Minimax multiple testing procedures for localising an abrupt change in a Poisson process with a known baseline intensity

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1 Introduction

Considering a Poisson process observed on a bounded fixed interval, we are interested in the problem of localising an abrupt change in its distribution, characterized by a jump in its intensity. The present work precisely aims at proposing a non asymptotic minimax multiple testing set-up to construct minimax localisation procedures. This work can be viewed as a second step in an off-line change-point analysis after having detected the change-point. The question of detecting a jump in the intensity of a Poisson process was studied by various author. In particular, the minimax multiple testing framework developed in this work carries on with the results and the methodology developed in [Fromont et al., 2022].

Let us consider a (possibly inhomogeneous) Poisson process $N = (N_t)_{t\in[0,1]}$ observed on the interval $[0, 1]$, with intensity λ defined with respect to some measure Λ on $[0, 1]$, and whose distribution is denoted by P_λ. As in [Fromont et al., 2011], [Fromont et al., 2013] and [Fromont et al., 2022], we assume that the measure Λ satisfies $d\Lambda(t) = Ldt$, where L is a known positive number. Note that when L is an integer, this assumption amounts to considering the Poisson process N as L pooled i.i.d. Poisson processes with the same intensity λ, with respect to dt: L can therefore be seen as a growing number when comparisons with asymptotic existing results in other frequentist models are needed. For all a and b in $[0, 1]$, $N(a, b]$ denotes the number of points of the process fallen in the interval $(a, b]$.

After having detected a jump in λ, the question of localising the jump location are here formulated as multiple testing problems, according to whether the jump size is known or not. In all the sequel, the baseline of the Poisson process is assumed to be known, equal to a positive constant function λ_0 on $[0,1]$. For the sake of simplicity, the constant function λ_0 and its value on $[0,1]$ are often confused in the following. We assume that the intensity λ belongs to $\mathcal{S}$ where $\mathcal{S}$ is the set of intensities defined as positive piecewise constant function with at most one jump.

We denote by $\mathcal{R}$ a multiple testing procedure associated to a collection of hypotheses $\mathcal{H}$ where H in $\mathcal{H}$ corresponds to a single hypothesis, and we set $\cap\mathcal{H} = \cap_{H\in\mathcal{H}} H$. Moreover, we define $\mathcal{T}(\lambda) = \{H \in \mathcal{H}, \lambda \in H\}$ the set of true hypotheses and $\mathcal{F}(\lambda) = \mathcal{H} \setminus \mathcal{T}(\lambda)$ the set of false hypotheses. Considering the usual metric d_2 of $\mathbb{L}_2([0,1])$, we introduce for all $r > 0$,

$$\mathcal{F}_r(\lambda) = \{H \in \mathcal{H} \ : \ d(\lambda, H) \geq r\}.$$

For a multiple testing procedure $\mathcal{R}$, we consider the Family-Wise Error Rate for the first kind error rate defined by

$$\mathrm{FWER}(\mathcal{R}) = \sup_{\lambda\in\mathcal{S}} P_\lambda\left(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing\right).$$

For α and β in $(0,1)$, a subspace $\mathcal{S}' \subset \mathcal{S}$ and a multiple testing procedure $\mathcal{R}$ whose FWER is controlled by α, we define the Family-Wise Separation Rate for the second kind error rate by

$$\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}') = \inf\{r > 0 : \inf_{\lambda\in\mathcal{S}'} P_\lambda\left(\mathcal{F}_r(\lambda) \subset \mathcal{R}\right) \geq 1 - \beta\}.$$

Our aim is to find the minimax Family-Wise Separation Rate of several multiple testing problems under different hypotheses in the knowledge of λ. For α and β in $(0,1)$ and some subspace $\mathcal{S}' \subset \mathcal{S}$, we compute

$$\begin{aligned}\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}') &= \inf_{\mathcal{R}\ :\ \mathrm{FWER}(\mathcal{R})\leq\alpha} \mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}') \\ &= \inf_{\mathcal{R}\ :\ \mathrm{FWER}(\mathcal{R})\leq\alpha} \inf\left\{r > 0 \ : \ \sup_{\lambda\in\mathcal{S}'} P_\lambda\left(\mathcal{F}_r(\lambda) \cap (\mathcal{H}\setminus\mathcal{R}) \neq \varnothing\right) \leq \beta\right\},\end{aligned}$$

where the infimum is taken over the multiple testing procedures that control the FWER by α over $\mathcal{S}$. To this end, we recall the next two lemmas, proved in [Fromont et al., 2016].

Lemma 1. *If $\mathcal{S}'' \subset \mathcal{S}'$ then* $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}'') \leq \mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}')$.

Lemma 2. *If the collection of hypothesis $\mathcal{H}$ is closed (i.e. for all H, H' in $\mathcal{H}$ we get $H \cap H'$ in $\mathcal{H}$), then*

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}') \geq \mathrm{mSR}^{\cap\mathcal{H}}_{\alpha,\beta}(\mathcal{S}') = \inf_{\phi_\alpha} \inf\left\{r \geq 0 : \sup_{\lambda\in\mathcal{S}',\ d(\lambda,\cap\mathcal{H})\geq r} P_\lambda(\phi_\alpha = 0) \leq \beta\right\}, \quad (1)$$

where the first infimum is taken over all the tests ϕ_α of level α for the null hypothesis (H_0) "$\lambda \in \cap\mathcal{H}$", that is $\sup_{\lambda\in\cap\mathcal{H}} P_\lambda(\phi_\alpha = 1) \leq \alpha$.

We will proceed in two steps: the computation of a lower bound and an upper bound for the minimax Family-Wise Separation Rate. As a first step, we make a correspondence between a multiple testing procedure $\mathcal{R}$ and an aggregated test $\bar{\phi}$ to apply the previous lemma and find a lower bound. More precisely, for each hypothesis H in $\mathcal{H}$, we build ϕ_H a simple test of the null hypothesis H against the alternative $\mathcal{H} \setminus H$. We define the related aggregated simple test $\bar{\phi}_{\mathcal{H}} = \sup_{H\in\mathcal{H}} \phi_H$ which rejects the null hypothesis $H_0 \subseteq \cap\mathcal{H}$ when at least a simple hypothesis H in $\mathcal{H}$ is rejected by ϕ_H. As a second step, we construct a non-asymptotic test that controls the FWER by α and achieves the minimax Family-Wise Separation Rate possibly up to a constant.

2 Minimax multiple test for detecting and localising a jump with a known change height

In this section, we focus on the case where the height of the possible jump from λ_0 in the intensity λ of the Poisson process N is known, equal to δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$. Hence, for $\lambda_0 > 0$ and δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$, we introduce the set $\mathcal{S}[\lambda_0, \delta^*]$ of intensities with one jump of height δ^* from λ_0 at location τ, that is

$$\mathcal{S}[\lambda_0, \delta^*] = \{\lambda : [0,1] \to (0, +\infty),\ \exists \tau \in (0,1),\ \forall t \in [0,1],\ \lambda(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau,1]}(t)\}, \quad (2)$$

and the set $\overline{\mathcal{S}}[\lambda_0, \delta^*] = \mathcal{S}[\lambda_0, \delta^*] \cup \{\lambda_0\}$ of intensities with at most one jump of height δ^* from λ_0 at location τ. By convention, we say that the intensity λ in $\overline{\mathcal{S}}[\lambda_0, \delta^*]$ is constant and equal to λ_0 when its jump location τ is equal to 1.

In the aim of localising the jump location, we consider for M in $\mathbb{N}^*$, the collection of hypotheses $\mathcal{H}_{M,\delta^*} = \{H_k[\lambda_0, \delta^*],\ k \in \{1, \ldots, M\}\}$ where for all $k \in \{1, \ldots, M\}$,

$$H_k[\lambda_0, \delta^*] = \{\lambda : [0,1] \to (0, +\infty),\ \exists \tau \in [k/M, 1],\ \forall t \in [0,1],\ \lambda(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau,1]}(t)\}.$$

Notice that for all k in $\{1, \ldots, M\}$, the hypothesis $H_k[\lambda_0, \delta^*]$ is included in $\overline{\mathcal{S}}[\lambda_0, \delta^*]$ and that $H_M[\lambda_0, \delta^*] = \{\lambda_0\}$. Moreover, since the hypotheses are nested in the sense that

$$H_M[\lambda_0, \delta^*] \subset H_{M-1}[\lambda_0, \delta^*] \subset \ldots \subset H_1[\lambda_0, \delta^*],$$

the collection $\mathcal{H}_{M,\delta^*}$ is closed under intersection (that is any intersection $H \cap H'$ of two hypotheses H and H' in $\mathcal{H}_{M,\delta^*}$ also belongs to $\mathcal{H}_{M,\delta^*}$). In particular $\bigcap \mathcal{H}_{M,\delta^*} = H_M[\lambda_0, \delta^*] = \{\lambda_0\}$ and then, for the considered multiple testing problem, given prescribed levels α and β in $(0,1)$, a lower bound for the (α, β)-minimax Family-Wise Separation Rate over $\mathcal{S}[\lambda_0, \delta^*]$ can be deduced from Lemma 2:

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, \delta^*]) \geq \mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}[\lambda_0, \delta^*]),$$

where $\mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}[\lambda_0, \delta^*])$ is the (α, β)-minimax Separation Rate over $\mathcal{S}[\lambda_0, \delta^*]$ for the problem of testing the null hypothesis "$\lambda = \lambda_0$" versus "$\lambda \neq \lambda_0$", defined by (1). Noticing that $\mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}[\lambda_0, \delta^*])$ corresponds to the minimax separation rate of the jump detection problem with known baseline and jump height studied in [Fromont et al., 2022], Proposition 14 in [Fromont et al., 2022] finally leads to the following lower bound.

Proposition 1 (Minimax lower bound). *Let α and β be fixed levels in $(0,1)$ such that $\alpha + \beta < 1$, $\lambda_0 > 0$ and δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$. For all M in $\mathbb{N}^*$ and for all $L \geq \lambda_0 \log C_{\alpha,\beta}/\delta^{*2}$,*

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, \delta^*]) \geq \sqrt{\frac{\lambda_0 \log C_{\alpha,\beta}}{L}}, \quad \text{where} \quad C_{\alpha,\beta} = 1 + 4(1 - \alpha - \beta)^2.$$

In order to prove that the above lower bound is sharp (possibly up to a constant), we secondly construct a minimax multiple testing procedure. Let us begin with a preliminary remark which provides a rough upper bound for the mFWSR.

Remark . Let $r > 0$ and M in $\mathbb{N}^*$. Recall that for λ in $\mathcal{S}[\lambda_0, \delta^*]$, $\mathcal{F}_r(\lambda) = \{H_k[\lambda_0, \delta^*] \in \mathcal{H}_{M,\delta^*},\ d_2(\lambda, H_k[\lambda_0, \delta^*]$ with $d_2(\lambda, H_k[\lambda_0, \delta^*]) = |\delta^*|\sqrt{k/M - \tau}\mathbb{1}_{\tau \leq k/M}$ for all k in $\{1, \dots, M\}$. One has the following straightforward assertion

$$\forall \lambda \in \mathcal{S}[\lambda_0, \delta^*],\ \mathcal{F}_r(\lambda) = \varnothing \iff r \geq |\delta^*|. \tag{3}$$

In particular, for all multiple testing procedure $\mathcal{R}$,

$$\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}[\lambda_0, \delta^*]) \leq |\delta^*|. \tag{4}$$

Therefore, with the lower bound established in Proposition 1, we get that for all M in $\mathbb{N}^*$ and for all $L \geq \lambda_0 \log C_{\alpha,\beta}/\delta^{*2}$,

$$\sqrt{\frac{\lambda_0 \log C_{\alpha,\beta}}{L}} \leq \mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, \delta^*]) \leq |\delta^*| \quad \text{with } C_{\alpha,\beta} = 1 + 4(1 - \alpha - \beta)^2.$$

A noteworthy fact highlights by the previous inequality is that $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, \delta^*])$ is bounded by quantities which does not depend on the choice of M.

