(\rho_j,c_j\right)\right\}.
\end{aligned} \tag{4.24}$$

We now will estimate the right hande side of the above inequality. We begin with

$$\sum_{j=0}^{J-1} p_j(t)\left(u_{j+\frac{1}{2}}(t)-u_{j-\frac{1}{2}}(t)\right) \leq \left[2\sum_{j=0}^{J-1}\frac{p_j^2(t)}{\mu(c_j)}\Delta x_j\right]^{\frac{1}{2}}\left[\frac{1}{2}\sum_{j=0}^{J-1}\mu(c_j)\left(\frac{u_{j+\frac{1}{2}}(t)-u_{j-\frac{1}{2}}(t)}{\Delta x_j}\right)^2\Delta x_j\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{\eta}\sum_{j=0}^{J-1}\frac{p_j^2(t)}{\mu(c_j)}\Delta x_j+\eta E_{H_1}(t), \tag{4.25}$$

for any $\eta>0$.

Next, we observe that owing to (4.10) and (4.21) we have that

$$\int_0^t\sum_{j=0}^{J-1}\left(\rho_j\partial_\rho p(\rho_j,c_j)-\frac{p(\rho_j,c_j)}{2}\right)\frac{1}{\Delta x_j}(u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}})^2 \tag{4.26}$$
$$\leq \sup_{(c,\rho)\in\left[\underline{c^0},\overline{c^0}\right]\times\left[C^1_{ini}(t)^{-1},C^1_{ini}(t)\right],}\left\{\frac{2\rho\partial_\rho p(\rho,c)-p(\rho,c)}{2\mu(c)}\right\}E_0 \tag{4.27}$$

where $\underline{c^0}=\inf_j c_j(0)$ and $\overline{c^0}=\sup_j c_j(0)$. Let us now conseider the last term of (4.24). Denoting

$$\sigma_j=\mu(c_j)\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_j}-p(\rho_j,c_j) \tag{4.28}$$

we observe that

$$\frac{d}{dt}\sum_{\ell=j}^{q-1}\rho_{\ell+\frac{1}{2}}\dot{u}_{\ell+\frac{1}{2}}\Delta x_{\ell+\frac{1}{2}}-\sigma_q=-\sigma_j.$$

Multiplying the last relation with $\dfrac{\Delta x_q}{\mu(c_q)}$ and summing over $q$ leads to

$$\sum_{q=0}^{J-1}\frac{\Delta x_q}{\mu(c_q)}\sum_{\ell=j}^{q-1}\rho_{\ell+\frac{1}{2}}\dot{u}_{\ell+\frac{1}{2}}\Delta x_{\ell+\frac{1}{2}}-\sum_{q=0}^{J-1}\frac{\sigma_q}{\mu(c_q)}\Delta x_q=-\sigma_j\sum_{q=0}^{J-1}\frac{\Delta x_q}{\mu(c_q)}.$$

Observe that using (4.28) and the periodicity we have that

$$\sum_{q=0}^{J-1}\frac{\sigma_q}{\mu(c_q)}\Delta x_q=\sum_{q=0}^{J-1}\frac{p(\rho_q,c_q)}{\mu(c_q)}\Delta x_q\leq\mathcal{C}(\gamma_+,\gamma_-)\frac{E_0}{\min\limits_{j\in\overline{0,J-1}}\mu\left(c_j^0\right)}$$

($\mathcal{C}(\gamma_+,\gamma_-)$ has been indroduced in (4.18)). Thus, we have that for all $j\in\overline{0,J-1}$

$$\sum_{q=0}^{J-1}\frac{\Delta x_q}{\mu(c_q)}|\sigma_j|\leq\mathcal{C}(\gamma_+,\gamma_-)\frac{E_0}{\min\limits_{j\in\overline{0,J-1}}\mu\left(c_j^0\right)}+\frac{(M_0)^{1/2}}{\min\limits_{j\in\overline{0,J-1}}\mu\left(c_j^0\right)}\left(\sum_{\ell=0}^{J-1}\rho_{\ell+\frac{1}{2}}\left|\dot{u}_{\ell+\frac{1}{2}}\right|^2\Delta x_{\ell+\frac{1}{2}}\right)^{\frac{1}{2}}$$

so that we obtain

$$\sup_j|\sigma_j(t)|\leq\frac{\max\limits_{j\in\overline{0,J-1}}\mu\left(c_j^0\right)}{\min\limits_{j\in\overline{0,J-1}}\mu\left(c_j^0\right)}\left\{\mathcal{C}(\gamma_+,\gamma_-)E_0+(M_0)^{1/2}\left(\sum_{\ell=0}^{J-1}\rho_{\ell+\frac{1}{2}}(t)\left|\dot{u}_{\ell+\frac{1}{2}}(t)\right|^2\Delta x_{\ell+\frac{1}{2}}(t)\right)^{\frac{1}{2}}\right\} \tag{4.29}$$

for any $t$. From the above inequality we infer that

$$\int_0^t\sum_{j=0}^{J-1}\frac{1}{\Delta x_j(\tau)}\left[u_{j+\frac{1}{2}}(\tau)-u_{j-\frac{1}{2}}(\tau)\right]^2\sigma_j(\tau)\,d\tau$$

$$\leq \int_0^t \sup_{j\in\overline{0,J-1}} \frac{|\sigma_j(\tau)|}{\mu(c_j(\tau))} \sum_{j=0}^{J-1} \frac{\mu(c_j(\tau))}{\Delta x_j(\tau)} \left[u_{j+\frac{1}{2}}(\tau) - u_{j-\frac{1}{2}}(\tau)\right]^2 d\tau$$

$$\leq \eta \int_0^t \left(\sup_{j\in\overline{0,J-1}} \frac{|\sigma_j(\tau)|}{\mu(c_j(\tau))}\right)^2 d\tau + \frac{1}{4\eta}\int_0^t \left(\sum_{j=0}^{J-1} \frac{\mu(c_j(\tau))}{\Delta x_j(\tau)} \left[u_{j+\frac{1}{2}}(\tau) - u_{j-\frac{1}{2}}(\tau)\right]^2\right)^2 d\tau$$

$$\leq 2\eta \frac{\max_{j\in\overline{0,J-1}} (\mu\left(c_j^0\right))^2}{\min_{j\in\overline{0,J-1}} \mu\left(c_j^0\right)^4} \int_0^t \left\{\mathcal{C}(\gamma_+,\gamma_-)^2 E_0^2 + M_0 \left(\sum_{\ell=0}^{J-1} \rho_{\ell+\frac{1}{2}}(\tau) \left|\dot{u}_{\ell+\frac{1}{2}}(\tau)\right|^2 \Delta x_{\ell+\frac{1}{2}}(\tau)\right)\right\} d\tau$$

$$+ \frac{2}{4\eta}\int_0^t E_{H_1}(\tau) \sum_{j=0}^{J-1} \frac{\mu(c_j(\tau))}{\Delta x_j(\tau)} \left[u_{j+\frac{1}{2}}(\tau) - u_{j-\frac{1}{2}}(\tau)\right]^2 d\tau, \tag{4.30}$$

for any $\eta > 0$.

**Remark 4.2.** *From now on, as keeping the exact dependence of constants becomes cumbersome, we will denote by $C(t)$ a generic increasing function depending on the initial data throught $M_0$, $E_0$ ,$\overline{\rho^0}$, $\underline{\rho^0}$, $\overline{c^0}$, $\overline{c^0}$, $\mu_-$, $\mu_+$ and $\left\|(u^0_{j+\frac{1}{2}})_{j\in\overline{0,J-1}}\right\|_{H^1_{disc}}$.*

Gathering the estimates (4.24), (4.25), (4.26) and (4.30) and taking $\eta$ sufficiently small we obtain that

$$E_{H_1}(t) \leq C(t) + \frac{1}{4\eta}\int_0^t E_{H_1}(\tau) \sum_{j=0}^{J-1} \frac{\mu(c_j(\tau))}{\Delta x_j(\tau)} \left[u_{j+\frac{1}{2}}(\tau) - u_{j-\frac{1}{2}}(\tau)\right]^2 d\tau.$$

Then using that

$$\sum_{j=0}^{J-1} \frac{\mu(c_j(\tau))}{\Delta x_j(\tau)} \left[u_{j+\frac{1}{2}}(\tau) - u_{j-\frac{1}{2}}(\tau)\right]^2 \in L^1(0,T)$$

because of the control $E(t) = E_0$ with $E$ given by (3.2), Grönwall's inequality implies that for all $t \geq 0$

$$E_{H_1}(t) \leq C(t).$$

Taking in consideration the estimate (4.29) along with the previous inequality we obtain that

$$\int_0^t \left[\sup_j |\sigma_j(\tau)|\right]^2 d\tau \leq C(t). \tag{4.31}$$

Note that, as the pressure is bounded, we also obtain that

$$\int_0^t \left[\sup_j \left|\frac{u_{j+\frac{1}{2}}(\tau) - u_{j-\frac{1}{2}}(\tau)}{\Delta x_j(\tau)}\right|\right]^2 d\tau \leq C(t). \tag{4.32}$$

### 4.6 Control of the second Hoff energy functional defined in (4.23)

Using that

$$\frac{d}{dt}(\rho_{j+1/2}\Delta x_{j+1/2}) = 0$$

we take the time derivative in the momentum equation to get

$$\rho_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}\ddot{u}_{j+\frac{1}{2}}+\dot{p}_{j+1}-\dot{p}_{j}=\frac{d}{dt}\left\{\mu\left(c_{j+1}\right)\frac{u_{j+\frac{3}{2}}-u_{j+\frac{1}{2}}}{\Delta x_{j+1}}-\mu\left(c_{j}\right)\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_{j}}\right\}.$$

Multiplying the above relation with $\dot{u}_{j+\frac{1}{2}}$, we write that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{J-1}\rho_{j+\frac{1}{2}}|\dot{u}_{j+\frac{1}{2}}|^{2}\Delta x_{j+\frac{1}{2}}+\sum_{j=0}^{J-1}\dot{u}_{j+\frac{1}{2}}\left(\dot{p}_{j+1}-\dot{p}_{j}\right)\\&=\sum_{j=0}^{J-1}\dot{u}_{j+\frac{1}{2}}\frac{d}{dt}\left\{\mu\left(c_{j+1}\right)\frac{u_{j+\frac{3}{2}}-u_{j+\frac{1}{2}}}{\Delta x_{j+1}}-\mu\left(c_{j}\right)\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_{j}}\right\}.\end{aligned}\tag{4.33}$$

Observe that

$$\sum_{j=0}^{J-1}\dot{u}_{j+\frac{1}{2}}\left(\dot{p}_{j+1}-\dot{p}_{j}\right)=-\sum_{j=0}^{J-1}\left(\dot{u}_{j+\frac{1}{2}}-\dot{u}_{j-\frac{1}{2}}\right)\dot{p}_{j}=\sum_{j=0}^{J-1}\frac{\rho_{j}\partial_{\rho}p\left(\rho_{j},c_{j}\right)}{\Delta x_{j}}\left(u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}\right)\left(\dot{u}_{j+\frac{1}{2}}-\dot{u}_{j-\frac{1}{2}}\right).\tag{4.34}$$

Next, we see that

$$\begin{aligned}&\sum_{j=0}^{J-1}\dot{u}_{j+\frac{1}{2}}\frac{d}{dt}\left\{\mu\left(c_{j+1}\right)\frac{u_{j+\frac{3}{2}}-u_{j+\frac{1}{2}}}{\Delta x_{j+1}}-\mu\left(c_{j}\right)\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_{j}}\right\}\\&=-\sum_{j=0}^{J-1}\left(\dot{u}_{j+\frac{1}{2}}-\dot{u}_{j-\frac{1}{2}}\right)\frac{d}{dt}\left\{\mu\left(c_{j}\right)\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_{j}}\right\}\\&=-\sum_{j=0}^{J-1}\mu\left(c_{j}\right)\frac{\left(\dot{u}_{j+\frac{1}{2}}-\dot{u}_{j-\frac{1}{2}}\right)^{2}}{\Delta x_{j}}+\sum_{j=0}^{J-1}\left(\dot{u}_{j+\frac{1}{2}}-\dot{u}_{j-\frac{1}{2}}\right)\left(u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}\right)\frac{1}{(\Delta x_{j})^{2}}\frac{d\Delta x_{j}}{dt}\mu(c_{j})\\&=-\sum_{j=0}^{J-1}\mu\left(c_{j}\right)\frac{\left(\dot{u}_{j+\frac{1}{2}}-\dot{u}_{j-\frac{1}{2}}\right)^{2}}{\Delta x_{j}}+\sum_{j=0}^{J-1}\frac{\left(\dot{u}_{j+\frac{1}{2}}-\dot{u}_{j-\frac{1}{2}}\right)}{\Delta x_{j}}\frac{\left(u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}\right)^{2}}{\Delta x_{j}}\mu(c_{j}).\end{aligned}\tag{4.35}$$

Recall the notation introduced in (3.4). Multiply the relation (4.33) with $\min\{1,t\}$, integrate in time and using the previous two relations we arrive at

$$\begin{aligned}E_{H_{2}}\left(t\right)=&\int_{0}^{1}\sum_{j=0}^{J-1}\rho_{j+\frac{1}{2}}\left(\tau\right)|\dot{u}_{j+\frac{1}{2}}\left(\tau\right)|^{2}\Delta x_{j+\frac{1}{2}}\left(\tau\right)d\tau\\&-\int_{0}^{t}\min\left(1,\tau\right)\sum_{j=0}^{J-1}\frac{\rho_{j}\left(\tau\right)\partial_{\rho}p\left(\rho_{j}\left(\tau\right),c_{j}\left(\tau\right)\right)}{\Delta x_{j}\left(\tau\right)}\left(u_{j+\frac{1}{2}}\left(\tau\right)-u_{j-\frac{1}{2}}\left(\tau\right)\right)\left(\dot{u}_{j+\frac{1}{2}}\left(\tau\right)-\dot{u}_{j-\frac{1}{2}}\left(\tau\right)\right)d\tau\\&+\int_{0}^{t}\min\left(1,\tau\right)\sum_{j=0}^{J-1}\frac{\left(\dot{u}_{j+\frac{1}{2}}\left(\tau\right)-\dot{u}_{j-\frac{1}{2}}\left(\tau\right)\right)}{\Delta x_{j}\left(\tau\right)}\frac{\left(u_{j+\frac{1}{2}}\left(\tau\right)-u_{j-\frac{1}{2}}\left(\tau\right)\right)^{2}}{\Delta x_{j}\left(\tau\right)}d\tau.\end{aligned}\tag{4.36}$$

In the following we estimate the terms appearing in the right hand side above.