To construct a multiple test whose β-Family-Wise separation rate over $\mathcal{S}[\lambda_0, \delta^*]$ achieves, possibly up to a multiplicative constant, the minimax lower bound for $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, \delta^*])$ established in Proposition 1, we use the closure method of [Marcus et al., 1976] applied to single tests of the null hypotheses $H_k[\lambda_0, \delta^*]$ in $\mathcal{H}_{M,\delta^*}$ versus the alternatives $\mathcal{S}[\lambda_0, \delta^*] \setminus H_k[\lambda_0, \delta^*]$ respectively. For k in $\{1, \dots, M\}$, let $\phi_{1,k}$ be the test defined by

$$\phi_{1,k}(N) = \mathbb{1}_{S_{\delta^*,k}(N) > s_{\delta^*,k}(1-\alpha)}, \tag{5}$$

where $S_{\delta^*,k}$ is defined by

$$S_{\delta^*,k}(N) = \sup_{t \in (0, k/M)} \left(\mathrm{sgn}(\delta^*) \left(N\left(t, \frac{k}{M}\right] - \lambda_0 L \left(\frac{k}{M} - t \right) \right) - \frac{|\delta^*|}{2} L \left(\frac{k}{M} - t \right) \right), \tag{6}$$

and $s_{\delta^*,k}(u)$ is the u-quantile of $S_{\delta^*,k}$ under $H_k[\lambda_0, \delta^*]$.
These simple tests, which take the knowledge of the change height δ^* into account, are inspired of the one considered in Section 2.5.1 of [Fromont et al., 2022] which achieves the minimax separation rate of the simple testing problem of detecting a single change point in the intensity of a Poisson process when the jump height is known.

We then define $\hat{k}_1 = \sup\{k' \in \{1, \dots, M\},\ \phi_{1,k'} = 0\}$, leading to the definition of our multiple testing procedure $\mathcal{R}_1$:

$$\mathcal{R}_1 = \{H_k[\lambda_0, \delta^*] :\ k \geq \hat{k}_1 + 1\} \tag{7}$$

whith the convention $\sup \varnothing = -\infty$.

Studying the FWSR of the proposed multiple test from a nonasymptotic point of view necessarily involves to quantify or at least to derive a sharp upper bound of the quantiles $s_{\delta^*,k}(u)$. The following lemma, deduced from an early result of [Pyke, 1959], allows in particular to see that the quantile of the supremum of a homogeneous Poisson process with intensity $\xi L > 0$ (with respect to the Lebesgue measure on $[0, +\infty)$) and with some well chosen drift can be upper bounded by a positive constant not depending on L. This will be a key point of the proof of Theorem 1 below, providing an upper bound for the FWSR of our multiple test $\mathcal{R}_1$ over $\mathcal{S}[\lambda_0, \delta^*]$.

Lemma 3. *Let $L \geq 1$, $\xi > 0$ and $\sigma > 0$. Let $(N_t^\xi)_{t\geq 0}$ be an homogeneous Poisson process with a constant intensity $\xi L > 0$ defined with respect to the Lebesgue measure. Then for all u in $(0,1)$, the u-quantile of $\sup_{t\geq 0}(N_t^\xi - (\xi+\sigma)Lt)$, denoted by $q_\xi(u,\sigma)$, is a positive constant which does not depend on L.*

Notice that when t tends to infinity $(N_t^\xi - (\xi+\sigma)Lt)$ tends to $-\infty$ almost surely which ensures that $\sup_{t\geq 0}(N_t^\xi - (\xi+\sigma)Lt)$ is finite almost surely. Therefore, $\mathbb{P}(\sup_{t\geq 0}(N_t^\xi - (\xi+\sigma)Lt) > x)$ tends to 0 when x tends to $+\infty$. Gathering this remark with the equation (7) in [Pyke, 1959] then straightforwardly leads to the above result.
We can now state the main result of this section, showing that our multiple test $\mathcal{R}_1$ has a Family-Wise error rate over $\overline{\mathcal{S}}[\lambda_0,\delta^*]$ controlled by α, and is minimax over $\mathcal{S}[\lambda_0,\delta^*]$.

Theorem 1 (Minimax upper bound)**.** *Let $L \geq 1$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$ and δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$. Then, there exists a constant $C(\alpha,\beta,\lambda_0,\delta^*) > 0$ such that the multiple testing procedure $\mathcal{R}_1$ defined by (7) satisfies for all M in $\mathbb{N}^*$,*

$$\mathrm{FWER}(\mathcal{R}_1) \leq \alpha, \quad \textit{and} \;\; \mathrm{FWSR}_\beta(\mathcal{R}_1, \mathcal{S}[\lambda_0,\delta^*]) \leq \min\left(|\delta^*|\;,\;\frac{C(\alpha,\beta,\lambda_0,\delta^*)}{\sqrt{L}}\right).$$

In particular, for all M in $\mathbb{N}^*$ and for all $L \geq (C(\alpha,\beta,\lambda_0,\delta^*)/\delta^*)^2$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,\delta^*])) \leq \frac{C(\alpha,\beta,\lambda_0,\delta^*)}{\sqrt{L}}.$$

This result, combined with its corresponding lower bound, shows that the minimax family-wise separation rate over the alternative set $\mathcal{S}[\lambda_0,\delta^*]$ is of parametric order $L^{-1/2}$ whatever the choice of M. For $M=1$, recall that the minimax family-wise separation rate is equal to the minimax separation rate since only one hypothesis is tested. In this case, we recover in particular the minimax separation rate established in Section 2.5.1 of [Fromont et al., 2022] dealing with the change-point detection problem with known baseline and jump height. By the way, the proof of Theorem 1 needs the condition on the distance between λ in $\mathcal{S}[\lambda_0,\delta^*]$ and $H_M[\lambda_0,\delta^*] = \{\lambda_0\}$ established in [Fromont et al., 2022, Proposition 15] to prove the upper bound for the minimax separation rate of jump detection problem. It is worth noting that the multiplicity of the hypotheses in our multiple testing framework does not affect the rate of the mFWSR which remains in the parametric order whatever the value of M.

3 Minimax multiple test for detecting and localising a jump with an unknown change height

In this section, we tackle the question of adaptation with respect to the change height and the jump location, and therefore introduce to this end a preliminary set for $\lambda_0 > 0$,

$$\mathcal{S}[\lambda_0] = \{\lambda : [0,1] \to (0,+\infty),\ \exists(\delta,\tau) \in \{(-\lambda_0,+\infty)\backslash\{0\}\}\times(0,1),\ \forall t \in [0,1],\ \lambda(t) = \lambda_0 + \delta\mathbb{1}_{(\tau,1]}(t)\}. \tag{8}$$

As in Section 2, a lower bound for the (α,β)-minimax Family-Wise separation rate over $\mathcal{S}[\lambda_0]$ is determined using Lemma 2 and the (α,β)-minimax Separation Rate dealing with simple testing problem for the null hypothesis (H_0) "$\lambda = \{\lambda_0\}$" versus the alternative (H_1) "$\lambda \in \mathcal{S}[\lambda_0]$". However, let us recall a lemma proved in [Fromont et al., 2022, Lemma 16] which underlines that the (α,β)-minimax Separation Rate over $\mathcal{S}[\lambda_0]$ is infinite.

Lemma 4. *Let α and β be fixed levels in $(0,1)$ such that $\alpha+\beta<1$. Considering the testing problem (H_0) "$\lambda = \{\lambda_0\}$" versus (H_1) "$\lambda \in \mathcal{S}[\lambda_0]$", with $\mathcal{S}[\lambda_0]$ defined by (8), one has*

$$\mathrm{mSR}^{\{\lambda_0\}}_{\alpha,\beta}\left(\mathcal{S}[\lambda_0]\right) = +\infty.$$

We therefore consider, for $R > \lambda_0$, the more suitable set of alternatives bounded by R, defined by:

$$\mathcal{S}[\lambda_0, R] = \{\lambda : [0,1] \to (0,R],\ \exists(\delta,\tau) \in \{(-\lambda_0, R-\lambda_0]\backslash\{0\}\}\times(0,1),\ \forall t \in [0,1],\ \lambda(t) = \lambda_0+\delta\mathbb{1}_{(\tau,1]}(t)\}.$$

In the aim of localising the jump location, we consider for M in $\mathbb{N}^*$, the collection of hypotheses $\mathcal{H}_{M,R} = \{H_k[\lambda_0, R],\ k \in \{1,\ldots,M\}\}$ where, for all k in $\{1,\ldots,M\}$,

$$H_k[\lambda_0, R] = \{\lambda :\ \exists(\delta,\tau) \in \{(-\lambda_0, R-\lambda_0]\backslash\{0\}\}\times[k/M, 1],\ \forall t \in [0,1],\ \lambda(t) = \lambda_0+\delta\mathbb{1}_{(\tau,1]}(t)\}.$$

In particular, for all k in $\{1,\ldots,M\}$, the single hypothesis $H_k[\lambda_0, R]$ is included in the set $\overline{\mathcal{S}}[\lambda_0, R] = \mathcal{S}[\lambda_0, R] \cup \{\lambda_0\}$ of intensities bounded by R with at most one jump. As in Section 2, we say that the intensity λ in $\overline{\mathcal{S}}[\lambda_0, R]$ is constant and equal to λ_0 when its jump location τ is equal to 1 and we notice that $H_M[\lambda_0, R] = \{\lambda_0\}$.

The collection $\mathcal{H}_{M,R}$ is closed under intersection because the hypotheses are nested: $H_M[\lambda_0, R] \subset H_{M-1}[\lambda_0, R] \subset \ldots \subset H_1[\lambda_0, R]$, and in particular $\bigcap \mathcal{H}_{M,R} = H_M[\lambda_0, R] = \{\lambda_0\}$. Then we get by Lemma 2 that for α and β in $(0,1)$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, R]) \geq \mathrm{mSR}^{\{\lambda_0\}}_{\alpha,\beta}(\mathcal{S}[\lambda_0, R]),$$

where $\mathrm{mSR}^{\{\lambda_0\}}_{\alpha,\beta}(\mathcal{S}[\lambda_0, R])$ is the (α,β)-minimax Separation Rate over $\mathcal{S}[\lambda_0, R]$ which corresponds to the minimax separation rate of the jump detection problem with known baseline studied in Section 2.5.1 of [Fromont et al., 2022]. Therefore, Proposition 17 in [Fromont et al., 2022] ensures the following lower bound.

Proposition 2 (Minimax lower bound)**.** *Let α and β be fixed levels in $(0,1)$ such that $\alpha+\beta<1/2$, $\lambda_0>0$ and $R>\lambda_0$. There exists $L_0(\alpha,\beta,\lambda_0,R)>0$ such that for all M in $\mathbb{N}^*$ and for all $L \geq L_0(\alpha,\beta,\lambda_0,R)$,*

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, R]) \geq \sqrt{\frac{\lambda_0 \log\log L}{L}}.$$

Let us now construct a multiple test whose β-Family-Wise separation rate over $\mathcal{S}[\lambda_0, R]$ achieves, possibly up to a multiplicative constant, the above minimax lower bound for $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, R])$.
We begin with a preliminary remark which provides a rough upper bound for the mFWSR.

Remark . Let $r>0$ and M in $\mathbb{N}^*$. Recall that for λ in $\mathcal{S}[\lambda_0, R]$,

$$\mathcal{F}_r(\lambda) = \{H_k[\lambda_0, R] \in \mathcal{H}_{M,R},\ d_2(\lambda, H_k[\lambda_0, R]) \geq r\}$$

with $d_2(\lambda, H_k[\lambda_0, R]) = |\delta|\sqrt{k/M-\tau}\mathbb{1}_{\tau\leq k/M}$ for all k in $\{1,\ldots,M\}$. One has the following straightforward assertion

$$\forall \lambda \in \mathcal{S}[\lambda_0, R],\ \mathcal{F}_r(\lambda) = \varnothing \iff r \geq \lambda_0 \vee (R-\lambda_0). \tag{9}$$

In particular, for all multiple testing procedure $\mathcal{R}$,

$$\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}[\lambda_0, R]) \leq \lambda_0 \vee (R - \lambda_0). \tag{10}$$

Indeed, let $r > 0$ be such that $r \geq \lambda_0 \vee (R - \lambda_0)$. Then for all λ in $\mathcal{S}[\lambda_0, R]$, $\mathcal{F}_r(\lambda) = \varnothing$ and then $P_\lambda(\mathcal{F}_r(\lambda) \subset \mathcal{R}) = 1 \geq 1 - \beta$ for all multiple testing procedure $\mathcal{R}$ and for all λ in $\mathcal{S}[\lambda_0, R]$.
Therefore, with the lower bound established in Proposition 2, we get that for all M in $\mathbb{N}^*$ and for L large enough

$$\sqrt{\frac{\lambda_0 \log\log L}{L}} \leq \mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, R]) \leq \lambda_0 \vee (R - \lambda_0).$$

Again, it is noteworthy that $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, R])$ is bounded by quantities which does not depend on the choice of M.

Following the discussion of Section 2.2 in [Fromont et al., 2022], we define two different minimax multiple tests: the first one is based on counting statistics and the second one is based on quadratic statistics. Both statistics are inspired of the ones considered in Section 2.5.1 of [Fromont et al., 2022] which achieve the minimax separation rate of the jump detection problem. Both multiple testing procedures are defined according to the closure method of [Marcus et al., 1976] applied to single tests of the null hypotheses $H_k[\lambda_0, R]$ in $\mathcal{H}_{M,R}$ versus the alternatives $\mathcal{S}[\lambda_0, R] \setminus H_k[\lambda_0, R]$ respectively.
We consider the corrected level $u_\alpha = \alpha / \lfloor \log_2 L \rfloor$ which allows to define the two following multiple tests.
Let us begin with the minimax multiple testing procedure based on counting statistics and let k in $\{1, \ldots, M\}$. We then consider a simple test $\phi_{2,k}^{(1)}$ for the null hypothesis $H_k[\lambda_0, R]$ versus the alternative $\mathcal{S}[\lambda_0, R] \setminus H_k[\lambda_0, R]$ defined by

$$\begin{aligned}\phi_{2,k}^{(1)}(N) = \mathbb{1}&\left\{\max_{j\in\{1,\ldots,\lfloor \log_2 L\rfloor\}}\left(N\left(\frac{k}{M}(1-2^{-j}), \frac{k}{M}\right] - p_{\frac{\lambda_0 kL}{M2^j}}\left(1-\frac{u_\alpha}{2}\right)\right)>0\right\} \\ &\vee \mathbb{1}\left\{\max_{j\in\{1,\ldots,\lfloor \log_2 L\rfloor\}}\left(p_{\frac{\lambda_0 kL}{M2^j}}\left(\frac{u_\alpha}{2}\right) - N\left(\frac{k}{M}(1-2^{-j}), \frac{k}{M}\right]\right)>0\right\},\end{aligned} \tag{11}$$

where $p_\xi(u)$ stands for the u-quantile of the Poisson distribution of parameter ξ. We introduce $\hat{k}_2^{(1)} = \sup\{k' \in \{1, \ldots, M\},\ \phi_{2,k'}^{(1)} = 0\}$ and we define our multiple testing procedure $\mathcal{R}_2^{(1)}$ by

$$\mathcal{R}_2^{(1)} = \{H_k[\lambda_0, R] :\ k \geq \hat{k}_2^{(1)} + 1\}. \tag{12}$$

We establish in Theorem 2 that our multiple test $\mathcal{R}_2^{(1)}$ has a FWER over $\overline{\mathcal{S}}[\lambda_0, R]$ controlled by α and is minimax over $\mathcal{S}[\lambda_0, R]$.

Theorem 2 (Minimax upper bound)**.** *Let* $L \geq 3$, α *and* β *be fixed levels in* $(0, 1)$, $\lambda_0 > 0$, $R > \lambda_0$. *Then, there exists a constant* $C(\alpha, \beta, \lambda_0, R) > 0$ *such that the multiple testing procedure* $\mathcal{R}_2^{(1)}$ *defined by* (12) *satisfies for all* M *in* $\mathbb{N}^*$, $\mathrm{FWER}(\mathcal{R}_2^{(1)}) \leq \alpha$ *and*

$$\mathrm{FWSR}_\beta(\mathcal{R}_2^{(1)}, \mathcal{S}[\lambda_0, R]) \leq \min\left(\max(\lambda_0, R - \lambda_0)\ ,\ C(\alpha, \beta, \lambda_0, R)\sqrt{\frac{\log\log L}{L}}\right).$$

In particular, there exists $L_0(\alpha,\beta,\lambda_0,R) > 0$ such that, for all M in $\mathbb{N}^*$ and for all $L \geq L_0(\alpha,\beta,\lambda_0,R)$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R]) \leq C(\alpha,\beta,\lambda_0,R)\sqrt{\frac{\log\log L}{L}}.$$

Let us turn now to the second minimax multiple testing procedure, based on quadratic statistics. For k in $\{1,\ldots,M\}$, we consider a simple test $\phi^{(2)}_{2,k}$ for the null hypothesis $H_k[\lambda_0,R]$ versus the alternative $\mathcal{S}[\lambda_0,R] \setminus H_k[\lambda_0,R]$ defined by

$$\phi^{(2)}_{2,k}(N) = \mathbb{1}_{\max_{j\in\{1,\ldots,\lfloor \log_2 L\rfloor\}}\left(T_{j,k}(N)-t_{j,k}(1-u_\alpha)\right)>0}, \tag{13}$$

where for all j in $\{1,\ldots,\lfloor \log_2 L\rfloor\}$,

$$\begin{aligned} T_{j,k}(N) =& \frac{2^j M}{L^2 k}\left(N\left(\frac{k}{M}\left(1-\frac{1}{2^j}\right),\frac{k}{M}\right]^2 - N\left(\frac{k}{M}\left(1-\frac{1}{2^j}\right),\frac{k}{M}\right]\right) \\ & -\frac{2\lambda_0}{L}N\left(\frac{k}{M}\left(1-\frac{1}{2^j}\right),\frac{k}{M}\right] + \frac{\lambda_0^2 k}{2^j M}, \end{aligned} \tag{14}$$

and $t_{j,k}(u)$ is the u-quantile of $T_{j,k}$ under $H_k[\lambda_0,R]$.

We set $\hat{k}^{(2)}_2 = \sup\{k' \in \{1,\ldots,M\},\ \phi^{(2)}_{2,k'} = 0\}$ and we define our multiple testing procedure $\mathcal{R}^{(2)}_2$ by

$$\mathcal{R}^{(2)}_2 = \{H_k[\lambda_0,R]:\ k \geq \hat{k}^{(2)}_2 + 1\}. \tag{15}$$

We establish in Theorem 3 an alternate minimax upper bound for $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R])$ using the multiple testing procedure $\mathcal{R}^{(2)}_2$.

Theorem 3 (Alternate minimax upper bound)**.** *Let $L \geq 3$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$, $R > \lambda_0$. Then, there exists a constant $C(\alpha,\beta,\lambda_0,R) > 0$ such that the multiple testing procedure $\mathcal{R}^{(2)}_2$ defined by* (15) *satisfies for all M in $\mathbb{N}^*$, $\mathrm{FWER}(\mathcal{R}^{(2)}_2) \leq \alpha$ and*

$$\mathrm{FWSR}_\beta(\mathcal{R}^{(2)}_2,\mathcal{S}[\lambda_0,R]) \leq \min\left(\max(\lambda_0, R-\lambda_0)\ ,\ C(\alpha,\beta,\lambda_0,R)\sqrt{\frac{\log\log L}{L}}\right).$$

In particular, there exists $L_0(\alpha,\beta,\lambda_0,R) > 0$ such that, for all M in $\mathbb{N}^*$ and for all $L \geq L_0(\alpha,\beta,\lambda_0,R)$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R]) \leq C(\alpha,\beta,\lambda_0,R)\sqrt{\frac{\log\log L}{L}}.$$

This result, combined with its corresponding lower bound, brings out a phase transition in the minimax family-wise separation rate orders, from the parametric order $1/\sqrt{L}$ (see Section 2) to $\sqrt{\log\log L/L}$. This means that adaptation with respect to both location and height of the abrupt change has an unavoidable logarithmic cost, while adaptation to only the change height does not cause any additional price. A comparable phase transition has already been observed in the particular case where $M = 1$ dealing with the change-point detection problem investigated in [Fromont et al., 2022]. By the way, the proof of Theorem 3 needs the condition on the distance between λ in $\mathcal{S}[\lambda_0, R]$ and $H_M[\lambda_0, R] = \{\lambda_0\}$ established in [Fromont et al., 2022, Proposition 18] to prove the upper bound for the minimax separation rate of the simple testing problem of detecting an abrupt change in the intensity of a Poisson process. It is worth noting again that the multiplicity of the hypotheses in our multiple testing framework does not affect the rate of the mFWSR which remains in the order $\sqrt{\log\log L/L}$ whatever the value of M.

4 Links between minimax multiple testing procedures and minimal length of confidence intervals for the jump localisation

In the sequel, the subset $\overline{\mathcal{S}}$ stands for $\overline{\mathcal{S}}[\lambda_0, \delta^*]$ or $\overline{\mathcal{S}}[\lambda_0, R]$, $\mathcal{H}$ to the related collection of hypotheses $\mathcal{H}_{M,\delta^*}$ or $\mathcal{H}_{M,R}$ and H_k to the corresponding hypothesis $H_k[\lambda_0, \delta^*]$ or $H_k[\lambda_0, R]$ for all k in $\{1, \ldots, M\}$ with M a non-zero integer. Given the observation of an inhomogeneous Poisson process $N = (N_t)_{t\in[0,1]}$ with an unknown intensity λ in $\overline{\mathcal{S}}$ defined with respect to the measure Λ, a $(1 - \varepsilon)$-confidence interval for the rupture location τ on $\overline{\mathcal{S}}$ is a real interval I_ϵ not depending on the unknown parameters such that

$$\inf_{\lambda\in\overline{\mathcal{S}}} P_\lambda(\tau \in I_\epsilon) \geq 1 - \varepsilon.$$

Since λ belongs to $\overline{S}$, recall that we set $\tau = 1$ by convention if $\lambda = \lambda_0$ on $[0, 1]$. Notice that the interval I_ε can be defined from an estimator $\hat{\tau}$ of τ and could depends on known parameters of the problem. The relationships between confidence intervals for the rupture localisation τ on $\overline{\mathcal{S}}$ and multiple testing procedures are summarized in the following lemmas.

First, it is well known that an α-level test for a single null hypothesis is intrinsically related to a $(1 - \alpha)$-confidence region, and that a similar correspondence can be made between a FWER controlling procedure and a confidence interval (see for example [Arlot et al., 2010]).

Lemma 5. *Let M in $\mathbb{N}^*$. From any multiple procedure $\mathcal{R}$ on $\mathcal{H}$ such that* $\mathrm{FWER}(\mathcal{R}) \leq \alpha$, *one can build a $(1 - \alpha)-$confidence interval I_α for τ on $\overline{\mathcal{S}}$ defined by $I_\alpha = \{x \in [0, 1] : x \leq ((\sup\{k \in \{1, \ldots, M\} : H_k \notin \mathcal{R}\} + 1)/M) \wedge 1\}$. Conversely, from any $(1-\alpha)$-confidence region I_α for τ on $\overline{\mathcal{S}}$, one can build a multiple testing procedure $\mathcal{R}$ on $\mathcal{H}$ such that* $\mathrm{FWER}(\mathcal{R}) \leq \alpha$ *which is defined by $\mathcal{R} = \{H_k \in \mathcal{H},\ k/M > \sup\{x \in [0, 1],\ x \in I_\alpha\}\}$.*

However, the confidence intervals build with FWER controlling procedures seem to be large and the multiple tests defined from confidence intervals too conservative. We are then interested in finding the smallest confidence interval for the estimation of the jump location of the intensity.
We consider $\mathcal{D}[0, 1]$ the space of càdlàg functions on $[0, 1]$ and $\mathcal{MD}$ the set of all the measurable real functions defined on $\mathcal{D}[0, 1]$ taking values in $[0, 1]$. We then focus on confidence intervals of the form $(\phi(N) - a, \phi(N) + b]$ where $a, b \geq 0$ and ϕ in $\mathcal{MD}$. One may think about $\phi(N)$ as an estimator of τ. We define the minimal length of $(1 - \varepsilon)$-confidence intervals for the estimator $\phi(N)$ of τ with λ in $\overline{\mathcal{S}}$ by

$$\mathcal{L}_\varepsilon(\phi, \overline{\mathcal{S}}) = \inf\{a + b : a, b > 0, \inf_{\lambda\in\overline{\mathcal{S}}} P_\lambda(\tau \in (\phi(N) - a, \phi(N) + b]) \geq 1 - \varepsilon\},$$

and the minimal length of $(1-\varepsilon)$-confidence intervals for τ with λ in $\overline{\mathcal{S}}$ by

$$\mathcal{L}_\varepsilon(\overline{\mathcal{S}}) = \inf\{\mathcal{L}_\varepsilon(\phi, \overline{\mathcal{S}}) : \phi \in \mathcal{MD}\}.$$

4.1 Confidence interval for the jump localisation when the change height is known

The following lemma gives bounds for the minimal length of confidence intervals for the estimation problem of τ when the change height is known.