The first term is bounded by the first energy:

$$\int_{0}^{1}\sum_{j=0}^{J-1}\rho_{j+\frac{1}{2}}\left(\tau\right)|\dot{u}_{j+\frac{1}{2}}\left(\tau\right)|^{2}\Delta x_{j+\frac{1}{2}}\left(\tau\right)d\tau\leq E_{H_{1}}\left(t\right).\tag{4.37}$$

The second term is treated as follows

$$
\begin{aligned}
&-\int_0^t \min(1,\tau)\sum_{j=0}^{J-1}\frac{\rho_j(\tau)\,\partial_\rho p(\rho_j(\tau),c_j(\tau))}{\Delta x_j(\tau)}\left(u_{j+\frac12}(\tau)-u_{j-\frac12}(\tau)\right)\left(\dot u_{j+\frac12}(\tau)-\dot u_{j-\frac12}(\tau)\right)d\tau\\
&\le \frac{1}{4\eta}C(t)+\eta\int_0^t \min(1,\tau)\sum_{j=0}^{J-1}\left(\frac{\dot u_{j+\frac12}(\tau)-\dot u_{j-\frac12}(\tau)}{\Delta x_j(\tau)}\right)^2\Delta x_j(\tau)\,d\tau
\end{aligned}
\tag{4.38}
$$

for any $\eta>0$, thanks to (4.32) and (4.19). Now we concentrate on the last term appearing in (4.36). Using Cauchy's inequality we obtain

$$
\begin{aligned}
&\int_0^t \min\{1,\tau\}\sum_{j=0}^{J-1}\frac{\dot u_{j+\frac12}(\tau)-\dot u_{j-\frac12}(\tau)}{\Delta x_j(\tau)}\frac{\left(u_{j+\frac12}(\tau)-u_{j-\frac12}(\tau)\right)^2}{\Delta x_j(\tau)}d\tau\\
&\le \eta\int_0^t\sum_{j=0}^{J-1}\min\{1,\tau\}\,\mu(c_j(\tau))\frac{\left|\dot u_{j+\frac12}(\tau)-\dot u_{j-\frac12}(\tau)\right|^2}{\Delta x_j(\tau)}\\
&+\frac{1}{4\eta}\int_0^t\sum_{j=0}^{J-1}\frac{1}{\mu(c_j(\tau))}\left|\frac{u_{j+\frac12}(\tau)-u_{j-\frac12}(\tau)}{\Delta x_j(\tau)}\right|^4\Delta x_j(\tau)\,d\tau\\
&\le \eta\int_0^t\sum_{j=0}^{J-1}\min\{1,\tau\}\,\mu(c_j(\tau))\frac{\left|\dot u_{j+\frac12}(\tau)-\dot u_{j-\frac12}(\tau)\right|^2}{\Delta x_j(\tau)}\\
&+\frac{1}{4\eta}\min_{j\in\overline{0,J-1}}\frac{1}{\mu^2(c_j(0))}\times\int_0^t\sup_{j\in\overline{0,J-1}}\left|\frac{u_{j+\frac12}(\tau)-u_{j-\frac12}(\tau)}{\Delta x_j(\tau)}\right|^2\sum_{j=0}^{J-1}\mu(c_j(\tau))\left|\frac{u_{j+\frac12}(\tau)-u_{j-\frac12}(\tau)}{\Delta x_j(\tau)}\right|^2\Delta x_j(\tau)\,d\tau\\
&\le \eta\int_0^t\sum_{j=0}^{J-1}\min\{1,\tau\}\,\mu(c_j(\tau))\frac{\left|\dot u_{j+\frac12}(\tau)-\dot u_{j-\frac12}(\tau)\right|^2}{\Delta x_j(\tau)}\\
&+\frac{1}{4\eta}\min_{j\in\overline{0,J-1}}\frac{1}{\mu^2(c_j(0))}\times E_{H_1}(t)\times\int_0^t\sup_{j\in\overline{0,J-1}}\left|\frac{u_{j+\frac12}(\tau)-u_{j-\frac12}(\tau)}{\Delta x_j(\tau)}\right|^2.
\end{aligned}
$$

Owing to (4.32),

$$
\int_0^t\sup_{j\in\overline{0,J-1}}\left|\frac{u_{j+\frac12}(\tau)-u_{j-\frac12}(\tau)}{\Delta x_j(\tau)}\right|^2\le C(t)
$$

so that the last term is bounded.

Gathering the information above, we get

$$
E_{H_2}(t)\le C(t)+\int_0^t\left(\sup_j|(Du)_j(\tau)|\right)^2E_{H_2}(\tau)\,d\tau
$$

where $(Du)_j=\frac{u_{j+\frac12}-u_{j-\frac12}}{\Delta x_j}$. Recalling that

$$
\left(\sup_j|(Du)_j(\tau)|\right)^2\in L^1(0,T)
$$

by (4.32), then using Grönwall's inequality, we get that

$$
E_{H_2}(t)\le C(t).
$$

Proposition 3.1 is proved.

## 5 Proof of Proposition 3.3

In this section we prove the result announced in Proposition 3.3. We begin by analyzing the density $\hat{\rho}_J$ and the volume fraction (or mass fraction, since at this level they are equal, and equal to 1 or 0) $\hat{c}_J$, then we look at the continuous velocity $\hat{u}_J$ to conclude with the constraint $\hat{\sigma}_J$.

**Estimates for the density $\hat{\rho}_J$ and volume fraction $\hat{c}_J$.** First of all, obviously $\hat{c}_J, \hat{\rho}_J$ belong to $L^\infty\left((0,T)\times\mathbb{T}^1\right)$ for all $T>0$ as it can be seen from estimate (4.19) and the fact that for all $j$, $c_j$ is constant in time so that we infer the following:

$$\begin{cases} \hat{c}_J(t,x)\in\{0,1\}, \\ \dfrac{1}{C^1_{ini}(t)} \leq \hat{\rho}_J(t,x) \leq C^1_{ini}(t). \end{cases}$$

Next, we obviously have that

$$\int_0^1 H\left(\hat{\rho}_J(t,x),\hat{c}_J(t,x)\right)dx = \sum_{j=0}^{J-1}\int_{x_j(t)}^{x_{j+1}(t)} H\left(\hat{\rho}_J(t,x),\hat{c}_J(t,x)\right)dx = \sum_{j=0}^{J-1} H\left(\rho_j,c_j\right)\Delta x_j. \tag{5.1}$$

**Estimates for the continuous velocity $\hat{u}_J$.** In the following lines we analyze the continuous velocity $\hat{u}_J$.
First we observe the following estimate for the total kinetic energy.

$$\begin{aligned} &\int_0^1 \hat{\rho}_J\left|\hat{u}_J\right|^2(t,x)\,dx \\ &= \sum_{j=0}^{J-1}\rho_j(t)\int_{x_{j-\frac12}}^{x_{j+\frac12}}\left|\hat{u}_J\right|^2(t,x)\,dx \leq 2\sum_{j=0}^{J-1}\rho_j(t)\left|u_{j+\frac12}(t)\right|^2\frac{\Delta x_j(t)}{3} + 2\sum_{j=0}^{J-1}\rho_j(t)\left|u_{j-\frac12}(t)\right|^2\frac{\Delta x_j(t)}{3} \\ &\leq \frac23\sum_{j=0}^{J-1}\left(\rho_j(t)\Delta x_j(t)+\rho_{j+1}(t)\Delta x_{j+1}(t)\right)\left|u_{j+\frac12}(t)\right|^2 = \frac43\sum_{j=0}^{J-1}\rho_{j+\frac12}(t)\left|u_{j+\frac12}(t)\right|^2\Delta x_{j+\frac12}(t) \leq \frac83 E_0. \end{aligned} \tag{5.2}$$

As $\hat{u}_J(t,\cdot)$ is piecewise linear, it posseses a weak derivative that is piecewise constant, more precisely

$$\partial_x\hat{u}_J(t,\cdot) = \frac{u_{j+\frac12}(t)-u_{j-\frac12}(t)}{\Delta x_j(t)} \text{ on } \left[x_{j-\frac12}(t),x_{j+\frac12}(t)\right)$$

for all $t\geq 0$. We infer that

$$\int_0^t\int_0^1 \mu\left(\hat{c}_J(\tau,x)\right)\left|\partial_x\hat{u}_J(\tau,x)\right|^2dxd\tau = \int_0^t\sum_{j=0}^{J-1}\mu\left(c_j(\tau)\right)\frac{\left|u_{j+\frac12}(\tau)-u_{j-\frac12}(\tau)\right|^2}{\Delta x_j(\tau)} \leq E_0. \tag{5.3}$$

Putting together (5.1), (5.2) and (5.3) we obtain the estimate on the second line of (2.14).
According to (4.32) we see that

$$\int_0^t\left(\sup_{x\in[0,1]}\left|\partial_x\hat{u}_J(t,x)\right|\right)^2 \leq C(t). \tag{5.4}$$

Let us observe that for all $t\geq 0$ and $x\in\left[x_{j-\frac{1}{2}}\left(t\right),x_{j+\frac{1}{2}}\left(t\right)\right)$ we have

$$\begin{aligned}\partial_t\hat{u}_J\left(t,x\right)&=\frac{x-x_{j-\frac{1}{2}}\left(t\right)}{\Delta x_j\left(t\right)}\dot{u}_{j+\frac{1}{2}}\left(t\right)+\frac{x_{j+\frac{1}{2}}\left(t\right)-x}{\Delta x_j\left(t\right)}\dot{u}_{j-\frac{1}{2}}\left(t\right)\\&-\frac{x-x_{j-\frac{1}{2}}\left(t\right)}{\Delta x_j\left(t\right)^2}\frac{d\Delta x_j}{dt}u_{j+\frac{1}{2}}\left(t\right)-\frac{x_{j+\frac{1}{2}}\left(t\right)-x}{\Delta x_j\left(t\right)^2}\frac{d\Delta x_j}{dt}u_{j-\frac{1}{2}}\left(t\right)\\&=\frac{x-x_{j-\frac{1}{2}}\left(t\right)}{\Delta x_j\left(t\right)}\dot{u}_{j+\frac{1}{2}}\left(t\right)+\frac{x_{j+\frac{1}{2}}\left(t\right)-x}{\Delta x_j\left(t\right)}\dot{u}_{j-\frac{1}{2}}\left(t\right)-\hat{u}_J\left(t,x\right)\frac{\left(u_{j+\frac{1}{2}}\left(t\right)-u_{j-\frac{1}{2}}\left(t\right)\right)}{\Delta x_j\left(t\right)}\\&=\frac{x-x_{j-\frac{1}{2}}\left(t\right)}{\Delta x_j\left(t\right)}\dot{u}_{j+\frac{1}{2}}\left(t\right)+\frac{x_{j+\frac{1}{2}}\left(t\right)-x}{\Delta x_j\left(t\right)}\dot{u}_{j-\frac{1}{2}}\left(t\right)-\hat{u}_J\left(t,x\right)\partial_x\hat{u}_J\left(t,x\right)\end{aligned}$$

We infer that

$$\begin{aligned}\left\|\partial_t\hat{u}_J\left(t,\cdot\right)\right\|_{L^2_x}^2&\leq\frac{8}{3}\sum_{j=0}^{J-1}\left|\dot{u}_{j+\frac{1}{2}}\left(t\right)\right|^2\Delta x_{j+\frac{1}{2}}(t)+2\left\|\hat{u}_J\left(t,\cdot\right)\right\|_{L^\infty_x}^2\left\|\partial_x\hat{u}_J\left(t,\cdot\right)\right\|_{L^2_x}^2\\&\leq\frac{8}{3}\sum_{j=0}^{J-1}\left|\dot{u}_{j+\frac{1}{2}}\left(t\right)\right|^2+2\left\|\partial_x\hat{u}_J\left(t,\cdot\right)\right\|_{L^2_x}^4\end{aligned}$$

from which we recover that

$$\int_0^t\left\|\partial_t\hat{u}_J\left(\tau,\cdot\right)\right\|_{L^2_x}^2+\min\left\{1,t\right\}\left\|\partial_t\hat{u}_J\left(t,\cdot\right)\right\|_{L^2_x}^2\leq C\left(t\right).\tag{5.5}$$

From (5.4) and (5.5) we get the fourth estimate (third line) of (2.14).

**The Cauchy stress quantity $\hat{\sigma}_J$.** Let us observe that we have

$$\partial_x\hat{\sigma}_J\left(t,x\right)=\frac{\sigma_{j+1}-\sigma_j}{\Delta x_{j+\frac{1}{2}}}\text{ for }x\in\left[x_j,x_{j+1}\right),\quad\text{for any }j.$$

Thus, owing to the fact that the Hoff-energy functionals are bounded we get that

$$\int_0^t\int_0^1\left|\partial_x\hat{\sigma}_J\left(\tau,x\right)\right|^2dxd\tau+\min\left\{1,t\right\}\int_0^1\left|\partial_x\hat{\sigma}_J\left(t,x\right)\right|^2dx\tag{5.6}$$

$$=\int_0^t\sum_{j=0}^{J-1}\left|\rho_{j+\frac{1}{2}}\left(\tau\right)\dot{u}_{j+\frac{1}{2}}\left(\tau\right)\right|^2\Delta x_{j+\frac{1}{2}}\left(\tau\right)d\tau+\min\left\{1,t\right\}\sum_{j=0}^{J-1}\left|\rho_{j+\frac{1}{2}}\left(t\right)\dot{u}_{j+\frac{1}{2}}\left(t\right)\right|^2\Delta x_{j+\frac{1}{2}}\left(t\right)\leq C\left(t\right).\tag{5.7}$$

It turns out that we can recover an estimate for $\left(\partial_x\hat{\sigma}_J\right)_J$ in $L_t^{4/3-}L_x^\infty$ uniformly in $J$. The argument goes as follows: for any $j$ and $k$ we have that

$$\begin{aligned}\left|\dot{u}_{j+\frac{1}{2}}\left(t\right)\right|^2&\leq\left|\dot{u}_{k+\frac{1}{2}}\left(t\right)\right|^2+\sum_{\ell=\min\{k,j\}+1}^{\max\{k,j\}}\left|\dot{u}_{\ell+\frac{1}{2}}\left(t\right)-\dot{u}_{\ell-\frac{1}{2}}\left(t\right)\right|\left|\dot{u}_{\ell+\frac{1}{2}}\left(t\right)+\dot{u}_{\ell-\frac{1}{2}}\left(t\right)\right|\\&\leq\left|\dot{u}_{k+\frac{1}{2}}\left(t\right)\right|^2+\sum_{\ell=0}^{J-1}\left|\dot{u}_{\ell+\frac{1}{2}}\left(t\right)-\dot{u}_{\ell-\frac{1}{2}}\left(t\right)\right|\left|\dot{u}_{\ell+\frac{1}{2}}\left(t\right)+\dot{u}_{\ell-\frac{1}{2}}\left(t\right)\right|\\&\leq\left|\dot{u}_{k+\frac{1}{2}}\left(t\right)\right|^2\end{aligned}$$

$$+\frac{\left\{\min\left\{1,t\right\}\sum_{\ell=0}^{J-1}\frac{\mu(c_j)\left|\dot{u}_{\ell+\frac{1}{2}}(t)-\dot{u}_{\ell-\frac{1}{2}}(t)\right|^2}{\Delta x_\ell}\right\}^{\frac{1}{2}}}{\inf\limits_{j=\overline{0,J-1}}\mu\left(c_j^0\right)^{\frac{1}{2}}\min\left\{1,t\right\}^{\frac{1}{2}}}\left\{2\sum_{\ell=0}^{J-1}\left|\dot{u}_{\ell+\frac{1}{2}}\left(t\right)\right|^2\Delta x_{\ell+\frac{1}{2}}\right\}^{\frac{1}{2}} \text{ for any } t>0,$$

where we have used that

$$2\Delta x_{\ell+\frac{1}{2}}=\Delta x_\ell+\Delta x_{\ell+1}.$$

The previous inequality implies that

$$\begin{aligned}\left|\dot{u}_{j+\frac{1}{2}}\left(t\right)\right| &\leq \left|\dot{u}_{k+\frac{1}{2}}\left(t\right)\right| \\ &+\frac{\left\{\min\left\{1,t\right\}\sum_{\ell=0}^{J-1}\frac{\mu(c_j)\left|\dot{u}_{\ell+\frac{1}{2}}(t)-\dot{u}_{\ell-\frac{1}{2}}(t)\right|^2}{\Delta x_\ell}\right\}^{\frac{1}{4}}}{\inf\limits_{j=\overline{0,J-1}}\mu\left(c_j^0\right)^{\frac{1}{4}}\min\left\{1,t\right\}^{\frac{1}{4}}}\left\{2\sum_{\ell=0}^{J-1}\left|\dot{u}_{\ell+\frac{1}{2}}\left(t\right)\right|^2\Delta x_{\ell+\frac{1}{2}}\right\}^{\frac{1}{4}}.\end{aligned}$$

Multiply the above inequality with $\Delta x_{k+\frac{1}{2}}$ and take the sum over $k\in\overline{0,J-1}$ in order to obtain that

$$\begin{aligned}\left|\dot{u}_{j+\frac{1}{2}}\left(t\right)\right| &\leq \sum_{k=0}^{J-1}\left|\dot{u}_{k+\frac{1}{2}}\left(t\right)\right|\Delta x_{k+\frac{1}{2}} \\ &+\frac{\left\{\min\left\{1,t\right\}\sum_{\ell=0}^{J-1}\frac{\mu(c_j)\left|\dot{u}_{\ell+\frac{1}{2}}(t)-\dot{u}_{\ell-\frac{1}{2}}(t)\right|^2}{\Delta x_\ell}\right\}^{\frac{1}{4}}}{\inf\limits_{j=\overline{0,J-1}}\mu\left(c_j^0\right)^{\frac{1}{4}}\min\left\{1,t\right\}^{\frac{1}{4}}}\left\{2\sum_{\ell=0}^{J-1}\left|\dot{u}_{\ell+\frac{1}{2}}\left(t\right)\right|^2\Delta x_{\ell+\frac{1}{2}}\right\}^{\frac{1}{4}}.\end{aligned}$$

From the previous inequality and the fact that the two functionals $E_{H_1}$ and $E_{H_2}$ are bounded and the fact that the density is bounded by below it follows that for all $r\in[1,\frac{4}{3})$

$$\int_0^t\left(\sup_{j\in\overline{0,J-1}}\left|\dot{u}_{j+\frac{1}{2}}\left(\tau\right)\right|\right)^r d\tau\leq C\left(t\right)$$

which obviously implies that

$$\begin{aligned}&\int_0^t(\sup_{x\in[0,1]}\left|\partial_x\hat{\sigma}_J\left(\tau,x\right)\right|)^r d\tau \\ &=\int_0^t(\sup_{j\in\overline{0,J-1}}\left|\frac{\sigma_{j+1}\left(\tau\right)-\sigma_j\left(\tau\right)}{\Delta x_{j+\frac{1}{2}}\left(\tau\right)}\right|)^r d\tau=\int_0^t(\sup_{j\in\overline{0,J-1}}\left|\rho_{j+\frac{1}{2}}\left(\tau\right)\dot{u}_{j+\frac{1}{2}}\left(\tau\right)\right|)^r d\tau\leq C\left(t\right)\end{aligned}$$

for all $r\in[1,\frac{4}{3})$. Together with (5.7), this gives the desired estimate.

**Time derivative of the Cauchy stress $\hat{\sigma}_J$.** The time derivative of $\hat{\sigma}_J$ is given by

$$\partial_t\hat{\sigma}_J\left(t,x\right)=\frac{x-x_j}{\Delta x_{j+\frac{1}{2}}}\dot{\sigma}_{j+1}+\frac{x_{j+1}-x}{\Delta x_{j+\frac{1}{2}}}\dot{\sigma}_j$$

$$
\begin{aligned}
&-\frac{u_{j+\frac{1}{2}}+u_{j-\frac{1}{2}}}{2\Delta x_{j+\frac{1}{2}}}\sigma_{j+1}+\frac{u_{j+\frac{3}{2}}+u_{j+\frac{1}{2}}}{2\Delta x_{j+\frac{1}{2}}}\sigma_j\\
&-\frac{x-x_j}{(\Delta x_{j+\frac{1}{2}})^2}\left(\frac{u_{j+\frac{3}{2}}-u_{j-\frac{1}{2}}}{2}\right)\sigma_{j+1}-\frac{x_{j+1}-x}{(\Delta x_{j+\frac{1}{2}})^2}\left(\frac{u_{j+\frac{1}{2}}-u_{j-\frac{3}{2}}}{2}\right)\sigma_j\\
&=T_{1j}+T_{2j}+T_{3j},
\end{aligned}
$$

for all $t\geq 0$ and $x\in[x_j(t),x_{j+1}(t))$ and $j\in\overline{0,J-1}$. We begin by treating the terms related to $T_{1j}$. First, we write that

$$
\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\left|\frac{x_{j+1}-x}{\Delta x_{j+\frac{1}{2}}}\dot{\sigma}_j\right|^2\leq\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}|\dot{\sigma}_j|^2
$$

$$
\begin{aligned}
2\leq&\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\mu^2(c_j)\left|\frac{\dot{u}_{j+\frac{1}{2}}-\dot{u}_{j-\frac{1}{2}}}{\Delta x_j}\right|^2+2\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\mu^2(c_j)\left|\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_j}\right|^4+2\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\left|\frac{d}{dt}p(c_j,\rho_j)\right|^2\\
\leq&\,2\max_{c\in[0,\overline{c}]}\mu(c)\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\mu(c_j)\left|\frac{\dot{u}_{j+\frac{1}{2}}-\dot{u}_{j-\frac{1}{2}}}{\Delta x_j}\right|^2+2\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\mu^2(c_j)\left|\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_j}\right|^4\\
&+2\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}|\rho_j\partial_\rho p(c_j,\rho_j)|^2\left|\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_j}\right|^2
\end{aligned}
$$

Concerning the first term in the right hand side above, observe that

$$
\int_0^t\min\{1,\tau\}\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\mu(c_j(\tau))\left|\frac{\dot{u}_{j+\frac{1}{2}}(\tau)-\dot{u}_{j-\frac{1}{2}}(\tau)}{\Delta x_j(\tau)}\right|^2\leq E_{H_2}(t)\,. \tag{5.8}
$$

Next, we have

$$
\begin{aligned}
\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\mu^2(c_j)\left|\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_j}\right|^4&=\sum_{j=0}^{J-1}\mu^2(c_j)\left|\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_j}\right|^4\Delta x_{j+\frac{1}{2}}\\
&\leq\max_{c\in[0,\overline{c}]}\mu(c)\,(\sup_{j\in\overline{0,J-1}}\left|D(u)_j(t)\right|)^2\sum_{j=0}^{J-1}\mu(c_j)\left|\frac{u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}}}{\Delta x_j}\right|^2\Delta x_{j+\frac{1}{2}}\\
&\leq\max_{c\in[0,\overline{c}]}\mu(c)\,(\sup_{j\in\overline{0,J-1}}\left|D(u)_j(t)\right|)^2E_{H_1}(t)\,,
\end{aligned}
$$

so that

$$
\int_0^t\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\mu^2(c_j(\tau))\left|\frac{u_{j+\frac{1}{2}}(\tau)-u_{j-\frac{1}{2}}(\tau)}{\Delta x_j(\tau)}\right|^4\leq\max_{c\in[0,\overline{c}]}\mu(c)\,C(t)\int_0^t(\sup_{j\in\overline{0,J-1}}\left|D(u)_j(\tau)\right|)^2d\tau.
$$

Thus, according to (4.32), we have

$$
\int_0^t\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\mu^2(c_j(\tau))\left|\frac{u_{j+\frac{1}{2}}(\tau)-u_{j-\frac{1}{2}}(\tau)}{\Delta x_j(\tau)}\right|^4d\tau\leq C(t)\,. \tag{5.9}
$$

At last, thanks to the uniform bounds on the density, we also have

$$\int\limits_0^t \sum_{j=0}^{J-1} \int\limits_{x_j}^{x_{j+1}} \left|\rho_j(\tau)\,\partial_\rho p\left(c_j(\tau),\rho_j(\tau)\right)\right|^2 \left|\frac{u_{j+\frac{1}{2}}(\tau)-u_{j-\frac{1}{2}}(\tau)}{\Delta x_j(\tau)}\right|^2$$

$$\leq C(t)\sum_{j=0}^{J-1}\int\limits_{x_j}^{x_{j+1}} \mu\left(c_j(\tau)\right)\left|\frac{u_{j+\frac{1}{2}}(\tau)-u_{j-\frac{1}{2}}(\tau)}{\Delta x_j(\tau)}\right|^2 \leq C(t). \tag{5.10}$$

Putting together (5.8), (5.9) and (5.10) we get that

$$\int\limits_0^t \min\{1,\tau\}\sum_{j=0}^{J-1}\left|T_{1j}(\tau)\right|^2 d\tau \leq C(t). \tag{5.11}$$

Next, in order to treat the terms related to $T_{2j}$ we write that

$$-\frac{u_{j+\frac{1}{2}}+u_{j-\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}}\sigma_{j+1}+\frac{u_{j+\frac{3}{2}}+u_{j+\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}}\sigma_j=-\left(u_{j+\frac{1}{2}}+u_{j-\frac{1}{2}}\right)\frac{\sigma_{j+1}-\sigma_j}{\Delta x_{j+\frac{1}{2}}}-\frac{u_{j+\frac{3}{2}}+u_{j-\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}}\sigma_j.$$

Next, we remark that

$$\int_0^t\int_{x_j}^{x_{j+1}}\sum_{j=0}^{J-1}\left(u_{j+\frac{1}{2}}(\tau)+u_{j-\frac{1}{2}}(\tau)\right)^2\left|\frac{\sigma_{j+1}(\tau)-\sigma_j(\tau)}{\Delta x_{j+\frac{1}{2}}(\tau)}\right|^2$$

$$\leq 2\sup_t\sup_j\left|u_{j+\frac{1}{2}}(t)\right|^2(t)\int_0^t\sum_{j=0}^{J-1}\left|\frac{\sigma_{j+1}(\tau)-\sigma_j(\tau)}{\Delta x_{j+\frac{1}{2}}(\tau)}\right|^2\Delta x_{j+\frac{1}{2}}(\tau)\leq C(t).$$

The second term is estimated as follows

$$\int_0^t\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\frac{1}{(\Delta x_{j+\frac{1}{2}}(\tau))^2}\left|u_{j+\frac{3}{2}}(\tau)-u_{j-\frac{1}{2}}(\tau)\right|^2\left|\sigma_j(\tau)\right|^2 d\tau$$

$$\leq\int_0^t(\max_j\left|\sigma_j(\tau)\right|)^2 d\tau\sup_{\tau\in[0,t]}\sum_{j=0}^{J-1}\frac{1}{\Delta x_{j+\frac{1}{2}}(\tau)}\left|u_{j+\frac{3}{2}}(\tau)-u_{j-\frac{1}{2}}(\tau)\right|^2\leq C(t).$$

Using the last two inequalities we get that

$$\int\limits_0^t\int\limits_{x_j}^{x_{j+1}}\sum_{j=0}^{J-1}\left|T_{2j}(\tau)\right|^2 d\tau\leq C(t). \tag{5.12}$$

Finally, let us take care of the terms related to $T_{3j}$. Observe that

$$\int_0^t\int_{x_j}^{x_{j+1}}\sum_{j=0}^{J-1}\frac{\left|x_{j+1}(\tau)-x\right|^2}{(\Delta x_{j+\frac{1}{2}}(\tau))^4}\left|u_{j+\frac{3}{2}}(\tau)-u_{j-\frac{1}{2}}(\tau)\right|^2\left|\sigma_j\right|^2$$

$$\leq\int_0^t\sum_{j=0}^{J-1}\frac{1}{\Delta x_{j+\frac{1}{2}}(\tau)}\left|u_{j+\frac{3}{2}}(\tau)-u_{j-\frac{1}{2}}(\tau)\right|^2\left|\sigma_j\right|^2$$

$$\leq\sup_{\tau\in[0,t]}\sum_{j=0}^{J-1}\frac{1}{\Delta x_{j+\frac{1}{2}}}\left|u_{j+\frac{3}{2}}-u_{j-\frac{1}{2}}\right|^2\int_0^t(\max_j\left|\sigma_j\right|)^2\leq C(t).$$

We get that

$$\int_0^t \int_{x_j}^{x_{j+1}} \sum_{j=0}^{J-1} |T_{3j}(\tau)|^2 \, d\tau \leq C(t) \,. \tag{5.13}$$

Combining the estimates (5.11), (5.12) and (5.13) we obtain that

$$\int_0^t \int_0^1 \min\{1,\tau\} \left|\partial_t \hat{\sigma}_J(\tau,x)\right|^2 dx \leq C(t) \,.$$

This conclude the proof of the "uniform estimates" part of Proposition 3.3.