Lemma 6. *Let $L \geq 1$, α and β in $(0,1)$, $\lambda_0 > 0$, δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$ and M in $\mathbb{N}^*$. Consider first $a, b > 0$ and ϕ in $\mathcal{MD}$ such that*

$$\inf_{\lambda \in \overline{\mathcal{S}}[\lambda_0, \delta^*]} P_\lambda(\tau \in (\phi(N) - a, \phi(N) + b]) \geq 1 - \alpha. \tag{16}$$

The multiple testing procedure $\mathcal{R}$ on $\mathcal{H}_{M,\delta^*}$ defined by $\mathcal{R} = \{H_k \in \mathcal{H}_{M,\delta^*},\ k/M > \phi(N)+b\}$ is satisfying

$$\mathrm{FWER}(\mathcal{R}) \leq \alpha, \quad \textit{and} \;\; \mathrm{FWSR}_\alpha(\mathcal{R}, \mathcal{S}[\lambda_0, \delta^*]) \leq |\delta^*|\sqrt{a+b}.$$

In particular, we get for all M in $\mathbb{N}^*$,

$$\mathcal{L}_\alpha(\overline{\mathcal{S}}[\lambda_0, \delta^*]) \geq \frac{\mathrm{mFWSR}_{\alpha,\alpha}(\mathcal{S}[\lambda_0, \delta^*])^2}{\delta^{*2}}.$$

Conversely, consider $r > 0$ be such that $r \geq \mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, \delta^*])$ and a multiple testing procedure $\mathcal{R}$ on $\mathcal{H}_{M,\delta^*}$ satisfying the two inequalities $\mathrm{FWER}(\mathcal{R}) \leq \alpha$ and $\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}[\lambda_0, \delta^*]) \leq r$. Set $\hat{k} = \sup\{k \in \{1, \dots, M\},\ H_k \notin \mathcal{R}\}$ and $\hat{\tau} = \hat{k}/M$. The interval $I_{r,\delta^*,M,\mathcal{R}}$ defined by $I_{r,\delta^*,M,\mathcal{R}} = (\hat{\tau} - r^2/\delta^{*2}, \hat{\tau} + 1/M]$ then satisfies

$$\inf_{\lambda \in \overline{\mathcal{S}}[\lambda_0, \delta^*]} P_\lambda(\tau \in I_{r,\delta^*,M,\mathcal{R}}) \geq 1 - \alpha - \beta.$$

In particular, we get that for all M in $\mathbb{N}^*$,

$$\mathcal{L}_{\alpha+\beta}(\overline{\mathcal{S}}[\lambda_0, \delta^*]) \leq \frac{1}{M} + \frac{\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, \delta^*])^2}{\delta^{*2}}.$$

This lemma, combined with Proposition 1 and Theorem 1, allows us to construct a minimal confidence interval for the jump localisation from a multiple testing procedure. Assume that we observe a Poisson process $N = (N_t)_{t\in[0,1]}$ on the interval $[0,1]$, with intensity λ in $\overline{\mathcal{S}}[\lambda_0, \delta^*]$ with respect to some measure Λ on $[0,1]$, where λ_0, δ^* and L are known parameters. We shall estimate the jump location τ using a multiple testing procedure. For α in $(0,1/2)$, if we assume that $L \geq \max(\lambda_0 \log C_{\alpha,\beta}, C\left(\alpha/2, \alpha/2, \lambda_0, \delta^*\right)^2)/\delta^{*2}$, Lemma 6, Proposition 1 and Theorem 1 ensure that for all M in $\mathbb{N}^*$,

$$\frac{\lambda_0 \log C_\alpha}{L\delta^{*2}} \leq \mathcal{L}_\alpha(\overline{\mathcal{S}}[\lambda_0, \delta^*]) \leq \frac{1}{M} + \frac{C\left(\alpha/2, \alpha/2, \lambda_0, \delta^*\right)^2}{L\delta^{*2}}$$

where $C_\alpha = 1 + 4(1-2\alpha)^2$ and $C\left(\alpha/2, \alpha/2, \lambda_0, \delta^*\right)$ is defined in Theorem 1.

We then aim at providing a confidence interval for τ of minimal length. To this end, we consider for $L \geq (C(\alpha/2, \alpha/2, \lambda_0, \delta^*)/\delta^*)^2$, the collection of hypotheses $\mathcal{H}_{M,\delta^*} = \{H_k[\lambda_0, \delta^*],\ k \in \{1, \dots, M\}\}$ with

$$M = \left\lfloor \frac{L\delta^{*2}}{C(\alpha/2, \alpha/2, \lambda_0, \delta^*)^2} \right\rfloor. \tag{17}$$

We then apply our multiple testing procedure $\mathcal{R}_1$ introduced in Section 2. Recall that we define

$$\hat{k}_1 = \sup\{k' \in \{1, \dots, M\},\ \phi_{1,k'} = 0\},$$

with $\phi_{1,k}$ defined by (5), leading to the definition of our multiple testing procedure $\mathcal{R}_1$:

$$\mathcal{R}_1 = \{H_k[\lambda_0, \delta^*] :\ k \geq \hat{k}_1 + 1\}.$$

We introduce an estimator of the location of the intensity jump by

$$\hat{\tau} = \frac{\hat{k}_1}{M}. \tag{18}$$

Corollary 1. *Let α in $(0, 1/2)$, $\lambda_0 > 0$ and δ^* in $(-\lambda_0, +\infty)\backslash\{0\}$. For $L \geq (C(\alpha/2, \alpha/2, \lambda_0, \delta^*)/\delta^*)^2$ and M defined by (17), the interval*

$$\left(\hat{\tau} - \frac{1}{M},\ \hat{\tau} + \frac{1}{M}\right]$$

is a $(1-\alpha)$-confidence interval for the jump localisation on $\overline{\mathcal{S}}[\lambda_0, \delta^*]$ which achieves, up to a constant, the minimal length of such confidence intervals.

4.2 Confidence interval for the jump localisation when the change height is unknown

When the change height δ^* is an unknown parameter, the notion of minimal confidence interval is quite irrelevant. Indeed, the minimal length for confidence intervals on the whole space $\overline{\mathcal{S}}[\lambda_0, R]$ is almost always equal to 1 and then, the interval $[0, 1]$ is a confidence interval of minimal length.

Lemma 7. *Let α in $(0, 1/2)$, $\lambda_0 > 0$ and $R > \lambda_0$. Then for all $L \geq 1$,*

$$\mathcal{L}_\alpha(\overline{\mathcal{S}}[\lambda_0, R]) = 1.$$

The Lemma 7 invites us to consider a smaller set for the intensities λ of the Poisson process. Therefore, we focus on intensities whose jumps are large enough. For $\lambda_0 > 0$ and Δ in $(0, \lambda_0)$ we introduce the set $\mathcal{S}_{\geq\Delta}[\lambda_0]$ of intensities whose jumps are at least of height Δ:

$$\mathcal{S}_{\geq\Delta}[\lambda_0] = \{\lambda : [0,1] \to (0, +\infty),\ \exists\delta \in \{(-\lambda_0, -\Delta] \cup [\Delta, +\infty)\},\ \exists\tau \in (0,1),\ \forall t \in [0,1],\ \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}(t)\}. \tag{19}$$

However, the following lemma underlines that the (α, β)-minimax Separation Rate over this preliminary alternative set is infinite and then, the arguments used with Lemma 2 provide an infinite (α, β)-minimax Family-Wise Separation Rate over $\mathcal{S}_{\geq\Delta}[\lambda_0]$.

Lemma 8. *Let α and β be fixed levels in $(0,1)$ such that $\alpha+\beta<1$. For $\lambda_0>0$ and Δ in $(0,\lambda_0)$, considering the testing problem (H_0) "$\lambda=\lambda_0$" versus (H_1) "$\lambda\in\mathcal{S}_{\geq\Delta}[\lambda_0]$" with $\mathcal{S}_{\geq\Delta}[\lambda_0]$ defined by (19), one has*

$$\mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}\left(\mathcal{S}_{\geq\Delta}[\lambda_0]\right)=+\infty.$$

We therefore introduce, for $\lambda_0>0$, $R>\lambda_0$ and Δ in $(0,\lambda_0\wedge(R-\lambda_0))$, the more suitable set $\mathcal{S}_{\geq\Delta}[\lambda_0,R]$ of intensities bounded by R whose jumps are at least of height Δ from λ_0:

$$\mathcal{S}_{\geq\Delta}[\lambda_0,R]=\{\lambda:[0,1]\to(0,R],\ \exists\delta\in\{(-\lambda_0,-\Delta]\cup[\Delta,R-\lambda_0]\},\ \exists\tau\in(0,1),\\ \forall t\in[0,1],\ \lambda(t)=\lambda_0+\delta\mathbb{1}_{(\tau,1]}(t)\}.$$

In the aim of localising the jump location, we consider for M in $\mathbb{N}^*$, the collection of hypotheses $\mathcal{H}_{M,\Delta,R}=\{H_k[\lambda_0,\Delta,R],\ k\in\{1,\dots,M\}\}$ where for all k in $\{1,\dots,M\}$,

$$H_k[\lambda_0,\Delta,R]=\{\lambda:\exists\delta\in\{(-\lambda_0,-\Delta]\cup[\Delta,R-\lambda_0]\},\ \exists\tau\in[k/M,1],\\ \forall t\in[0,1],\ \lambda(t)=\lambda_0+\delta\mathbb{1}_{(\tau,1]}(t)\}.$$

For all k in $\{1,\dots,M\}$, notice that the hypothesis $H_k[\lambda_0,\Delta,R]$ is included in the set $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]=\mathcal{S}_{\geq\Delta}[\lambda_0,R]\cup\{\lambda_0\}$ of intensities bounded by R and with at most a change whose height is lower bounded by Δ.
Since the set $\mathcal{S}_{\geq\Delta}[\lambda_0,R]$ includes $\mathcal{S}[\lambda_0,\Delta]$ defined in (2) for all Δ in $(0,\lambda_0\wedge(R-\lambda_0))$, the Lemma 1 combined with Proposition 1 leads to

Proposition 3 (Minimax lower bound)**.** *Let α and β be fixed levels in $(0,1)$ such that $\alpha+\beta<1$, $\lambda_0>0$, $R>\lambda_0$ and Δ in $(0,\lambda_0\wedge(R-\lambda_0))$. For all M in $\mathbb{N}^*$ and for all $L\geq\lambda_0\log C_{\alpha,\beta}/\Delta^2$,*

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}_{\geq\Delta}[\lambda_0,R])\geq\sqrt{\frac{\lambda_0\log C_{\alpha,\beta}}{L}}$$

where $C_{\alpha,\beta}=1+4(1-\alpha-\beta)^2$.

To define a multiple testing procedure whose β-Family-Wise Separation Rate over the set $\mathcal{S}_{\geq\Delta}[\lambda_0,R]$ achieves, possibly up to a multiplicative constant, the above minimax lower bound for $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}_{\geq\Delta}[\lambda_0,R])$, we construct for k in $\{1,\dots,M\}$ a simple test for the null hypothesis $H_k[\lambda_0,\Delta,R]$ versus the alternative $\mathcal{S}_{\geq\Delta}[\lambda_0,R]\setminus H_k[\lambda_0,\Delta,R]$. More precisely, we define the test $\phi_{3,k}$ for all k in $\{1,\dots,M\}$ by

$$\phi_{3,k}(N)=\mathbb{1}_{S_{\Delta,k}(N)>s_{\Delta,k}(1-\alpha/2)}\vee\mathbb{1}_{S_{-\Delta,k}(N)>s_{-\Delta,k}(1-\alpha/2)},\tag{20}$$

where for all δ in $\mathbb{R}^*$, $S_{\delta,k}(N)$ is defined in (6) by

$$S_{\delta,k}(N)=\sup_{t\in(0,k/M)}\left(\mathrm{sgn}(\delta)\left(N\left(t,\frac{k}{M}\right]-\lambda_0L\left(\frac{k}{M}-t\right)\right)-\frac{|\delta|}{2}L\left(\frac{k}{M}-t\right)\right),$$

and $s_{\Delta,k}(u)$ (respectively $s_{-\Delta,k}(u)$) is the u-quantile of $S_{\Delta,k}$ (respectively the u-quantile of $S_{-\Delta,k}$) under $H_k[\lambda_0,\Delta,R]$. These simple tests, which take the knowledge of the minimal value Δ of the change height into account, are closed to the ones considered in Section 2.