### 5.1 Limiting equations when $J \to \infty$

Consider discrete initial data verifying the hypothesis (2.6) along with the globally defined solution of the system of ODEs (2.1)-(2.4). Furthermore, consider the functions $(\hat{c}_J, \hat{\rho}_J, \hat{u}_J, \hat{\sigma}_J)$ given by (2.8), (2.9), (2.10) and (2.11). These function verify uniformly in $J$ the estimates announced in (2.14) and thus up to a subsequence we have that

$$\begin{cases} \hat{c}_J \rightharpoonup c \text{ weakly-}* \text{ in } L^\infty\left((0,T)\times\mathbb{T}^1\right), \\ \hat{\rho}_J \rightharpoonup \rho \text{ weakly-}* \text{ in } L^\infty\left((0,T)\times\mathbb{T}^1\right), \\ \hat{u}_J \rightharpoonup u \text{ weakly in } H^1\left((0,T)\times\mathbb{T}^1\right) \text{ such that} \\ \partial_x\hat{u}_J \rightharpoonup \partial_x u \text{ weakly-}* \text{ in } L^2\left((0,T);L^\infty\left(\mathbb{T}^1\right)\right), \\ \hat{\sigma}_J \rightharpoonup \sigma \text{ weakly in } H^1\left((\tau,T)\times\mathbb{T}^1\right) \text{ for all } \tau>0, \text{ such that} \\ \partial_x\hat{\sigma}_J \rightharpoonup \partial_x\sigma \text{ weakly in } L^2\left((0,T)\times\mathbb{T}^1\right). \end{cases} \tag{5.14}$$

In order to conclude the proof of Proposition 3.3 we have to obtain equations for $\rho$ and $u$. First of all we will prove Proposition 3.2. Let $\psi \in \mathcal{D}(\mathbb{R})$ and observe that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \hat{\rho}_J(t,x)\left(\partial_t\psi + \hat{u}_J\partial_x\psi\right)(\tau,x)\,dxd\tau \\
&= \int_0^t \sum_{j\in\mathbb{Z}} \int_{x_{j-\frac12}(\tau)}^{x_{j+\frac12}(\tau)} \hat{\rho}_J(\tau,x)\left(\partial_t\psi + \hat{u}_J\partial_x\psi\right)(\tau,x)\,dx\,d\tau = \int_0^t \sum_{j\in\mathbb{Z}} \int_{x_{j-\frac12}(\tau)}^{x_{j+\frac12}(\tau)} \rho_j(\tau)\left(\partial_t\psi + \hat{u}_J\partial_x\psi\right)(\tau,x)\,dx\,d\tau \\
&= \int_0^t \sum_{j\in\mathbb{Z}} \rho_j(\tau)\left(\int_{x_{j-\frac12}(\tau)}^{x_{j+\frac12}(\tau)} \partial_t\psi(\tau,x)\,dx + u_{j+\frac12}\psi\left(\tau,x_{j+\frac12}(\tau)\right) - u_{j-\frac12}\psi\left(\tau,x_{j-\frac12}(\tau)\right) - \int_{x_{j-\frac12}(\tau)}^{x_{j+\frac12}(\tau)} \partial_x\hat{u}_J\psi(\tau,x)\,dx\right)d\tau \\
&= \int_0^t \sum_{j\in\mathbb{Z}} \left(\int_{x_{j-\frac12}(\tau)}^{x_{j+\frac12}(\tau)} \partial_t\left(\psi(\tau,x)\rho_j(t)\right)dx + u_{j+\frac12}\psi\left(\tau,x_{j+\frac12}(\tau)\right) - u_{j-\frac12}\psi\left(\tau,x_{j-\frac12}(\tau)\right)\right)d\tau \\
&= \int_0^t \sum_{j\in\mathbb{Z}} \frac{d}{dt}\int_{x_{j-\frac12}(\tau)}^{x_{j+\frac12}(\tau)} \psi(\tau,x)\rho_j(\tau)\,dx\,d\tau = \int_0^t \frac{d}{dt}\sum_{j\in\mathbb{Z}} \int_{x_{j-\frac12}(t)}^{x_{j+\frac12}(t)} \psi(\tau,x)\hat{\rho}_J(\tau,x)\,dx\,d\tau = \int_0^t \frac{d}{dt}\int_{\mathbb{R}} \psi(\tau,x)\hat{\rho}_J(\tau,x)\,dx\,d\tau,
\end{aligned}$$

and, thus,

$$\int_0^t \int_{\mathbb{R}} \hat{\rho}_J(\tau,x)\left(\partial_t\psi + \hat{u}_J\partial_x\psi\right)(\tau,x)\,dx\,d\tau = \int_{\mathbb{R}} \psi(t,x)\hat{\rho}_J(t,x)\,dx - \int_{\mathbb{R}} \psi(0,x)\hat{\rho}_J(0,x)\,dx.$$

which is exactly the second equation of (3.5). The fact that $\hat{c}_J$ verifies the transport equation is proved in the same way. This concludes the proof of Proposition 3.2.

Obviously, using the information from (5.14) we infer that $\rho$ and $u$ verify the first equation from (3.8).

Let us now describe how to obtain an equation for $u$. Consider the auxiliary variables

$$\hat{m}_J(t,x) = \rho_{j+\frac{1}{2}}(t)\, u_{j+\frac{1}{2}}(t) \text{ if } x \in [x_j, x_{j+1}]$$
$$\hat{v}_J(t,x) = \frac{x_{j+1}-x}{\Delta x_{j+\frac{1}{2}}} u_j + \frac{x-x_j}{\Delta x_{j+\frac{1}{2}}} u_{j+1} \text{ if } x \in [x_j, x_{j+1}],$$

where we define $u_j$ by

$$u_j(t) = \frac{u_{j-\frac{1}{2}}(t) + u_{j+\frac{1}{2}}(t)}{2} \ \forall j \in \mathbb{Z}$$

(recall the definition of $\rho_{j+\frac{1}{2}}$ in (2.3)). It is rather straightforward to check that

$$\partial_t \hat{m}_J + \partial_x(\hat{m}_J \hat{v}_J) + \partial_x \hat{\sigma}_J = 0 \tag{5.15}$$

and that

$$\hat{m}_J \in L^\infty\left((0,T);L^2\left(\mathbb{T}^1\right)\right), \ \hat{v}_J \in L^2\left((0,T);H^1\left(\mathbb{T}^1\right)\right).$$

Therefore we can extract subsequences

$$\hat{m}_J \rightharpoonup_* m \text{ weakly-}\star \text{ in } L^\infty((0,T);L^2(\mathbb{T}^1)) \text{ and}$$
$$\hat{v}_J \rightharpoonup v \text{ weakly in } L^2\left((0,T);H^1\left(\mathbb{T}^1\right)\right).$$

Using (5.15) and the uniform bounds for $(\hat{v}_J, \hat{m}_j, \hat{\sigma}_J)_J$ we obtain

$$\partial_t \hat{m}_J \in L^2\left((0,T);H^{-1}\left(\mathbb{T}^1\right)\right)$$

uniformly, so that (see [27], Lemma 5.1., page 12)

$$\partial_t m + \partial_x(mv) + \partial_x \sigma = 0.$$

It remains to identify $u$ with $v$ and $m$ with $\rho u$. Let us observe that

$$\hat{u}_J(t, x_j(t)) = \frac{u_{j-\frac{1}{2}}(t) + u_{j-\frac{1}{2}}(t)}{2} = \hat{v}_J(t, x_j(t)),$$

along with

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \hat{\rho}_J \hat{u}_J = \rho_j \Delta x_j \frac{u_{j-\frac{1}{2}} + u_{j+\frac{1}{2}}}{2}. \tag{5.16}$$

Since the computations are cumbersome yet straightforward we skip the details.

Next using that $(\partial_x \hat{\sigma}_J)_J$ is uniformly bounded in $L^2\left((0,T)\times\mathbb{T}^1\right)$ along with the fact that $\left(\partial_t \frac{1}{\mu(\hat{c}_J)}\right)_J$ is bounded in $L^2\left((0,T);H^{-1}\left(\mathbb{T}^1\right)+L^2\left(\mathbb{T}^1\right)\right)$ one can conclude that

$$\partial_x u - \left\langle \hat{c}_J \frac{p_+(\hat{\rho}_J)}{\mu_+} + (1-\hat{c}_J)\frac{p_-(\hat{\rho}_J)}{\mu_-} \right\rangle = \lim_{J\to\infty} \frac{\hat{\sigma}_J}{\mu(\hat{c}_J)} = \left\langle \frac{1}{\mu(\hat{c}_J)} \right\rangle \sigma.$$

This concludes the proof of Proposition 3.3.

## 6 Proof of Proposition 3.5

We naturally divide the proof of Proposition 3.5 into two parts. First, we prove that the limiting measures verify the equations (3.12) while in a second time we will prove that knowing that if at initial time $\Theta$ has the special structure (2.17) then we can propagate this structure, i.e. (2.18) holds for all time $t > 0$. This property will go along with the fact that the quantities $(\alpha_+, \alpha_-, \rho_+, \rho_-)$ satisfy (1.1) with $u$. Let us consider $b(x, \xi, \eta) \in C_c^1(\mathbb{T}_x^1 \times \mathbb{R}_\xi \times [0,1])$. For all $N \geq 0$ we write

$$\hat{\rho}_J^N(t) \stackrel{def.}{:=} F_N * \hat{\rho}_J(t)$$

where $F_N$ is the Fejér kernel

$$F_N(x) = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)}\right)^2.$$

For $t \in [0,T]$ and $p \in [1, \infty)$ we have that

$$\begin{cases} \lim_{N\to\infty} \left\|\hat{\rho}_J^N(t) - \hat{\rho}_J(t)\right\|_{L^p(\mathbb{T}^1)} = 0, \\ \lim_{N\to\infty} \left\|\hat{\rho}_J^N - \hat{\rho}_J\right\|_{L^p([0,T]\times\mathbb{T}^1)} = 0. \end{cases} \tag{6.1}$$

Let us apply $F_N$ to the second transport equation in (3.5) and write that

$$\partial_t \hat{\rho}_J^N + \partial_x\left(\hat{\rho}_J^N \hat{u}_J\right) = r_N(\hat{\rho}_J, \hat{u}_J), \tag{6.2}$$

where $r_N(\hat{\rho}_J, \hat{u}_J) := \partial_x((F_N * \hat{\rho}_J)\hat{u}_J) - \partial_x(F_N * (\hat{\rho}_J \hat{u}_J))$ satisfies (see [14, Lemma II.1]):

$$\lim_{N\to\infty} \|r_N(\hat{\rho}_J, \hat{u}_J)\|_{L^2([0,T]\times\mathbb{T}^1)} = 0. \tag{6.3}$$

Similarly, with the first transport equation of (3.5), we obtain:

$$\partial_t \hat{c}_J^N + \hat{u}_J \partial_x \hat{c}_J^N = r_N(\hat{c}_J, \hat{u}_J) - \hat{c}_J^N \partial_x \hat{u}_J + (c\partial_x u)_J^N, \tag{6.4}$$

with $r_N$ satisfying also (6.3). We multiply (6.2) with $\partial_2 b(x, \hat{\rho}_J^N, \hat{c}_J^N)$ and (6.4) with $\partial_3 b(x, \hat{\rho}_J^N, \hat{c}_J^N)$ and we write that

$$\begin{aligned} \partial_t b(x, \hat{\rho}_J^N, \hat{c}_J^N) &+ \partial_x\left(\hat{u}_J\, b(x, \hat{\rho}_J^N, c_J^N)\right) - \hat{u}_J \partial_1 b\left(x, \hat{\rho}_J^N, \hat{c}_J^N\right) \\ &+ \left(\hat{\rho}_J^N \partial_2 b\left(x, \hat{\rho}_J^N, \hat{c}_J^N\right) - b\left(x, \hat{\rho}_J^N, \hat{c}_J^N\right)\right)\partial_x \hat{u}_J \\ &= r_N(\hat{\rho}_J, \hat{u}_J)\,\partial_2 b(x, \hat{\rho}_J^N, \hat{c}_J^N) + r_N(\hat{c}_J, \hat{u}_J)\partial_3 b(x, \hat{\rho}_J^N, \hat{c}_J^N) \\ &\quad - \left[\hat{c}_J^N \partial_x \hat{u}_J - (c\partial_x u)_J^N\right]\partial_3 b(x, \hat{\rho}_J^N, \hat{c}_J^N) \end{aligned}$$

**Remark 6.1.** *Let us mention that by $\partial_t b(x, \hat{\rho}_J^N, \hat{c}_J), \partial_x b(x, \hat{\rho}_J^N, \hat{c}_J)$ we understand the derivative with respect to time, space of the function*

$$(t, x) \to b(x, \hat{\rho}_J^N(t,x), \hat{c}_J(t,x))$$

*while when using numbers $\partial_k b(t, x, \hat{\rho}_J^N)$, $k \in \{1,2,3\}$ represents the derivative of $b$ with respect to its $k$th variable computed in $(x, \hat{\rho}_J^N(t,x), \hat{c}_J(t,x))$.*

In order to take advantage of the compactness properties of the stress $\hat{\sigma}_J = \mu(\hat{c}_J)\partial_x \hat{u}_J - p(\hat{\rho}_J, \hat{c}_J)$, see (3.6), we rewrite the above equation as

$$
\begin{aligned}
&\partial_t b(x, \hat{\rho}_J^N, \hat{c}_J^N) + \partial_x \left(\hat{u}_J b(x, \hat{\rho}_J^N, \hat{c}_J^N)\right) - \hat{u}_J \partial_1 b(x, \hat{\rho}_J^N, \hat{c}_J^N) \\
&\qquad + \frac{1}{\mu(\hat{c}_J^N)} \left(\hat{\rho}_J^N \partial_2 b(x, \hat{\rho}_J^N, \hat{c}_J^N) - b(x, \hat{\rho}_J^N, \hat{c}_J^N)\right) \hat{\sigma}_J \\
&\qquad + \frac{1}{\mu(\hat{c}_J^N)} \left(\hat{\rho}_J^N \partial_2 b(x, \hat{\rho}_J^N, \hat{c}_J^N) - b(x, \hat{\rho}_J^N, \hat{c}_J^N)\right) p\left(\hat{\rho}_J^N, \hat{c}_J^N\right) \\
&\qquad\qquad = r_N\left(\hat{\rho}_J, \hat{u}_J\right) \partial_2 b(x, \hat{\rho}_J^N, \hat{c}_J^N) + r_N(\hat{c}_J, \hat{u}_J)\partial_3 b(x, \hat{\rho}_J^N, \hat{c}_J^N) \\
&\qquad\qquad\quad - \left[\hat{c}_J^N \partial_x \hat{u}_J - (c\partial_x u)_J^N\right] \partial_3 b(x, \hat{\rho}_J^N, \hat{c}_J^N).
\end{aligned}
\tag{6.5}
$$

Owing to (6.1), we get that, up to the extraction of a subsequence,

$$
\left\{
\begin{array}{c}
(\hat{\rho}_J^N, \hat{c}_J^N) \to (\hat{\rho}_J, \hat{c}_J) \text{ a.e. } [0,T] \times \mathbb{T}^1, \\
(\hat{\rho}_J^N(T), \hat{c}_J^N(T)) \to (\hat{\rho}_J(T), \hat{c}_J(T)) \text{ a.e. } \mathbb{T}^1, \\
(\hat{\rho}_J^N(0), \hat{c}_J^N(0)) \to (\hat{\rho}_J(0), \hat{c}_J(0)) \text{ a.e. } \mathbb{T}^1.
\end{array}
\right.
\tag{6.6}
$$

Hence, by applying a dominated convergence argument, we obtain that the left-hand side of (6.5) converges in $\mathcal{D}'((0,T) \times \mathbb{T}^1)$ to

$$
\begin{aligned}
&\partial_t b(x, \hat{\rho}_J, \hat{c}_J) + \partial_x \left(\hat{u}_J b(x, \hat{\rho}_J, \hat{c}_J)\right) - \hat{u}_J \partial_1 b(x, \hat{\rho}_J, \hat{c}_J) \\
&\qquad + \frac{1}{\mu(\hat{c}_J)} \left(\hat{\rho}_J \partial_2 b(x, \hat{\rho}_J, \hat{c}_J) - b(x, \hat{\rho}_J, \hat{c}_J)\right) \hat{\sigma}_J \\
&\qquad + \frac{1}{\mu(\hat{c}_J)} \left(\hat{\rho}_J \partial_2 b(x, \hat{\rho}_J, \hat{c}_J) - b(x, \hat{\rho}_J, \hat{c}_J)\right) p\left(\hat{\rho}_J, \hat{c}_J\right)
\end{aligned}
$$

For the right-hand side, we apply (6.3) together with the regularity $\partial_x \hat{u}_J \in L^\infty_{\mathrm{loc}}((0,T) \times \mathbb{T}^1)$ to yield that:

$$
\lim_{N \to \infty} \|\hat{c}_J^N \partial_x \hat{u}_J - (c\partial_x u)_J^N\|_{L^2_{\mathrm{loc}}((0,T)\times\mathbb{T})} = 0.
$$

This entails that:

$$
\begin{aligned}
&\partial_t b(x, \hat{\rho}_J, \hat{c}_J) + \partial_x \left(\hat{u}_J b(x, \hat{\rho}_J, \hat{c}_J)\right) - \hat{u}_J \partial_1 b(x, \hat{\rho}_J, \hat{c}_J) \\
&\qquad + \frac{1}{\mu(\hat{c}_J)} \left(\hat{\rho}_J \partial_2 b(x, \hat{\rho}_J, \hat{c}_J) - b(x, \hat{\rho}_J, \hat{c}_J)\right) \hat{\sigma}_J \\
&\qquad + \frac{1}{\mu(\hat{c}_J)} \left(\hat{\rho}_J \partial_2 b(x, \hat{\rho}_J, \hat{c}_J) - b(x, \hat{\rho}_J, \hat{c}_J)\right) p\left(\hat{\rho}_J, \hat{c}_J\right) = \sum_{i=1}^{5} I_i = 0.
\end{aligned}
$$

Let us now integrate in space this equality. We first observe that the integration of the second quantity vanishes. Then using the definition of the sequences of measures $\Theta_J$ i.e.(3.9) namely

$$
< \Theta_J(t), b > = \int_{\mathbb{T}^1} b(x, \hat{\rho}_J(t,x), \hat{c}_J(t,x)) = \int_{\mathbb{T}^1_x \times \mathbb{R}_\xi \times [0,1]} b \, d\Theta_J(t)
$$

we get that:

$$
\partial_t \Theta_J + \partial_x \left(\hat{u}_J \Theta_J\right) - \partial_\xi \left( \left( \frac{\xi \hat{\sigma}_J}{\mu(\eta)} + \frac{\xi p\left(\eta, \xi\right)}{\mu(\eta)} \right) \Theta_J \right) - \left( \frac{\hat{\sigma}_J}{\mu\left(\eta\right)} + \frac{p\left(\eta, \xi\right)}{\mu(\eta)} \right) \Theta_J = 0.
\tag{6.7}
$$

Note that

$$
\int_{\mathbb{T}^1} b(x, \hat{\rho}_J(t,x), \hat{c}_J(t,x)) \, dx = < \Theta_J(t), b >
$$

and remark that the second term in (6.7) comes from the integration in space of $I_3$. Indeed

$$-\int_{\mathbb{T}^1} \hat{u}_J \partial_1 b(x, \hat{\rho}_J(t,x), \hat{c}_J(t,x))dx = - < \hat{\Theta}_j(t), \hat{u}_J \partial_1 b >=< \partial_x(\Theta_J(t)\hat{u}_J, b > .$$

The integration in space of $I_4$ and $I_5$ playing with the $\zeta$ and $\eta$ variables provides the last two terms in (6.7). With the first statement, we obtain that, whatever $b \in C_c^1(\mathbb{T}^1 \times \mathbb{R}_\xi \times [0,1])$, the quantity $\partial_t b(x, \hat{\rho}_J, \hat{c}_J)$ is bounded in $L^\infty(0,T; H^{-1}(\mathbb{T}^1))$. By a standard Ascoli-Arzela argument, applying that the $\Theta_J$ have uniformly finite mass, we obtain that $\langle \Theta_J, b\rangle$ is precompact in $C([0,T])$. We can then use that the $\Theta_J$ have compact support (uniformly in $\mathbb{N}$) to extract a limit for a denumerable set of $b$ and combine with a density argument to obtain that the $\Theta_J$ converge (up to the extraction of a subsequence) in $C_w([0,\infty); \mathcal{M}_+(\mathbb{T}_x^1 \times \mathbb{R}_\xi \times [0,1])$.

We are now in position to pass to the limit $J \to \infty$ in this last equation. For this, we note that $\partial_t \hat{u}_J$ is bounded in $L^2((0,T) \times \mathbb{T}^1)$ so that by a classical Ascoli-Arzela argument we have that (up to the extraction of a subsequence) $\hat{u}_J$ converges to $u$ in $L^2((0,T); C(\mathbb{T}^1))$. Consequently:

$$\Theta_J \, \hat{u}_J \to \Theta u \text{ in } \mathcal{D}'((0,T) \times \mathbb{T}^1 \times \mathbb{R}_\xi \times [0,1]).$$

Concerning the remaining terms, the only difficulty lies in passing to the limit in the produce $\hat{\sigma}_J \Theta_J$. For this, we note that $\partial_t \hat{\rho}_J$ is bounded in $L^\infty((0,T); H^{-1}(\mathbb{T}^1))$ while $\hat{\sigma}_J$ is bounded in $L^2((0,T); H^1(\mathbb{T}^1))$. By a classical compensated compactness argument (see [7, Lemma 10]), we obtain that $\Theta$ satisfies (3.12). This concludes the first part of Proposition 3.5.

## 7 Proof of Proposition 3.6 and end of proof of Theorem 1

The objective of this section is to prove Theorem 1. We follow the approach from [6] and construct explicit solutions to the limit system (1.1). Afterwards, using the classical uniqueness result for transport equations with measure initial data, we may identify the limit measure with the particular one we have constructed. At first, we note that the limiting velocity field and stress field have the regularity:

$$u \in C([0,T]; L^2(\mathbb{T}^1)) \cap L^2(0,T; W^{1,\infty}(\mathbb{T}^1)), \partial_x \sigma \in L^{\frac{4}{3}-}\left([0,T]\,; L^\infty\left(\mathbb{T}^1\right)\right).$$

Consequently, classical arguments for semilinear hyperbolic problems yield that, given

$$(\alpha_-^0, \alpha_+^0, \rho_-^0, \rho_+^0) \in L^\infty(\mathbb{T}^1; \mathbb{R}^4)$$

such that

$$0 \leq \min(\alpha_-^0, \alpha_+^0, \rho_-^0, \rho_+^0) \text{ and } \alpha_-^0 + \alpha_+^0 = 1,$$

there exists a unique solution $(\tilde{\alpha}_-, \tilde{\alpha}_+, \tilde{\rho}_-, \tilde{\rho}_+) \in L^\infty((0,T) \times \mathbb{T}^1) \cap C([0,T]; L^1(\mathbb{T}^1))$ to

$$\begin{cases} \partial_t \tilde{\rho}_\pm + u \partial_x \tilde{\rho}_\pm + \tilde{\rho}\left(\sigma + p_+\left(\tilde{\rho}_\pm\right)\right) = 0, \\ \partial_t \tilde{\alpha}_\pm + u \partial_x \tilde{\alpha}_\pm = \dfrac{\tilde{\alpha}_+}{\mu_+}\left(F_+ - \sigma\right), \end{cases}$$

where

$$F_\pm = -\mu_\pm \partial_x u + p_\pm(\tilde{\rho}_\pm),$$

with initial condition $(\alpha_-^0, \alpha_+^0, \rho_-^0, \rho_+^0)$. We note that the solution, which can be obtained via a standard fixed point argument, is a priori defined only locally. However, noticing that

$$0 \leq \tilde{\rho}_\pm(t,x) \leq \rho_{\pm,0}\,(x) \exp\left(\int_0^t \|\sigma\|_{L^\infty}\right) \tag{7.1}$$

and owing to the uniform bounds for $\|\sigma\|_{L^1(0,T;L^\infty(\mathbb{T}^1))}$, we may extend the local solutions $\rho_\pm$ to global ones. A similar argument shows that $\tilde{\alpha}_\pm$ can be defined globally.