We then define $\hat{k}_3 = \sup\{k' \in \{1, \ldots, M\},\ \phi_{3,k'}(N) = 0\}$, leading to the definition of our multiple testing procedure $\mathcal{R}_3$:

$$\mathcal{R}_3 = \{H_k[\lambda_0, \Delta, R] :\ k \geq \hat{k}_3 + 1\}. \tag{21}$$

The following lemma, deduced from a result of Loader [Loader, 1990], ensures that the quantile of a homogeneous Poisson process with intensity $\xi L > 0$ and with a well chosen drift is an increasing function of ξ. This will be a key point of the proof of Theorem 4 below, providing an upper bound for the FWSR of our multiple test $\mathcal{R}_3$ over $\mathcal{S}_{\geq \Delta}[\lambda_0, R]$.

Lemma 9. *Let $\xi > 0$ and $\sigma > 0$. For all u in $(0, 1)$, the function $\xi \mapsto q_\xi(u, \sigma)$ is increasing, where $q_\xi(u, \sigma)$ is defined in Lemma 3.*

Comment. Let $(N_t^\xi)_{t\geq 0}$ be a homogeneous Poisson process with a constant intensity $\xi L > 0$. For all $x \geq 0$, an Abel transform on the exact expression of $\mathbb{P}(\sup_{t\geq 0}(N_t^\xi - (\xi + \sigma)Lt) > x)$ given in [Loader, 1990, Theorem 2.2] shows that $\lambda \mapsto \mathbb{P}(\sup_{t\geq 0}(N_t^\xi - (\xi + \sigma)Lt) > x)$ is increasing. The above result then follows easily.

The following theorem shows that the multiple test $\mathcal{R}_3$ is minimax over $\mathcal{S}_{\geq \Delta}[\lambda_0, R]$ and its proof follows essentially the same ideas of the proof of Theorem 1 with the argument given in Lemma 9. For the sake of clarity and completeness, the proof of Theorem 4 is detailed in Section 5.4.

Theorem 4 (Minimax upper bound)**.** *Let $L \geq 1$, α and β be fixed levels in $(0, 1)$, $\lambda_0 > 0$, $R > \lambda_0$ and Δ in $(0, \lambda_0 \wedge (R - \lambda_0))$. Then, there exists a constant $C(\alpha, \beta, \lambda_0, \Delta, R) > 0$ such that the multiple testing procedure $\mathcal{R}_3$ defined by (21) satisfies for all M in $\mathbb{N}^*$,*

$$\mathrm{FWER}(\mathcal{R}_3) \leq \alpha, \ \ \textit{and}\ \mathrm{FWSR}_\beta(\mathcal{R}_3, \mathcal{S}_{\geq\Delta}[\lambda_0, R]) \leq \min\left(\lambda_0 \vee (R - \lambda_0)\ ,\ \frac{C(\alpha, \beta, \lambda_0, \Delta, R)}{\sqrt{L}}\right).$$

In particular, for all M in $\mathbb{N}^*$ and for all $L \geq (C(\alpha, \beta, \lambda_0, \Delta, R)/(\lambda_0 \vee (R - \lambda_0)))^2$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}_{\geq\Delta}[\lambda_0, R])) \leq \frac{C(\alpha, \beta, \lambda_0, \Delta, R)}{\sqrt{L}}.$$

This result, combined with its corresponding lower bound, shows that the minimax family-wise separation rate over the alternative $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$ has the same parametric order $L^{-1/2}$ as the case where the height of the jump is known (see Section 2). Then, when the change-point location is unknown, it is the possibility for the jump height to be close to zero (and not the fact that it is unknown) which deteriorates the mFWSR with a logarithmic factor (see Section 3). For $M = 1$, the lower bound and the upper bound established in Proposition 3 and Theorem 4 complete the minimax study for the detection of an abrupt change in the intensity of a Poisson process from a known constant baseline considered in [Fromont et al., 2022]: when both location and height of the change are unknown but with a minimal known value for the jump height, the minimax separation rate of the simple testing problem (H_0) "$\lambda = \lambda_0$" versus (H_1) "$\lambda \in \mathcal{S}_{\geq\Delta}[\lambda_0, R]$" is of parametric order $\sqrt{1/L}$.

Now, let us turn back to the construction of a confidence interval of minimal length for the jump location. The following lemma gives bounds for the minimal length of confidence intervals for the estimation problem of τ when the change height is lower bounded by Δ.

Lemma 10. *Let $L \geq 1$, α and β in $(0,1)$, $\lambda_0 > 0$, $R > \lambda_0$, Δ in $(0, \lambda_0 \wedge (R - \lambda_0))$ and M in $\mathbb{N}^*$. Consider first $a, b > 0$ and ϕ in $\mathcal{MD}$ such that*

$$\inf_{\lambda \in \overline{\mathcal{S}}_{\geq \Delta}[\lambda_0, R]} P_\lambda(\tau \in (\phi(N) - a, \phi(N) + b]) \geq 1 - \alpha. \tag{22}$$

The multiple testing procedure $\mathcal{R}$ on $\mathcal{H}_{M,\Delta,R}$ defined by $\mathcal{R} = \{H_k \in \mathcal{H}_{M,\Delta,R},\ k/M > \phi(N) + b\}$ is satisfying

$$\mathrm{FWER}(\mathcal{R}) \leq \alpha, \quad \textit{and } \mathrm{FWSR}_\alpha(\mathcal{R}, \mathcal{S}_{\geq \Delta}[\lambda_0, R]) \leq (\lambda_0 \wedge (R - \lambda_0))\sqrt{a + b}.$$

In particular, we get that for all M in $\mathbb{N}^*$,

$$\mathcal{L}_\alpha(\overline{\mathcal{S}}_{\geq \Delta}[\lambda_0, R]) \geq \frac{\mathrm{mFWSR}_{\alpha,\alpha}(\mathcal{S}_{\geq \Delta}[\lambda_0, R])^2}{(\lambda_0 \wedge (R - \lambda_0))^2}.$$

Conversely, consider $r > 0$ be such that $r \geq \mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}_{\geq \Delta}[\lambda_0, R])$ and a multiple testing procedure $\mathcal{R}$ on $\mathcal{H}_{M,\Delta,R}$ satisfying the two inequalities $\mathrm{FWER}(\mathcal{R}) \leq \alpha$ and $\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}_{\geq \Delta}[\lambda_0, R]) \leq r$. Set $\hat{k} = \sup\{k \in \{1, \ldots, M\},\ H_k \notin \mathcal{R}\}$ and $\hat{\tau} = \hat{k}/M$. The interval $I_{r,\Delta,M,\mathcal{R}}$ defined by $I_{r,\Delta,M,\mathcal{R}} = (\hat{\tau} - r^2/\Delta^2, \hat{\tau} + 1/M]$ then satisfies

$$\inf_{\lambda \in \overline{\mathcal{S}}_{\geq \Delta}[\lambda_0, R]} P_\lambda(\tau \in I_{r,\Delta,M,\mathcal{R}}) \geq 1 - \alpha - \beta.$$

In particular, we get that for all M in $\mathbb{N}^*$,

$$\mathcal{L}_{\alpha+\beta}(\overline{\mathcal{S}}_{\geq \Delta}[\lambda_0, R]) \leq \frac{1}{M} + \frac{\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}_{\geq \Delta}[\lambda_0, R])^2}{\Delta^2}.$$

This lemma, combined with Proposition 3 and Theorem 4, allows us to construct a minimal confidence interval for the jump localisation from a multiple testing procedure. Assume that we observe a Poisson process $N = (N_t)_{t \in [0,1]}$ on the interval $[0,1]$, with intensity λ in $\overline{\mathcal{S}}_{\geq \Delta}[\lambda_0, R]$ with respect to some measure Λ on $[0,1]$, where λ_0, Δ, R and L are known parameters. We shall estimate the jump location τ using a multiple testing procedure.
For α in $(0, 1/2)$, if we assume that $L \geq \max(\lambda_0 \log C_{\alpha,\beta}/\Delta^2, C\left(\alpha/2, \alpha/2, \lambda_0, \Delta, R\right)^2/(\lambda_0^2 \vee (R - \lambda_0)^2)$, Lemma 10, Proposition 3 and Theorem 4 ensure that for all M in $\mathbb{N}^*$,

$$\frac{\lambda_0 \log C_\alpha}{L(\lambda_0^2 \wedge (R - \lambda_0)^2)} \leq \mathcal{L}_\alpha(\overline{\mathcal{S}}_{\geq \Delta}[\lambda_0, R]) \leq \frac{1}{M} + \frac{C\left(\alpha/2, \alpha/2, \lambda_0, \Delta, R\right)^2}{L\Delta^2},$$

where $C_\alpha = 1 + 4(1 - 2\alpha)^2$ and $C\left(\alpha/2, \alpha/2, \lambda_0, \Delta, R\right)$ is defined in Theorem 4.
We then aim at providing a confidence interval for τ of minimal length. To this end, we consider for $L \geq (C\left(\alpha/2, \alpha/2, \lambda_0, \Delta, R\right)/\Delta)^2$, the collection of hypotheses $\mathcal{H}_{M,\Delta,R} = \{H_k[\lambda_0, \Delta, R],\ k \in \{1, \ldots, M\}\}$ with

$$M = \left\lfloor \frac{L\Delta^2}{C\left(\alpha/2, \alpha/2, \lambda_0, \Delta, R\right)^2} \right\rfloor. \tag{23}$$

We then apply our multiple testing procedure $\mathcal{R}_3$ defined by (21) and we introduce an estimator of the location of the intensity jump by

$$\hat{\tau} = \frac{\hat{k}_3}{M}. \tag{24}$$

Corollary 2. *Let α in $(0,1/2)$, $\lambda_0 > 0$, $R > 0$ and Δ in $(0, \lambda_0 \wedge (R-\lambda_0))$. For $L \geq (C(\alpha/2, \alpha/2, \lambda_0, \Delta, R)/\Delta)^2$ and M defined by (23), the interval*

$$\left(\hat{\tau} - \frac{1}{M},\ \hat{\tau} + \frac{1}{M}\right]$$

is a $(1-\alpha)$-confidence interval for the jump localisation on $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0, R]$ which achieves, up to a constant, the minimal length of such confidence intervals.

5 Proofs

We define for all $x \geq 0$ the positive, continuous and non-decreasing function g by

$$g(x) = (1+x)\log(1+x) - x \tag{25}$$

which satisfies for every $x \geq 0$

$$g^{-1}(x) \leq \sqrt{2x} + 2x/3. \tag{26}$$

5.1 Proof of Theorem 1

By now, for $\lambda_0 > 0$, δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$ and M in $\mathbb{N}^*$, the simple hypothesis $H_k[\lambda_0, \delta^*]$ is simply written H_k for short. We begin with a lemma which gives an upper bound for the quantile $s_{\delta^*,k}(1-\alpha)$ of $S_{\delta^*,k}$ under H_k. It highlights in particular that we can bound this quantile by some constants which do not depend on k, L and M.

Lemma 11 (Control of the quantiles)**.** *Let α in $(0,1)$, $\lambda_0 > 0$ and δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$. For all M in $\mathbb{N}^*$, for all $L \geq 1$ and for all $k \in \{1, \dots, M\}$, one has*

$$\begin{cases} s_{\delta^*,k}(1-\alpha) \leq q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right) & \text{if } \delta^* > 0, \\ s_{\delta^*,k}(1-\alpha) \leq \frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0+\delta^*/2}\right)} & \text{if } -\lambda_0 < \delta^* < 0, \end{cases} \tag{27}$$

where $q_{\lambda_0}(1-\alpha, \delta^*/2)$ is defined in Lemma 3.