At this point, we define a measure on $\mathbb{T}^1 \times \mathbb{R}_\xi \times \mathbb{R}_\eta$ by the following formula:

$$\left\langle \bar{\Theta}\left(t\right), b\right\rangle \overset{def.}{:=} \int_{\mathbb{T}^1} \tilde{\alpha}_-\left(t,x\right) b(x,\tilde{\rho}_-(t,x),0) + \tilde{\alpha}_+\left(t,x\right) b(x,\tilde{\rho}_+(t,x),1) dx.$$

We observe that, for all $t \in [0,T]$, the measure $\bar{\Theta}(t)$ has compact support in $\mathbb{T}^1_x \times \mathbb{R}_\xi \times [0,1]$ and, given the system satisfied by $(\tilde{\alpha}_-, \tilde{\alpha}_+, \tilde{\rho}_-, \tilde{\rho}_+)$, one can check that the measure $\bar{\Theta}$ verifies the following equation

$$\partial_t \bar{\Theta} + \partial_x\left(u\,\bar{\Theta}\right) - \partial_\xi\left(\left(\frac{\xi\sigma}{\mu(\eta)} + \frac{\xi p(\eta,\xi)}{\mu(\eta)}\right)\bar{\Theta}\right) - \left(\frac{\sigma}{\mu\left(\eta\right)} + \frac{p\left(\eta,\xi\right)}{\mu(\eta)}\right)\bar{\Theta} = 0. \tag{7.2}$$

Moreover, we have that

$$\lim_{t\to 0}\langle\bar{\Theta}\left(t\right), b\rangle = \int_{\mathbb{T}^1} (\alpha_{-,0}\left(x\right) b(x,\rho_{-,0}(x),0) + \alpha_{+,0}\left(x\right) b\left(x,\rho_+(x),1\right)) dx = \langle\Theta(0), b\rangle.$$

Let use fix $C(T) \geq 1$ such that $\Theta$ and $\bar{\Theta}$ both have their support in $\mathbb{T}^1 \times [0, C(T)] \times [0,1]$. Considering $\chi$ a smooth function

$$\chi : \mathbb{R} \to [0,1] \text{ such that } \chi = 1 \text{ on } [0, C\left(T\right)] \tag{7.3}$$

we can write that $\Theta$ and $\bar{\Theta}$ are both solutions to

$$\begin{cases} \partial_t \bar{\Phi} + \partial_x\left(u\,\bar{\Phi}\right) - \partial_\xi\left(\left(\frac{\xi\sigma}{\mu(\eta)} + \frac{\xi p(\eta,\xi)}{\mu(\eta)}\right)\chi\left(\xi\right)\chi\left(\eta\right)\bar{\Phi}\right) - \left(\frac{\sigma}{\mu\left(\eta\right)} + \frac{p\left(\eta,\xi\right)}{\mu(\eta)}\right)\chi\left(\xi\right)\chi\left(\eta\right)\bar{\Phi} = 0, \\ \left\langle \bar{\Phi}_{|t=0}, b\right\rangle = \int_{\mathbb{T}^1} (\alpha_{-,0}\left(x\right) b(x,\rho_{-,0}(x),0) + \alpha_{+,0}\left(x\right) b\left(x,\rho_+(x),1\right)) dx \end{cases} \tag{7.4}$$

Let us observe that $\bar{\Phi}$ is transported by the field $V = (V_1, V_2, V_3)$ :

$$\begin{cases} V_1\left(t,x,\xi,\eta\right) = u\left(t,x\right), \\ V_2\left(t,x,\xi,\eta\right) = -\left(\frac{\xi\sigma\left(t,x\right)}{\mu(\eta)} + \frac{\xi p(\eta,\xi)}{\mu(\eta)}\right)\chi\left(\xi\right)\chi\left(\eta\right), \\ V_3\left(t,x,\xi,\eta\right) = 0. \end{cases}$$

The estimates that we obtained for $u, \sigma$ allow us to conclude that $V \in L^1([0,T]; L^\infty(\mathbb{T}^1))$. Uniqueness of solutions for transport equations with vector fields having this kind of regularity is then classical, so that we obtain that $\Theta(t) = \bar{\Theta}(t)$. This concludes the proof of Theorem 1.

# 8 Numerical illustrations

In this last section, we design two numerical schemes: one to approximate the continuous version of the mesoscopic system, that is to say

$$\begin{cases} \partial_t c + u\partial_x c = 0 \text{ with } c\left(1-c\right) = 0 \\ \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x((c\mu_+ + (1-c)\mu_-)\partial_x u) + \partial_x(cp_+(\rho) + (1-c)p_-(\rho)) = 0 \end{cases} \tag{8.1}$$

with

$$c|_{t=0} = c_0 \in \{0,1\}, \qquad \rho|_{t=0} = \rho_0, \qquad u|_{t=0} = u_0$$

and one to approximate the macroscopic system (1.1)–(1.4) with Cauchy datum. Later, we will denote

$$\mu(c,\rho) = c\mu_+ + (1-c)\mu_-, \qquad p(c,\rho) = cp_+(\rho) + (1-c)p_-(\rho).$$

In all the following, for the consiseness of the notation, $\alpha$ will stand for $\alpha_+$ (thus $\alpha_-$ will be understood as $1-\alpha$).

## 8.1 Mesoscopic discretization

The numerical scheme we design here consists in a “brute force" discretization of System (8.1) where $c_0(1-c_0) = 0$. We will consider a time discretization of the semi-discrete scheme managed in Theorem (1). Let us recall here the ideas. As the fluids have to remain pure (not mixed) in every cell, because, for modeling reasons, we want to use only the pure pressure laws (the mixture pressure law being unknown at this stage), the length of each pure zone has to be larger than a cell (and, more precisely, has to be large as an integer number of cells). Here, in the numerical tests, we choose to consider a numerical initial condition such that the fluid changes from one cell to the other (but of course this is not a restriction). The problem to achieve the aim here comes from the so-called numerical diffusion: the discretization of the $\partial_t c + u\partial_x c = 0$ with a stable scheme usually brings a certain amount of diffusion, the effect of which being not to preserve the important feature $c(t,\cdot)(1-c(t,\cdot)) = 0 \quad a.e.$. In order to pass over this phenomenon, we consider a *Lagrangian*, or *pseudo-Lagrangian*[2] scheme in which the cells follow the fluid in its transport, namely in which the edges of every cell moves at the fluid velocity. In this Lagrangian frame, the equation for the mass fraction is $D_t c = 0$ (recall that $D_t = \partial_t + u\partial_x$).

The spirit of the proposed scheme is the one of staggered schemes: it can be seen as a modification of the schemes in [25] and [18], this modification being that the present scheme is more explicit (precisely, the nonlinearity are time-discretized in a backward Euler way) and that it is a pseudo-Lagrange scheme. Staggered schemes are schemes in which different unknowns are associated to different points or cells in the mesh (for example, the density and the velocity, here). At last, this scheme is a time discretization of the semi-discrete scheme (2.1) that was proposed to determine the limit macroscopic system.

The discretization is the following. Let $J \in \mathbb{N}\setminus\{0\}$ be the number of cells in $[0,1)$. Let $(x^0_{j-1/2})_{j=1}^J$ be the collection of cell interface positions at time 0. One assumes $0 \le x^0_{j-1/2} < x^0_{j+1/2} < 1$ for any $j = 1,\dots,J-1$.

In order to take into account the fact that the problem under consideration is posed on $\mathbb{T}$ in a simple manner, i.e. without taking care of the cells and quantities on the boundary, we extend all the data over $\mathbb{R}$ and $\mathbb{Z}$ by periodicity.

The cells themselves are denoted by $\omega_j^0 = [x^0_{j-1/2}, x^0_{j+1/2})$ for $j \in \mathbb{Z}$. We denote by $\Delta x_j^n = x^n_{j+1/2} - x^n_{j-1/2}$ their length. The maximum length of these cells is intended to be small (and to tend to 0 as $J$ tends to $\infty$ to reach convergence). We also will need the distance between two centers of consecutive cells: $\Delta x^n_{j+1/2} = (\Delta x^n_j + \Delta x^n_{j+1})/2$.

Each time step of the scheme, given a discrete datum $\left(x^n_{j-1/2}, \rho^n_j, c^n_j, u^n_{j-1/2}\right)_{j\in\mathbb{Z}}$, consists in defining

[2] It can be called pseudo-Lagrangian because, although the solution is actually expressed in the classical Euler variable, the scheme strongly uses the Langrange formulation of the system.

appropriately $\Delta t^n > 0$ and constructing $\left(x^{n+1}_{j-1/2}, \rho^{n+1}_j, c^{n+1}_j, u^{n+1}_{j-1/2}\right)_{j\in\mathbb{Z}}$ by the formula

$$\begin{cases} \rho^n_{j+1/2} = \dfrac{\Delta x^n_j \rho^n_j + \Delta x^n_{j+1}\rho^n_{j+1}}{\Delta x^n_j + \Delta x^n_{j+1}}, \quad j\in\mathbb{Z}, \\ c^{n+1}_j = c^n_j, \quad j \in \mathbb{Z}, \\ \rho^n_{j+1/2}\Delta x^n_{j+1/2} u^{n+1}_{j+1/2} = \rho^n_{j+1/2}\Delta x^n_{j+1/2}u^n_{j+1/2} - \Delta t^n \left(p(c^n_{j+1},\rho^n_{j+1}) - p(c^n_j,\rho^n_j)\right) \\ \qquad\qquad + \Delta t^n \left(\mu(c^n_{j+1},\rho^n_{j+1})\dfrac{u^{n+1}_{j+3/2} - u^{n+1}_{j+1/2}}{\Delta x^n_{j+1}} - \mu(c^n_j,\rho^n_j)\dfrac{u^{n+1}_{j+1/2} - u^{n+1}_{j-1/2}}{\Delta x^n_j}\right), \quad j\in\mathbb{Z}, \\ x^{n+1}_{j+1/2} = x^n_{j+1/2} + \Delta t^n u^{n+1}_{j+1/2}, \quad j\in\mathbb{Z}, \\ \Delta x^{n+1}_j = x^{n+1}_{j+1/2} - x^{n+1}_{j-1/2}, \quad j \in\mathbb{Z}, \\ \Delta x^{n+1}_{j+1/2} = \dfrac{\Delta x^{n+1}_j + \Delta x^{n+1}_{j+1}}{2}, \quad j\in\mathbb{Z}, \\ \rho^{n+1}_j = \rho^n_j \dfrac{\Delta x^n_j}{\Delta x^{n+1}_j}, \quad j\in\mathbb{Z}. \end{cases} \tag{8.2}$$

In the system above,

- The first equation defines a density associated to the nodes $x^n_{j+1/2}$, density that is used in the third equation,
- The second equation is a (non-diffusive) discretization of $D_t c = 0$,
- The third equation is the discretization of $\partial_t \rho u + \partial_x(\rho u^2 + p) = \partial_x(\mu\partial_x u)$: indeed notice that thanks to the last equation of the system, this third equation rewrites

$$\rho^{n+1}_{j+1/2}\Delta x^{n+1}_{j+1/2} u^{n+1}_{j+1/2} = \rho^n_{j+1/2}\Delta x^n_{j+1/2}u^n_{j+1/2} - \Delta t^n \left(p(c^n_{j+1},\rho^n_{j+1}) - p(c^n_j,\rho^n_j)\right) + \Delta t^n \left(\mu(c^n_{j+1},\rho^n_{j+1})\frac{u^{n+1}_{j+3/2} - u^{n+1}_{j+1/2}}{\Delta x^n_{j+1}} - \mu(c^n_j,\rho^n_j)\frac{u^{n+1}_{j+1/2} - u^{n+1}_{j-1/2}}{\Delta x^n_j}\right), \quad j\in\mathbb{Z},$$

  which is consistent with the partial differential equation,
- The fourth equation is the translation of the mesh,
- Fifth and sixth equations redefine quantities that are used in the scheme,
- The last equation expresses the conservation of mass in a material volume $\partial_t\rho + \partial_x \rho u = 0$).

It is possible to prove that if the time step $\Delta t^n$ is sufficently small, $x^n_{j-1/2} < x^n_{j+1/2}$ for all $j$ implies $x^{n+1}_{j-1/2} < x^{n+1}_{j+1/2}$ for all $j$.

## 8.2 Macroscopic discretization

For the macroscopic homogenized system (1.1), we use the same type of scheme. The only difference is that the volume fraction of fluid + does not satisfy $\alpha_+(1-\alpha_+) = 0$ but

$$D_t\alpha = \frac{\alpha(1-\alpha)}{\alpha\mu_- + (1-\alpha)\mu_+}(p_+(\rho_+) - p_-(\rho_-) - (\mu_+ - \mu_-)\partial_x u).$$

In the following we choose to discretize this equation in a forward Euler way (but a backward Euler scheme has also been tested and validated):

$$\alpha_j^{n+1} = \alpha_j^n + \Delta t \frac{\alpha_j^n(1-\alpha_j^n)}{\alpha_j^n\mu_- + (1-\alpha_j^n)\mu_+}\left(p_+(\rho_{+,j}^n) - p_-(\rho_{-,j}^n) - (\mu_+ - \mu_-)\frac{u_{j+1/2}^{n+1} - u_{j-1/2}^{n+1}}{x_{j+1/2}^{n+1} - x_{j-1/2}^{n+1}}\right). \tag{8.3}$$

All the other variables are approximated in a very standard and natural way:

$$\left\{\begin{array}{l}
\rho_{j+1/2}^n = \dfrac{\Delta x_j^n \rho_j^n + \Delta x_{j+1}^n \rho_{j+1}^n}{\Delta x_j^n + \Delta x_{j+1}^n}, \quad j \in \mathbb{Z}, \\
p_j^n = \dfrac{\alpha_j^n p_+(\rho_{+,j}^n)\mu_- + (1-\alpha_j^n)p_-(\rho_{-,j}^n)\mu_+}{\alpha_j^n\mu_- + (1-\alpha_j^n)\mu_+}, \quad j \in \mathbb{Z}, \\
\mu_j^n = \dfrac{\mu_+\mu_-}{\alpha_j^n\mu_- + (1-\alpha_j^n)\mu_+}, \quad j \in \mathbb{Z}, \\
c_j^{n+1} = c_j^n, \quad j \in \mathbb{Z}, \\
\rho_{j+1/2}^n \Delta x_{j+1/2}^n u_{j+1/2}^{n+1} = \rho_{j+1/2}^n \Delta x_{j+1/2}^n u_{j+1/2}^n - \Delta t^n \left(p_{j+1}^n - p_j^n\right) \\
\qquad\qquad + \Delta t^n \left(\mu_{j+1}^n \dfrac{u_{j+3/2}^{n+1} - u_{j+1/2}^{n+1}}{\Delta x_{j+1}^n} - \mu_j^n \dfrac{u_{j+1/2}^{n+1} - u_{j-1/2}^{n+1}}{\Delta x_j^n}\right), \quad j \in \mathbb{Z}, \\
x_{j+1/2}^{n+1} = x_{j+1/2}^n + \Delta t^n u_{j+1/2}^{n+1}, \quad j \in \mathbb{Z}, \\
\Delta x_j^{n+1} = x_{j+1/2}^{n+1} - x_{j-1/2}^{n+1}, \quad j \in \mathbb{Z}, \\
\Delta x_{j+1/2}^{n+1} = \dfrac{\Delta x_j^{n+1} + \Delta x_{j+1}^{n+1}}{2}, \quad j \in \mathbb{Z}, \\
\rho_j^{n+1} = \rho_j^n \dfrac{\Delta x_j^n}{\Delta x_j^{n+1}}, \quad j \in \mathbb{Z}, \\
\rho_{+,j}^{n+1} = c_j^{n+1}\rho_j^{n+1}/\alpha_j^{n+1}, \quad j \in \mathbb{Z}, \\
\rho_{-,j}^{n+1} = (1-c_j^{n+1})\rho_j^{n+1}/(1-\alpha_j^{n+1}), \quad j \in \mathbb{Z}.
\end{array}\right. \tag{8.4}$$

In the following experiment, the values 0 and 1 are avoided for the volume fractions, so that the last two equations of System (8.4) have a sense. In a more general situation, one should replace these two equations and the volume fraction evolution (8.3) with a discretization of the (equivalent) equation on $\rho_+$

$$\partial_t \rho_+ + \partial_x \rho_+ u = \frac{\rho_+(1-\alpha)}{\alpha\mu_- + (1-\alpha)\mu_+}(\sigma_+ - \sigma_-)$$

and the symmetric equation on $\rho_-$.

### 8.3 Experiments

We propose two test-cases with $p_+(x) = x$ and $p_-(x) = x^2$. They are associated with a Cauchy datum of Riemann type:

$$\left\{\begin{array}{l}
\alpha_0(x) = 1/2, \quad x \in \mathbb{T}_x, \\
\rho_+(x) = \rho_-(x) = \left\{\begin{array}{l} 1/8 \text{ if } x \in [0,1/4) \cup [3/4,1), \\ 2 \text{ if } x \in [1/4,3/4), \end{array}\right. \\
u(x) = 0, \quad x \in \mathbb{T}_x,
\end{array}\right.$$

and we propose to compare the numerical solutions obtained at time $t = 0.1$ with 1000 cells

- with the homogenized scheme of Section 8.2,
- and with the mesoscopic scheme of Section 8.1 by setting

$$(\alpha_j^0, \rho_{+,j}^0, \rho_{-,j}^0) = \left\{\begin{array}{l} (1, \rho_j^0, 0) \text{ if } j \text{ is even}, \\ (0, 0, \rho_j^0) \text{ if } j \text{ is odd}, \end{array}\right.$$

  and with a mesh with constant space step, which indeed corresponds in the weak limit to $\alpha = 1/2$. Note that the pressure is largely oscillating in this initial condition for the mesoscopic system.

In the first test, we take $\mu_+ = \mu_- = 0.1$ while in the second one we choose $\mu_+ = 0.1$ and $\mu_- = 0.02$. Figures 1 to 5 allow to compare the density, velocity, pressure and volume fraction. We observe a very good agreement between the mesoscopic and the macroscopic results. Note that for the mesoscopic computation, we consider that there is only one density and one pressure, thus these quantities oscillate very fast (at the scale of the cell, which is the scale of the mixture). We observe, especially on the zoom of the density proposed by Figure 2, that these oscillations occur between two functions that are very close to $\rho_+$ and $\rho_-$ computed by the macroscopic scheme. With the mesoscopic scheme, the volume fraction of fluid + should oscillate between 0 and 1. In order to evaluate a volume fraction of + in the limit mixture, what we here (Figure 5) call $\alpha_j^n$ is computed by

$$\alpha_j^n = \frac{c_j(x_{j+1/2}^n - x_{j-1/2}^n) + c_{j-1}(x_{j-1/2}^n - x_{j-3/2}^n)/2 + c_{j+1}(x_{j+3/2}^n - x_{j+1/2}^n)/2}{x_{j+1}^n - x_{j-1}^n}$$

(recall that $c_j$ is equal to 0 or 1 and does not depend on the time index).

The organization and the comments for the case with different viscosities, from Figure 6 to Figure 10, are the same.

### 8.3.1 Case with equal viscosities

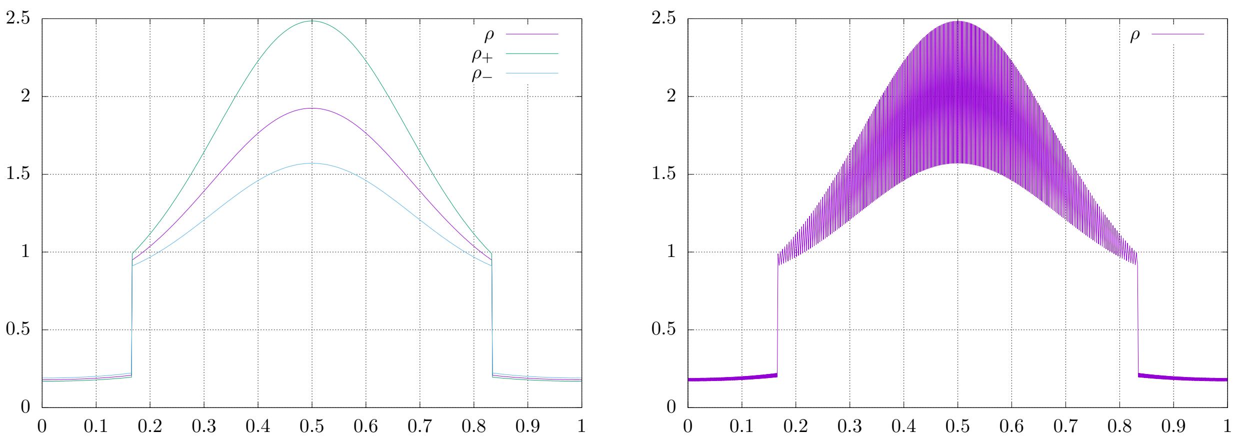



Figure 1: Densities. On the left, the 3 densities of the mixture, on the right, the density of the unmixed fluid.

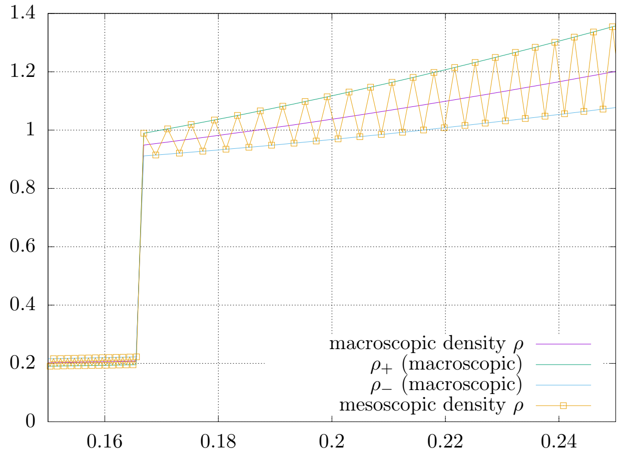
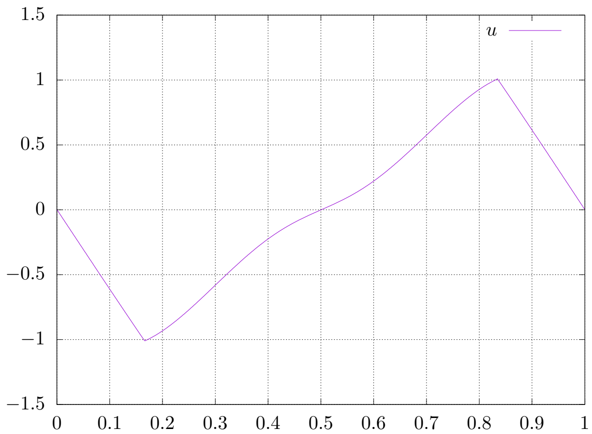



Figure 2: Densities. Zoom of the preceding figures.



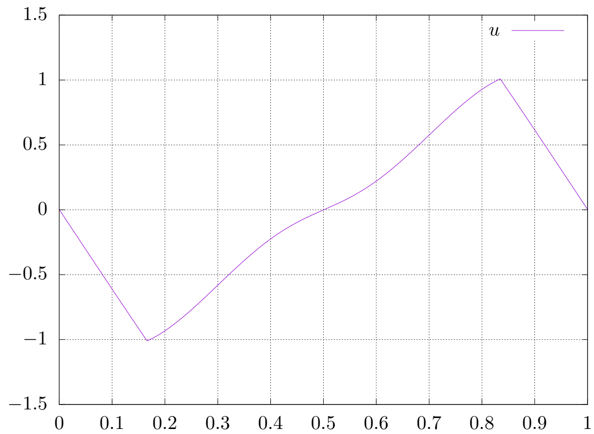




Figure 3: Velocities. On the left, the velocity of the mixture, on the right, the velocity of the unmixed fluid.

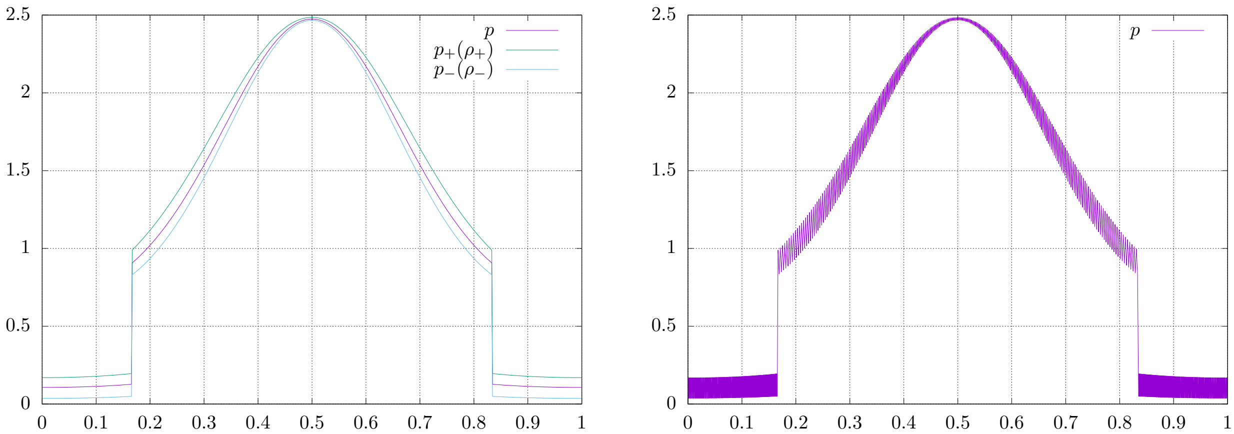



Figure 4: Pressures. On the left, the 3 pressures in the mixture, on the right, the pressure in the unmixed fluid.

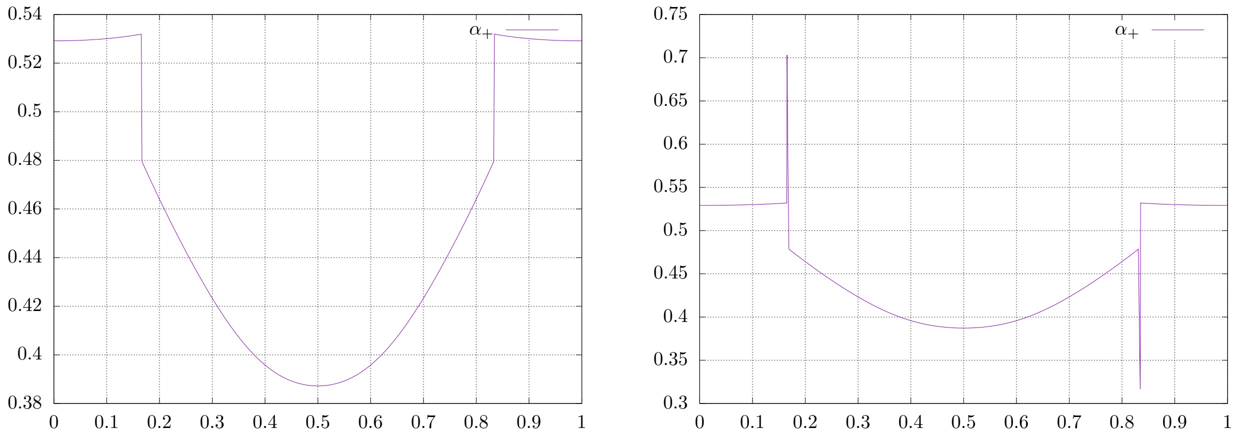



Figure 5: Volume fractions. On the left, the volume fraction $\alpha_+$ in the mixture, on the right, the estimate of the volume fraction in the unmixed fluid.

### 8.3.2 Case with different viscosities

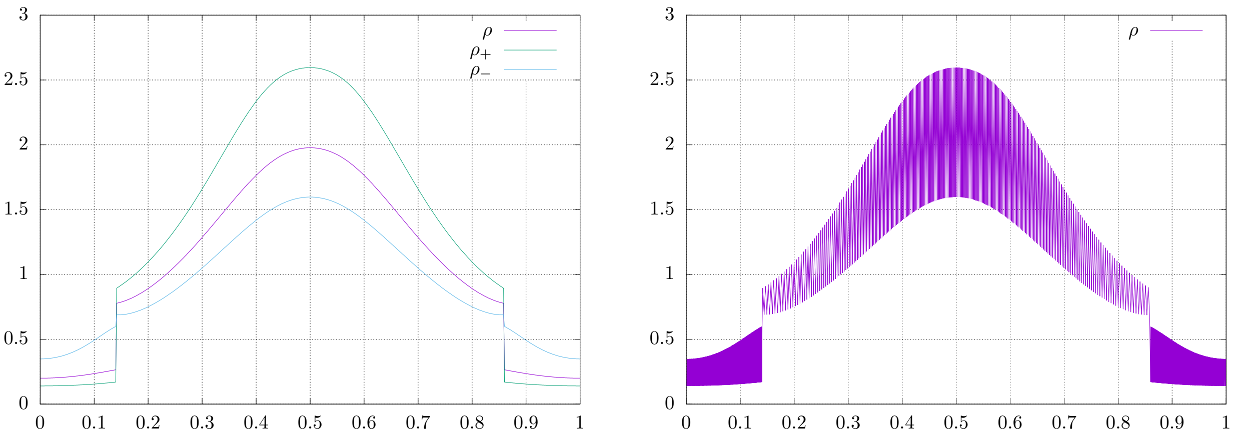



Figure 6: Densities. On the left, the 3 densities of the mixture, on the right, the density of the unmixed fluid.