Proof of Lemma 11. Let k in $\{1, \dots, M\}$. Under (H_k), N is a homogeneous Poisson process on $[0, k/M]$ of intensity λ_0 with respect to the measure Λ, and since the processes $(N(t, k/M])_{t\in(0,k/M)}$ and $(N(0, k/M - t])_{t\in(0,k/M)}$ are left continuous and have the same finite dimensional laws, one obtains

$$S_{\delta^*,k} \overset{d}{=} \sup_{t\in(0,k/M)} \left(\operatorname{sgn}(\delta^*)(N(0,t] - \lambda_0 Lt) - \frac{|\delta^*|}{2} Lt\right). \tag{28}$$

Assume first that $\delta^* > 0$. The equality (28) simply reads in this case

$$S_{\delta^*,k} \overset{d}{=} \sup_{t\in(0,k/M)} \left(N(0,t] - \left(\lambda_0 + \frac{\delta^*}{2}\right) Lt\right),$$

and we get for all λ in H_k,

$$P_\lambda\left(S_{\delta^*,k} > q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right)\right) = \mathbb{P}\left(\sup_{t\in(0,k/M)}\left(N^{\lambda_0}(0,t] - \left(\lambda_0 + \frac{\delta^*}{2}\right)Lt\right) > q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right)\right)$$
$$\leq \mathbb{P}\left(\sup_{t\in[0,+\infty)}\left(N^{\lambda_0}(0,t] - \left(\lambda_0 + \frac{\delta^*}{2}\right)Lt\right) > q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right)\right),$$

where $(N_t^{\lambda_0})_{t\geq 0}$ is a homogeneous Poisson process of intensity $\lambda_0 L$ with respect to the Lebesgue measure on $\mathbb{R}^+$. By definition of $q_{\lambda_0}(1-\alpha, \delta^*/2)$ (see Lemma 3), this leads to $\sup_{\lambda\in H_k} P_\lambda\left(S_{\delta^*,k} > q_{\lambda_0}\left(1-\alpha, \delta^*/2\right)\right) \leq \alpha$ and the first part in (27) holds by definition of the quantile $s_{\delta^*,k}(1-\alpha)$.
Assume now that δ^* belongs to $(-\lambda_0, 0)$. For all $x > 0$ and for all λ in H_k,

$$P_\lambda\left(S_{\delta^*,k} > x\right) = P_\lambda\left(\sup_{t\in(0,k/M)}\left(\left(\lambda_0 - \frac{|\delta^*|}{2}\right)Lt\right) - N(0,t] > x\right)$$
$$= \mathbb{P}\left(\sup_{t\in(0,k/M)}\left(\left(\lambda_0 - \frac{|\delta^*|}{2}\right)Lt\right) - N^{\lambda_0}(0,t] > x\right)$$
$$\leq \mathbb{P}\left(\inf_{t\geq 0}\left(N^{\lambda_0}(0,t] - \left(\lambda_0 - \frac{|\delta^*|}{2}\right)Lt\right) < -x\right).$$

Theorem 3 and equation (15) in [Pyke, 1959] yield for all λ in H_k, $P_\lambda\left(S_{\delta^*,k} > x\right) \leq \exp\left(-\omega x\right)$ where ω is the largest real root of the equation $\lambda_0(1-e^{-\omega}) = \omega\left(\lambda_0 - |\delta^*|/2\right)$. A straightforward study of the function $x \mapsto \lambda_0(1-e^{-x}) - x\left(\lambda_0 - |\delta^*|/2\right)$ on $\mathbb{R}$ therefore ensures that ω satisfies $\omega > \log\left(\lambda_0/(\lambda_0 - |\delta^*|/2)\right)$. If we assume the inequality $x \geq -\log\alpha/\log\left(\lambda_0/(\lambda_0 - |\delta^*|/2)\right)$, then $\sup_{\lambda\in H_k} P_\lambda\left(S_{\delta^*,k} > x\right) \leq \alpha$ and the second part of (27) holds by definition of the quantile $s_{\delta^*,k}(1-\alpha)$. □

Let us turn back to the proof of Theorem 1 and first recall that for λ in $\mathcal{S}[\lambda_0, \delta^*]$, $\mathcal{T}(\lambda) = \{H_k \in \mathcal{H}_{M,\delta^*}\ ,\ \lambda \in H_k\}$ is the set of true hypotheses.

Proof of Theorem 1. For all k in $\{1, \ldots, M\}$, recall that H_k stands for $H_k[\lambda_0, \delta^*]$. We start with the control of FWER($\mathcal{R}_1$) over $\overline{\mathcal{S}}[\lambda_0, \delta^*]$, and for λ in $\overline{\mathcal{S}}[\lambda_0, \delta^*]$ we compute to this end

$$P_\lambda\left(\mathcal{R}_1 \cap \mathcal{T}(\lambda) \neq \varnothing\right) = P_\lambda\left(\exists k \in \{1, \ldots, \lfloor \tau M \rfloor\},\ k \geq \hat{k}_1 + 1,\ \lambda \in H_k\right)$$

because λ belongs to $H_{\lfloor \tau M \rfloor}$ and not to $H_{\lfloor \tau M \rfloor + 1}$. If $\tau < 1/M$ then $P_\lambda\left(\mathcal{R}_1 \cap \mathcal{T}(\lambda) \neq \varnothing\right) = 0$, and if $\tau \geq 1/M$ one has

$$\begin{aligned}P_\lambda\left(\mathcal{R}_1 \cap \mathcal{T}(\lambda) \neq \varnothing\right) &= P_\lambda(\hat{k}_1 + 1 \leq \lfloor \tau M \rfloor)\\ &= P_\lambda\left(\phi_{1,\lfloor \tau M \rfloor} = 1\right)\\ &= P_\lambda\left(S_{\delta^*,\lfloor \tau M \rfloor}(\delta^*) > s_{\delta^*,\lfloor \tau M \rfloor}(1-\alpha)\right)\\ &\leq \alpha,\end{aligned}$$

that is $\mathrm{FWER}(\mathcal{R}_1)$ is bounded by α.
Let us compute now an upper bound for $\mathrm{FWSR}_\beta(\mathcal{R}_1, \mathcal{S}[\lambda_0, \delta^*])$. We shall consider in our results the following constant

$$C(\alpha,\beta,\lambda_0,\delta^*) = 2\max\left(\sqrt{|\delta^*|}\left(\sqrt{q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)+\frac{\log(3/\beta)}{\log\left(\frac{\lambda_0+\delta^*}{\lambda_0+\delta^*/2}\right)}}\mathbb{1}_{\delta^*>0}\right.\right.$$
$$\left.\left.+\sqrt{q_{\lambda_0+\delta^*}\left(1-\frac{\beta}{3},\frac{|\delta^*|}{2}\right)+\frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0+\delta^*/2}\right)}}\mathbb{1}_{-\lambda_0<\delta^*<0}\right),\ 2\sqrt{\frac{3(\lambda_0+\delta^*)}{\beta}}\right), \tag{29}$$

where $q_{\lambda_0}(1-\alpha,\delta^*/2)$ for $\delta^*>0$ and $q_{\lambda_0+\delta^*}(1-\beta/3,|\delta^*|/2)$ for $-\lambda_0<\delta^*<0$ are two positive constants defined in Lemma 3. Recall that (4) leads to $\mathrm{FWSR}_\beta(\mathcal{R}_1, \mathcal{S}[\lambda_0, \delta^*]) \leq |\delta^*|$. Now, assume that $L > (C(\alpha,\beta,\lambda_0,\delta^*)/\delta^*)^2$ and let $r>0$ be such that

$$|\delta^*| > r \geq \frac{C(\alpha,\beta,\lambda_0,\delta^*)}{\sqrt{L}}, \tag{30}$$

where $C(\alpha,\beta,\lambda_0,\delta^*)$ is defined by (29).
Recall that for λ in $\mathcal{S}[\lambda_0,\delta^*]$, $\mathcal{F}_r(\lambda) = \{H_k \in \mathcal{H}_{M,\delta^*},\ d_2(\lambda,H_k)\geq r\}$ with $d_2(\lambda,H_k) = |\delta^*|\sqrt{k/M-\tau}\mathbb{1}_{\tau\leq k/M}$, and that to bound $\mathrm{FWSR}_\beta(\mathcal{R}_1, \mathcal{S}[\lambda_0, \delta^*])$ by r, it is sufficient to obtain $P_\lambda\left(\mathcal{F}_r(\lambda)\subset\mathcal{R}_1\right)\geq 1-\beta$ for all λ in $\mathcal{S}[\lambda_0,\delta^*]$.
We consider λ in $\mathcal{S}[\lambda_0,\delta^*]$ of the form $\lambda=\lambda_0+\delta^*\mathbb{1}_{(\tau,1]}$ with τ in $(0,1)$. If $\mathcal{F}_r(\lambda)=\varnothing$, we easily get $P_\lambda\left(\mathcal{F}_r(\lambda)\subset\mathcal{R}_1\right) = 1 \geq 1-\beta$. We therefore assume by now that λ is satisfying $\mathcal{F}_r(\lambda)\neq\varnothing$, and we define

$$k_r = \min\{\tau M < k' \leq M,\ \delta^{*2}\left(k'/M-\tau\right)\geq r^2\}. \tag{31}$$

By virtue of $\{\mathcal{F}_r(\lambda)\subset\mathcal{R}_1\} = \{k_r \geq \hat{k}_1+1\}$, we want to prove the following inequality

$$P_\lambda\left(\hat{k}_1 \geq k_r\right) \leq \beta$$

to obtain the expected result.
First, if $k_r = M$ then

$$\begin{aligned}
P_\lambda\left(\hat{k}_1 \geq k_r\right) &= P_\lambda\left(\hat{k}_1 = M\right)\\
&= P_\lambda\left(\phi_{1,M}=0\right)\\
&= P_\lambda\left(S_{\delta^*,M}\leq s_{\delta^*,M}(1-\alpha)\right)\\
&= P_\lambda\left(\sup_{t\in(0,1)}\left(\mathrm{sgn}(\delta^*)\left(N\left(t,1\right]-\lambda_0 L\left(1-t\right)\right)-\frac{|\delta^*|}{2}L\left(1-t\right)\right)\leq s_{\delta^*,M}(1-\alpha)\right).
\end{aligned}$$

By definition of k_r, since the condition (30) ensures in particular that

$$|\delta^*|\sqrt{1-\tau} \geq \frac{2}{\sqrt{L}}\max\left(\sqrt{\delta^* q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)}\mathbb{1}_{\delta*>0}+\sqrt{\frac{|\delta^*|\log(1/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0+\delta^*/2}\right)}}\mathbb{1}_{-\lambda_0<\delta^*<0}\ ,\ 2\sqrt{\frac{\lambda_0+\delta^*}{\beta}}\right),$$

we immediately obtain that

$$P_\lambda\left(\sup_{t\in(0,1)}\left(\operatorname{sgn}(\delta^*)\left(N\left(t,1\right]-\lambda_0 L\left(1-t\right)\right)-\frac{|\delta^*|}{2}L\left(1-t\right)\right)\le s_{\delta^*,M}(1-\alpha)\right)\le\beta,$$

according to the minimax study of the change-point detection done in [Fromont et al., 2022, Proposition 15] which involves the statistic $\sup_{t\in(0,1)}\left(\operatorname{sgn}(\delta^*)\left(N\left(t,1\right]-\lambda_0 L\left(1-t\right)\right)-|\delta^*|L(1-t)/2\right)$.