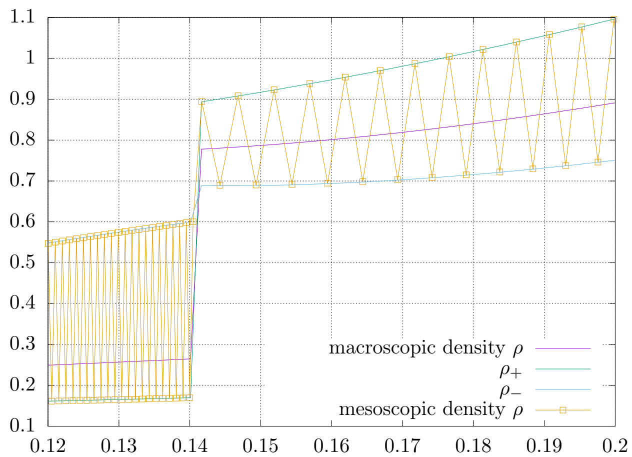



Figure 7: Densities. Zoom of the preceding figures.

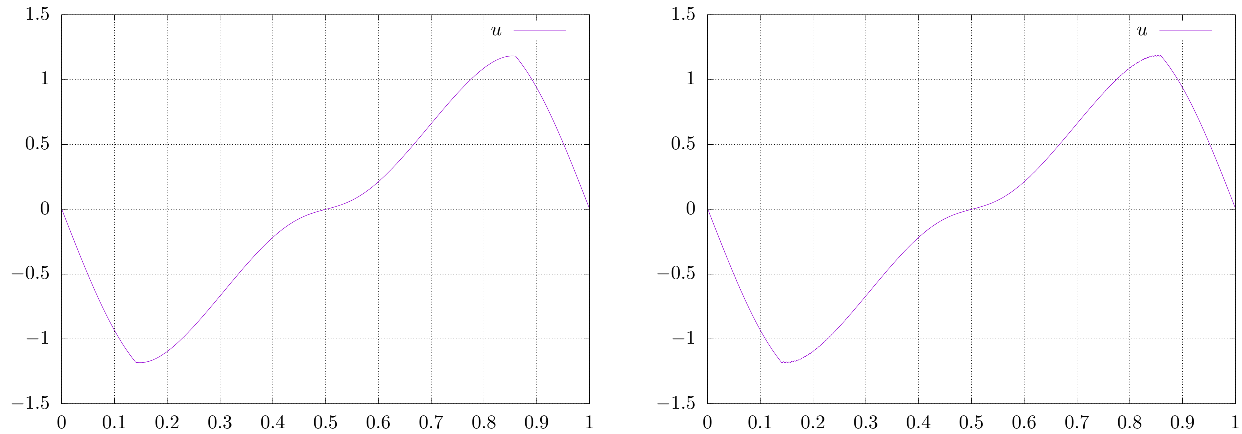



Figure 8: Velocities. On the left, the velocity of the mixture, on the right, the velocity of the unmixed fluid.

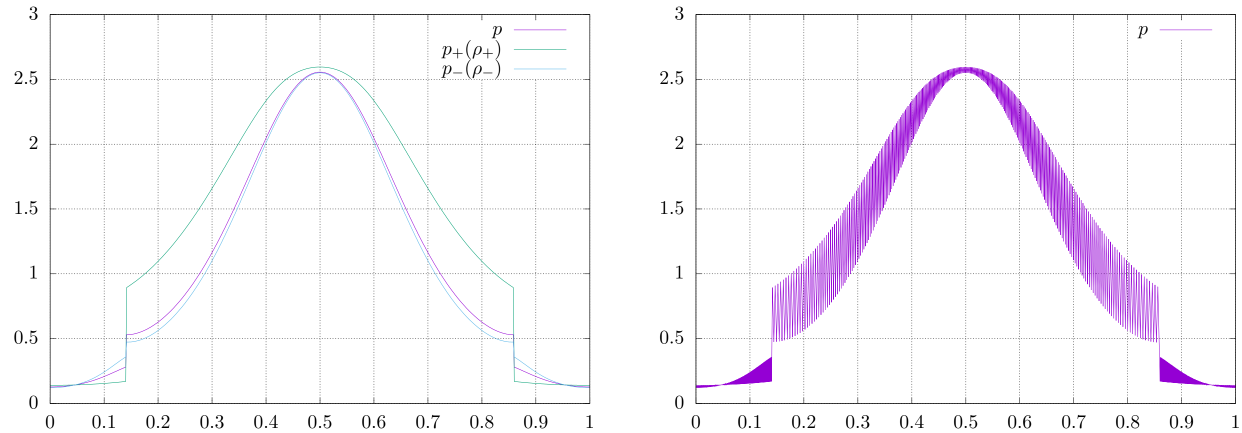



Figure 9: Pressures. On the left, the 3 pressures in the mixture, on the right, the pressure in the unmixed fluid.

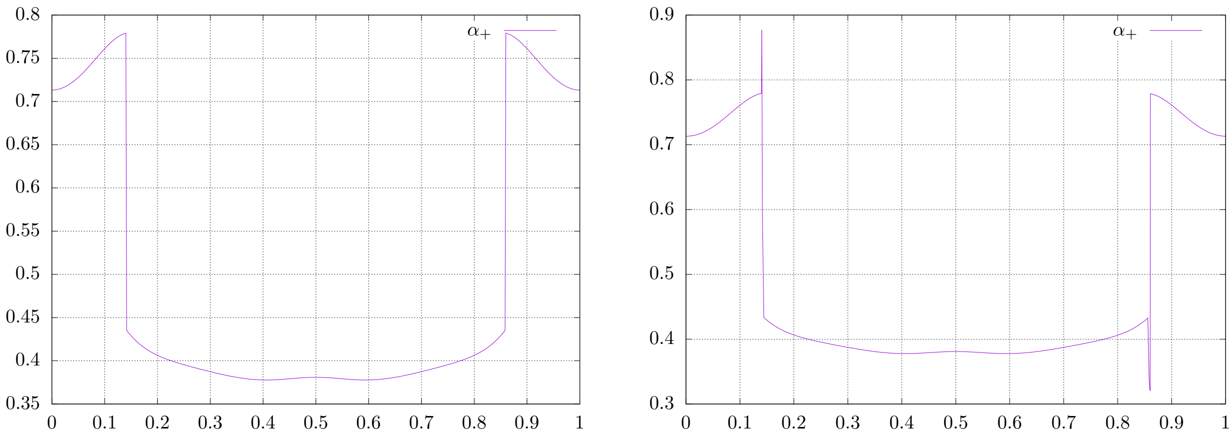



Figure 10: Volume fractions. On the left, the volume fraction $\alpha_+$ in the mixture, on the right, the estimate of the volume fraction in the unmixed fluid.

# 9 Appendix

## 9.1 Formal derivation of $(1.1)_1$

In this subsection we propose a formal discrete procedure to derive the relation $(1.1)_1$ which is a novelty procedure compared to the WKB approach presented in [6]. Equation $(1.1)_2$ with $(1.1)_3$ will follow the usual homogenized procedure using the almost continuity of the discretized stress across the interfaces.

Consider a situation where the fluids are separated (say, at a small scale $\varepsilon$), and a point $x(t) \in \mathbb{T}$ at an interface between fluid $+$ on its right and fluid $-$ on its left, for any time $t$. Denote by $x_+(t)$ the center of the zone of pure fluid $+$ on the right of $x(t)$, by $x_-(t)$ the center of the zone of pure fluid $-$ on the left of $x(t)$, and

$$\varepsilon_+(t) = x_+(t) - x(t), \varepsilon_-(t) = x(t) - x_-(t)$$

which are supposed to be small. We define $\alpha(t)$ by

$$\alpha(t) = \varepsilon_+(t)/(\varepsilon_+(t) + \varepsilon_-(t)).$$

Indeed this quantity represents the local (at point $x(t)$) volume fraction of fluid $+$. Obviously one has

$$D_t\varepsilon_+(t) = u(t, x_+(t)) - u(t, x(t))$$

and

$$D_t(\varepsilon_+ + \varepsilon_-)(t) = u(t, x_+(t)) - u(t, x_-(t)).$$

This allows to write

$$D_t\alpha(t) = \frac{(\varepsilon_+ + \varepsilon_-)D_t\varepsilon_+ - \varepsilon_+ D_t(\varepsilon_+ + \varepsilon_-)}{(\varepsilon_+ + \varepsilon_-)^2}$$
$$= \frac{\varepsilon_-(u(t, x_+(t)) - u(t, x(t))) - \varepsilon_+(u(t, x(t)) - u(t, x_-(t)))}{(\varepsilon_+ + \varepsilon_-)^2} \tag{9.1}$$

The regularity of the solution is expected to be the following: at any time $t$, the pressure and the space derivative of the velocity should be continuous in space in each pure region (namely, in $(x_- - \varepsilon_-, x_- + \varepsilon_-)$ and in $(x_+ - \varepsilon_+, x_+ + \varepsilon_+)$), but not at the point $x(t)$. At this point, what is expected is that the constraint $p - \mu\partial_x u$ is continuous (and this continuity in space stands for the law of reciprocal forces of Newton). In the case where the two viscosity coefficients are equal, the formal computation is straightforward. Thus we propose to begin by assuming this equality, and to obtain the general law for $\alpha$ in a second stage.

- Case where $\mu_+ = \mu_- = \mu$
  The continuity of the effective flux together with the regularity on pure zones expresses as

$$p_-(t) - \mu\frac{u(t,x(t)) - u(t,x_-(t))}{\varepsilon_-} = p_+(t) - \mu\frac{u(t,x_+(t)) - u(t,x(t))}{\varepsilon_+} + r(\varepsilon_- + \varepsilon_+),$$

  where $p_\pm(t)$ denotes $p_\pm(\rho(t,x_\pm(t)))$ and $r$ is a function such that $r(x) \to 0$ as $x \to 0^+$. This rewrites

$$p_+(t) - p_-(t) = \mu\frac{\varepsilon_-(u(t,x_+(t)) - u(t,x(t))) - \varepsilon_+(u(t,x(t)) - u(t,x_-(t)))}{\varepsilon_+\varepsilon_-} + r(\varepsilon_- + \varepsilon_+),$$

  and, thanks to (9.1) and letting $\varepsilon_\pm$ go to 0,

$$p_+(t) - p_-(t) = \mu\frac{(\varepsilon_+ + \varepsilon_-)^2}{\varepsilon_+\varepsilon_-}D_t\alpha_+ = \frac{\mu}{\alpha_+(1-\alpha_+)}D_t\alpha_+,$$

  which is exactly what is stated in this paper.

- Case where $\mu_+ \neq \mu_-$
  In this general case, it is convenient to define the approximate space derivatives of the velocity $d_-(t)$ and $d_+(t)$

$$d_-(t) = \frac{u(t,x(t)) - u(t,x_-(t))}{\varepsilon_-(t)}, \quad d_+(t) = \frac{u(t,x_+(t)) - u(t,x(t))}{\varepsilon_+(t)}.$$

  Equipped with this, we can rewrite (9.1) as

$$D_t\alpha(t) = \frac{\varepsilon_-\varepsilon_+}{(\varepsilon_- + \varepsilon_+)^2}(d_+(t) - d_-(t)).$$

  We would like to express the limit, as $\varepsilon_- + \varepsilon_+$ tends to 0, of the right-hand side term as a function of the limit quantities. Remark that $u$ is intended to converge strongly but $\partial_x u$ only weakly, thus $d_+(t)$ and $d_-(t)$ are not approximations of $\partial_x u(t,x(t))$: however $\dfrac{\varepsilon_-}{\varepsilon_- + \varepsilon_+}d_- + \dfrac{\varepsilon_+}{\varepsilon_- + \varepsilon_+}d_-$ is intended to converge toward $\partial_x u$. The limit of the right-hand side should be expressed as a function of the limit unknowns $\alpha_+$, $\alpha_-$, $p_+$, $p_-$, $\partial_x u$... We already know that $\dfrac{\varepsilon_-\varepsilon_+}{(\varepsilon_- + \varepsilon_+)^2}$ converges to $\alpha_+\alpha_-$. It remains to treat the term $d_+ - d_-$. As $\mu_+ d_+ - \mu_- d_-$ is intended to converge to $p_+ - p_-$, it is quite natural to try to write

$$d_+ - d_- = a(\mu_+ d_+ - \mu_- d_-) + (1 - a\mu_+)d_+ - (1 - a\mu_-)d_-$$

  with $a \in \mathbb{R}$ such that there exists $b \in \mathbb{R}$ satisfying

$$1 - a\mu_+ = b\alpha \quad \text{and} \quad 1 - a\mu_- = -b(1-\alpha),$$

  in which case one would have

$$d_+ - d_- \longrightarrow_{\varepsilon_- + \varepsilon_+ \to 0} a(p_+ - p-) + b\partial_x u.$$

The linear system in $a$ and $b$ has a unique solution, $a = \dfrac{1}{(1-\alpha)\mu_+ + \alpha\mu_-}$ and $b = \dfrac{\mu_- - \mu_+}{(1-\alpha)\mu_+ + \alpha\mu_-}$, which finally gives

$$D_t\alpha_+ = \frac{\alpha(1-\alpha)}{(1-\alpha)\mu_+ + \alpha\mu_-}(p_+ - p_- - (\mu_+ - \mu_-)\partial_x u),$$

which is exactly the first equation in (1.1)

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