Assume by now that $k_r\le M-1$, and we compute

$$\begin{aligned}
&P_\lambda\left(\hat{k}_1\ge k_r\right) &(32)\\
&=P_\lambda\left(\exists k\ge k_r,\phi_{1,k}=0\right)\\
&=P_\lambda\left(\exists k\ge k_r,\ S_{\delta^*,k}\le s_{\delta^*,k}(1-\alpha)\right)\\
&=P_\lambda\left(\exists k\ge k_r,\ \sup_{t\in(0,k/M)}\left(\operatorname{sgn}(\delta^*)\left(N\left(t,\frac{k}{M}\right]-\lambda_0 L\left(\frac{k}{M}-t\right)\right)-\frac{|\delta^*|}{2}L\left(\frac{k}{M}-t\right)\right)\le s_{\delta^*,k}(1-\alpha)\right)\\
&\le P_\lambda\left(\exists k\ge k_r,\ \operatorname{sgn}(\delta^*)\left(N\left(\tau,\frac{k}{M}\right]-\lambda_0 L\left(\frac{k}{M}-\tau\right)\right)-\frac{|\delta^*|}{2}L\left(\frac{k}{M}-\tau\right)\le s_{\delta^*,k}(1-\alpha)\right). &(33)
\end{aligned}$$

Assume first that $\delta^*>0$.
We use (27) in Lemma 11 to get

$$\begin{aligned}
P_\lambda\left(\hat{k}_1\ge k_r\right)&\le P_\lambda\left(\exists k\ge k_r,\ N\left(\tau,\frac{k}{M}\right]-\left(\lambda_0+\frac{\delta^*}{2}\right)L\left(\frac{k}{M}-\tau\right)\le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)\right)\\
&=P_\lambda\left(\inf_{k\in[k_r,M]}\left(N\left(\tau,\frac{k}{M}\right]-\left(\lambda_0+\frac{\delta^*}{2}\right)L\left(\frac{k}{M}-\tau\right)\right)\le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)\right)\\
&\le P_\lambda\left(\inf_{s\in[k_r/M,1]}\left(N\left(\tau,s\right]-\left(\lambda_0+\frac{\delta^*}{2}\right)L\left(s-\tau\right)\right)\le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)\right)\\
&=P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0+\frac{\delta^*}{2}\right)L\left(\frac{k_r}{M}-\tau\right)+\inf_{s\in(k_r/M,1]}Z_s\le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)\right)\\
&=P_\lambda\left(\inf_{s\in(k_r/M,1]}Z_s\le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)-N\left(\tau,\frac{k_r}{M}\right]+\left(\lambda_0+\frac{\delta^*}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\right).
\end{aligned}$$

where $Z_t=N\left(k_r/M,t\right]-\left(\lambda_0+\delta^*/2\right)\left(t-k_r/M\right)L$ for t in $(k_r/M,1]$. Let us write J for the interval

$$J=\left[(\lambda_0+\delta^*)\left(\frac{k_r}{M}-\tau\right)L\pm\sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}}\right]. \tag{34}$$

Using the total probability formula, we get

$$
\begin{aligned}
&P_\lambda\left(\hat{k}_1 \geq k_r\right) \\
&\leq P_\lambda\left(\inf_{s\in(k_r/M,1]} Z_s \leq q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - N\left(\tau,\frac{k_r}{M}\right] + \left(\lambda_0+\frac{\delta^*}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right) \\
&\quad + P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right] < (\lambda_0+\delta^*)\left(\frac{k_r}{M}-\tau\right)L - \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}}\right) \\
&\quad + P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right] > (\lambda_0+\delta^*)\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}}\right) \\
&\leq P_\lambda\left(\inf_{s\in(k_r/M,1]} Z_s \leq q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - N\left(\tau,\frac{k_r}{M}\right] + \left(\lambda_0+\frac{\delta^*}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right) \\
&\quad + \frac{2\beta}{3}
\end{aligned}
\tag{35}
$$

with the Bienayme-Chebyshev inequality. To compute this last probability, we consider a simple Poisson process $(N_t^{\lambda_0+\delta^*})_{t\geq 0}$ of intensity $(\lambda_0+\delta^*)L$ with respect to the Lebesgue measure on $\mathbb{R}^+$, which is the distribution of N_t for t greater than k_r/M. We then obtain

$$
\begin{aligned}
&P_\lambda\left(\inf_{s\in(k_r/M,1]} Z_s \leq q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - N\left(\tau,\frac{k_r}{M}\right] + \left(\lambda_0+\frac{\delta^*}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right) \\
&\leq P_\lambda\left(\inf_{s\in(k_r/M,1]} Z_s \leq q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}}\right) \\
&= \mathbb{P}\left(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta^*}(0,t] - \left(\lambda_0+\frac{\delta^*}{2}\right)Lt\right) \leq q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L \right.\\
&\qquad\qquad \left. + \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}}\right).
\end{aligned}
$$

By definition of k_r in (31), the condition (30) gives with (29)

$$
\delta^*\sqrt{\frac{k_r}{M}-\tau} \geq \frac{2}{\sqrt{L}}\max\left(\sqrt{\delta^*}\sqrt{q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) + \frac{\log(3/\beta)}{\log\left(\frac{\lambda_0+\delta^*}{\lambda_0+\delta^*/2}\right)}}\ ,\ 2\sqrt{3}\sqrt{\frac{\lambda_0+\delta^*}{\beta}}\right).
\tag{36}
$$

On the one hand, (36) leads to

$$
\delta^*\sqrt{k_r/M-\tau} \geq 2\sqrt{\delta^*/L}\sqrt{q_{\lambda_0}\left(1-\alpha,\delta^*/2\right) + \log(3/\beta)/\log\left(\left(\lambda_0+\delta^*\right)/\left(\lambda_0+\delta^*/2\right)\right)},
$$

and then $\delta^*\left(k_r/M-\tau\right)L \geq 4\left(q_{\lambda_0}\left(1-\alpha,\delta^*/2\right) + \log(3/\beta)/\log\left(\left(\lambda_0+\delta^*\right)/\left(\lambda_0+\delta^*/2\right)\right)\right)$. On the other hand, (36) yields also $\delta^*\sqrt{k_r/M-\tau} \geq 4\sqrt{3\left(\lambda_0+\delta^*\right)/(\beta L)}$ and then $\delta^*\left(k_r/M-\tau\right)L \geq 4\sqrt{3\left(\lambda_0+\delta^*\right)\left(k_r/M-\tau\right)L/\beta}$. This gives

$$
\delta^*\left(\frac{k_r}{M}-\tau\right)L \geq 4\max\left(q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) + \frac{\log(3/\beta)}{\log\left(\frac{\lambda_0+\delta^*}{\lambda_0+\delta^*/2}\right)}\ ,\ \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}}\right),
$$

and using the fact that $a+b \leq 2\max(a,b)$ for all $a,b \geq 0$, one obtains

$$q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}} \leq \frac{\log(\beta/3)}{\log\left(\frac{\lambda_0+\delta^*}{\lambda_0+\delta^*/2}\right)}, \quad (37)$$

which is equivalent to

$$\exp\left(\left(q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}}\right)\log\left(\frac{\lambda_0+\delta^*}{\lambda_0+\delta^*/2}\right)\right) \leq \frac{\beta}{3}. \quad (38)$$

Notice that since $\beta < 1$, (37) ensures

$$q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}} \leq 0. \quad (39)$$

We therefore may apply Theorem 3 and equation (15) in [Pyke, 1959] to obtain

$$\mathbb{P}\left(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta^*}(0,t] - \left(\lambda_0+\frac{\delta^*}{2}\right)Lt\right) \leq q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}}\right)$$
$$\leq \exp\left(\left(q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}}\right)\omega\right),$$

where ω is the largest real root of the equation $(\lambda_0+\delta^*)(1-e^{-\omega}) = \omega\left(\lambda_0+\delta^*/2\right)$. The root ω satisfies $\omega > \log\left((\lambda_0+\delta^*)/(\lambda_0+\delta^*/2)\right)$, and then

$$\mathbb{P}\left(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta^*}(0,t] - \left(\lambda_0+\frac{\delta^*}{2}\right)Lt\right) \leq q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}}\right)$$
$$\leq \exp\left(\left(q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}}\right)\log\left(\frac{\lambda_0+\delta^*}{\lambda_0+\delta^*/2}\right)\right),$$

which entails with (38)

$$\mathbb{P}\left(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta^*}(0,t] - \left(\lambda_0+\frac{\delta^*}{2}\right)Lt\right) \leq q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)\left(k_r/M-\tau\right)L}{\beta}}\right) \leq \frac{\beta}{3}.$$

Gathering this inequality with (35) leads finally to $P_\lambda\left(\hat{k}_1 \geq k_r\right) \leq \beta$.

Now, assume that $-\lambda_0 < \delta^* < 0$.
We proceed as in the case $\delta^* > 0$. Let us define to this end $X_t = N\,(k_r/M, t] - (\lambda_0 - |\delta^*|/2)\,(t - k_r/M)\,L$ for all t in $(k_r/M, 1]$. Applying the inequalities (27) and (32), we get

$$P_\lambda\left(\hat{k}_1 \geq k_r\right) \leq P_\lambda\left(\exists k \geq k_r,\ \left(\lambda_0 - \frac{|\delta^*|}{2}\right) L \left(\frac{k}{M} - \tau\right) - N\left(\tau, \frac{k}{M}\right] \leq \frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0 - |\delta^*|/2}\right)}\right)$$

$$\leq P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s) \leq \frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0 - |\delta^*|/2}\right)} + N\left(\tau, \frac{k_r}{M}\right] - \left(\lambda_0 - \frac{|\delta^*|}{2}\right)\left(\frac{k_r}{M} - \tau\right) L\right).$$

Using the interval J defined by (34), we obtain using the total probability formula and the Bienayme-Chebyshev inequality

$$\begin{aligned}
&P_\lambda\left(\hat{k}_1 \geq k_r\right) \\
&\leq P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s) \leq \frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0 - |\delta^*|/2}\right)} + N\left(\tau, \frac{k_r}{M}\right] - \left(\lambda_0 - \frac{|\delta^*|}{2}\right)\left(\frac{k_r}{M} - \tau\right) L\ ,\ N\left(\tau, \frac{k_r}{M}\right] \in J\right) \\
&\quad + P_\lambda\left(N\left(\tau, \frac{k_r}{M}\right] < (\lambda_0 + \delta^*)\left(\frac{k_r}{M} - \tau\right) L - \sqrt{\frac{3(\lambda_0 + \delta^*)\,(k_r/M - \tau)\,L}{\beta}}\right) \\
&\quad + P_\lambda\left(N\left(\tau, \frac{k_r}{M}\right] > (\lambda_0 + \delta^*)\left(\frac{k_r}{M} - \tau\right) L + \sqrt{\frac{3(\lambda_0 + \delta^*)\,(k_r/M - \tau)\,L}{\beta}}\right) \\
&\leq P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s) \leq \frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0 - |\delta^*|/2}\right)} + N\left(\tau, \frac{k_r}{M}\right] - \left(\lambda_0 - \frac{|\delta^*|}{2}\right)\left(\frac{k_r}{M} - \tau\right) L\ ,\ N\left(\tau, \frac{k_r}{M}\right] \in J\right) \\
&\quad + \frac{2\beta}{3}.
\end{aligned} \tag{40}$$

We conclude the proof giving an upper bound for this last probability. We compute

$$P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s) \le \frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)} + N\left(\tau,\frac{k_r}{M}\right] - \left(\lambda_0-\frac{|\delta^*|}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right)$$

$$\le P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s) \le \frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)} - \frac{|\delta^*|}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)\,(k_r/M-\tau)\,L}{\beta}}\right)$$

$$= P_\lambda\left(\sup_{s\in(k_r/M,1]} X_s \ge \frac{\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)} + \frac{|\delta^*|}{2}\left(\frac{k_r}{M}-\tau\right)L - \sqrt{\frac{3(\lambda_0+\delta^*)\,(k_r/M-\tau)\,L}{\beta}}\right)$$

$$\le \mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta^*}(0,t] - \left(\lambda_0-\frac{|\delta^*|}{2}\right)Lt\right) \ge \frac{\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)} + \frac{|\delta^*|}{2}\left(\frac{k_r}{M}-\tau\right)L - \sqrt{\frac{3(\lambda_0+\delta^*)\,(k_r/M-\tau)\,L}{\beta}}\right), \tag{41}$$

By definition of k_r in (31), the condition (30) gives with (29),

$$|\delta^*|\sqrt{\frac{k_r}{M}-\tau} \ge \frac{2}{\sqrt{L}}\max\left(\sqrt{|\delta^*|}\sqrt{q_{\lambda_0+\delta^*}\left(1-\frac{\beta}{3},\frac{|\delta^*|}{2}\right) + \frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)}}\ ,\ 2\sqrt{\frac{3(\lambda_0+\delta^*)}{\beta}}\right),$$

which ensures, as in the case $\delta^*>0$, that

$$\frac{\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)} + \frac{|\delta^*|}{2}\left(\frac{k_r}{M}-\tau\right)L - \sqrt{\frac{3(\lambda_0+\delta^*)\,(k_r/M-\tau)\,L}{\beta}} \ge q_{\lambda_0+\delta^*}\left(1-\frac{\beta}{3},\frac{|\delta^*|}{2}\right).$$

We therefore obtain with (41)

$$\mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta^*}(0,t] - \left(\lambda_0-\frac{|\delta^*|}{2}\right)Lt\right) \ge \frac{\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)} + \frac{|\delta^*|}{2}\left(\frac{k_r}{M}-\tau\right)L - \sqrt{\frac{3(\lambda_0+\delta^*)\,(k_r/M-\tau)\,L}{\beta}}\right)$$

$$\le \mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta^*}(0,t] - \left(\lambda_0-\frac{|\delta^*|}{2}\right)Lt\right) \ge q_{\lambda_0+\delta^*}\left(1-\frac{\beta}{3},\frac{|\delta^*|}{2}\right)\right) \le \frac{\beta}{3}$$

by definition of $q_{\lambda_0+\delta^*}(1-\beta/3,|\delta^*|/2)$ in Lemma 3. The proof is then complete using (40).

□

5.2 Proof of Theorem 2

For $\lambda_0 > 0$ and $R > \lambda_0$, the simple hypothesis $H_k[\lambda_0, R]$ is denoted H_k for short and recall that u_α stands for $\alpha/\lfloor \log_2 L \rfloor$. We begin by giving an upper bound for the u-quantile $p_\xi(u)$ of the Poisson distribution of parameter $\xi > 0$. The proof, based on the application of the Cramér-Chernov inequality to a Poisson random variable with parameter ξ, can be found in [Fromont et al., 2022, Lemma 46].

Lemma 12 (Quantile bounds for the Poisson distribution)**.** *The u-quantile $p_\xi(u)$ of the Poisson distribution with parameter ξ satisfies*

$$\left(\xi - \xi g^{-1}\left(\frac{\log(1/u)}{\xi}\right)\right) \vee 0 \leq p_\xi(u) \leq \xi + \xi g^{-1}\left(\frac{\log(1/(1-u))}{\xi}\right). \tag{42}$$

For τ in $(0,1)$, let us define now $k_\tau = \min\{k' \in \{1,\ldots,M\},\ k' > \tau M\}$ and

$$j_\tau(k) = \left\lceil -\log_2\left(1 - \frac{\tau M}{k}\right)\right\rceil \wedge \lfloor \log_2(L) \rfloor \tag{43}$$

for $k \geq k_\tau$, in order to get $j_\tau(k)$ in $\{1,\ldots,\lfloor \log_2(L)\rfloor\}$ as well as the inequalities

$$\frac{k}{M}\left(1 - \frac{1}{2^{j_\tau(k)-1}}\right) < \tau \leq \frac{k}{M}\left(1 - \frac{1}{2^{j_\tau(k)}}\right) \tag{44}$$

under the following condition

$$\left\lceil -\log_2\left(1 - \frac{\tau M}{k}\right)\right\rceil \leq \lfloor \log_2(L) \rfloor. \tag{45}$$

We shall consider also a cover of the interval $[k_\tau, M]$, denoted $\mathcal{P} = \cup_{i=0}^{\Psi-1}[x_i, x_{i+1})$ where Ψ in $\mathbb{N}^*$ is the cardinal of the cover $\mathcal{P}$ and the real $x_0 < \ldots < x_\Psi$ satisfy $x_0 = k_\tau$ and $x_\Psi > M$. Note that the two real x_i and x_{i+1} may depend on λ_0, τ, L, M and k_τ. We will write $\sum_{[c,d)\in\mathcal{P}}$ for the sum over each disjoint interval $[c,d)$ of the cover $\mathcal{P}$. Assuming (45), the key argument in the proof of Theorem 2 will consist on giving an upper bound for the probabilities $P_\lambda(\sup_{k\in\mathcal{P}}(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2) - N(k(1-2^{-j_\tau(k)})/M, k/M]) \geq 0)$ and $P_\lambda(\sup_{k\in\mathcal{P}}(N(k(1-2^{-j_\tau(k)})/M, k/M] - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2) \geq 0)$, instead of the probabilities $P_\lambda(\sup_{k\geq k_\tau}(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2) - N(k(1-2^{-j_\tau(k)})/M, k/M]) \geq 0)$ and $P_\lambda(\sup_{k\geq k_\tau}(N(k(1-2^{-j_\tau(k)})/M, k/M] - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2) \geq 0)$ in order to get more refined bounds. The following technical lemmas allow us to bound $P_\lambda(\sup_{k\in[c,d)}(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2) - N(k(1-2^{-j_\tau(k)})/M, k/M]) \geq 0)$ and $P_\lambda(\sup_{k\in[c,d)}(N(k(1-2^{-j_\tau(k)})/M, k/M] - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2)) \geq 0)$ for each interval $[c,d)$ of the cover $\mathcal{P}$, using an exponential inequality related to the oscillation modulus of some martingales developed in [Le Guével, 2021, Theorem 4].

Lemma 13. *Let $L \geq 3$, M in $\mathbb{N}^*$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$ and $R > \lambda_0$. For λ in $\mathcal{S}[\lambda_0, R]$, assume* (45) *and that the following inequality holds for all interval $[c,d)$ of the cover $\mathcal{P}$:*

$$\inf_{k\in[c,d)}\left(\frac{Lk}{M2^{j_\tau(k)}}\left(|\delta| - \lambda_0 g^{-1}\left(\frac{\log(2/u_\alpha)M2^{j_\tau(k)}}{\lambda_0 kL}\right)\right)\right) \geq 2\left(\frac{d}{M} - \tau\right)L(\lambda_0+\delta)g^{-1}\left(\frac{\log(2\Psi/\beta)}{(d/M-\tau)L(\lambda_0+\delta)}\right), \tag{46}$$

where the function g is defined by (25)*. Then, when $\delta > 0$*

$$\sum_{[c,d)\in\mathcal{P}} P_\lambda\left(\sup_{k\in[c,d)}\left(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2) - N(k(1-2^{-j_\tau(k)})/M, k/M]\right)\geq 0\right)\leq\beta,$$

and when $\delta < 0$

$$\sum_{[c,d)\in\mathcal{P}} P_\lambda\left(\sup_{k\in[c,d)}\left(N(k(1-2^{-j_\tau(k)})/M, k/M] - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2)\right)\geq 0\right)\leq\beta.$$

Proof of Lemma 13. Let λ in $\mathcal{S}[\lambda_0, R]$ be such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau,1]}$, where δ in $(-\lambda_0, R-\lambda_0]\setminus\{0\}$ and τ in $(0,1)$. Notice that for all k in $\{1,\ldots,M\}$ and j in $\{1,\ldots,\lfloor\log_2 L\rfloor\}$, the counting statistic $N(k(1-2^{-j})/M, k/M]$ can be written

$$N(k(1-2^{-j})/M, k/M] = M_{k(1-2^{-j})/M}^{k/M} + B_{k(1-2^{-j})/M,k/M}, \tag{47}$$

where for all $0\leq s<t$, $M_s^t = \int_s^t (dN_x - \lambda(x)Ldx)$ and $B_{s,t} = \int_s^t \lambda(x)Ldx$ is the bias of $N(s,t]$.

Assume first that $0<\delta\leq R-\lambda_0$. For $[c,d)$ in $\mathcal{P}$ we define

$$\varrho_{c,d}^{(1)} = \inf_{k\in[c,d)}\left(B_{k(1-2^{-j_\tau(k)})/M,k/M} - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2)\right),$$

and we prove that

$$\varrho_{c,d}^{(1)} \geq \inf_{k\in[c,d)}\left(\frac{Lk}{M2^{j_\tau(k)}}\left(\delta - \lambda_0 g^{-1}\left(\frac{\log(2/u_\alpha)M2^{j_\tau(k)}}{\lambda_0 kL}\right)\right)\right). \tag{48}$$

Let k in $[c,d)$ and we compute

$$\begin{aligned}
&B_{k(1-2^{-j_\tau(k)})/M,k/M} - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2)\\
&= (\lambda_0+\delta)L\frac{k}{M2^{j_\tau(k)}} - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2) \text{ with (44)}\\
&\geq \frac{Lk}{M2^{j_\tau(k)}}\left(\delta - \lambda_0 g^{-1}\left(\frac{\log(2/u_\alpha)M2^{M2^{j_\tau(k)}}}{\lambda_0 kL}\right)\right) \quad \text{with (42)},
\end{aligned}$$

that is (48). Notice in particular that the condition (46) ensures that for all k in $[c,d)$,

$$B_{k(1-2^{-j_\tau(k)})/M,k/M} - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2)\geq 0. \tag{49}$$

Using the condition (46) again we obtain

$$\varrho_{c,d}^{(1)}\geq 2\left(\frac{d}{M}-\tau\right)L(\lambda_0+\delta)g^{-1}\left(\frac{\log(2\Psi/\beta)}{(d/M-\tau)L(\lambda_0+\delta)}\right),$$

which is equivalent to

$$2\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg\left(\frac{\varrho_{c,d}^{(1)}}{2\left(d/M-\tau\right)L(\lambda_0+\delta)}\right)\right)\leq\frac{\beta}{\Psi}. \tag{50}$$

Finally, we get the following inequalities

$$
\begin{aligned}
&P_\lambda\left(\sup_{k\in[c,d)}\left(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2)-N(k(1-2^{-j_\tau(k)})/M,k/M]\right)\geq 0\right)\\
&\quad = P_\lambda\left(\sup_{k\in[c,d)}\left(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2)-B_{k(1-2^{-j_\tau(k)})/M,k/M}-M^{k/M}_{k(1-2^{-j_\tau(k)})/M}\right)\geq 0\right)\ \text{with (47)}\\
&\quad \leq P_\lambda\left(\exists k\in[c,d),\ |M^{k/M}_{k(1-2^{-j_\tau(k)})/M}|\geq B_{k(1-2^{-j_\tau(k)})/M,k/M}-p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2)\right)\ \text{with (49)}\\
&\quad \leq P_\lambda\left(\exists k\in[c,d),\ |M^{k/M}_{k(1-2^{-j_\tau(k)})/M}|\geq \varrho^{(1)}_{c,d}\right)\\
&\quad \leq P_\lambda\left(\sup_{s,t\in[\tau,d/M]}|M^t_s|\geq \varrho^{(1)}_{c,d}\right)\ \text{with (44)},
\end{aligned}
$$

and Theorem 4 in [Le Guével, 2021] leads to

$$
P_\lambda\left(\sup_{s,t\in[\tau,d/M]}|M^t_s|\geq \varrho^{(1)}_{c,d}\right)\leq 2\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg\left(\frac{\varrho^{(1)}_{c,d}}{2\,(d/M-\tau)\,L(\lambda_0+\delta)}\right)\right).
$$

Therefore (50) yields

$$
P_\lambda\left(\sup_{k\in[c,d)}\left(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2)-N(k(1-2^{-j_\tau(k)})/M,k/M]\right)\geq 0\right)\leq\frac{\beta}{\Psi},
$$

and the result follows summing over all the interval of the cover $\mathcal{P}$.
Assume now that $-\lambda_0<\delta<0$ and define for $[c,d)$ in $\mathcal{P}$

$$
\varrho^{(2)}_{c,d}=\inf_{k\in[c,d)}\left(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2)-B_{k(1-2^{-j_\tau(k)})/M,k/M}\right).
$$

As for the case where $\delta>0$, the equations (42) and (44) entails that for all k in $[c,d)$

$$
p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2)-B_{k(1-2^{-j_\tau(k)})/M,k/M}\geq\frac{Lk}{M2^{j_\tau(k)}}\left(|\delta|-\lambda_0 g^{-1}\left(\frac{\log(2/u_\alpha)M2^{j_\tau(k)}}{\lambda_0 kL}\right)\right),
$$

and we get

$$
\varrho^{(2)}_{c,d}\geq\inf_{k\in[c,d)}\left(\frac{Lk}{M2^{j_\tau(k)}}\left(|\delta|-\lambda_0 g^{-1}\left(\frac{\log(2/u_\alpha)M2^{j_\tau(k)}}{\lambda_0 kL}\right)\right)\right). \tag{51}
$$

The condition (46) ensures on the one hand that for all k in $[c,d)$

$$
p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2)-B_{k(1-2^{-j_\tau(k)})/M,k/M}\geq 0, \tag{52}
$$

and on the other hand that

$$
2\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg\left(\frac{\varrho^{(2)}_{c,d}}{2\,(d/M-\tau)\,L(\lambda_0+\delta)}\right)\right)\leq\frac{\beta}{\Psi}. \tag{53}
$$

Finally, we get the following inequalities

$$
\begin{aligned}
&P_\lambda\left(\sup_{k\in[c,d)}\left(N(k(1-2^{-j_\tau(k)})/M, k/M] - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2)\right)\geq 0\right)\\
&\quad\leq P_\lambda\left(\exists k\in[c,d),\ |M^{k/M}_{k(1-2^{-j_\tau(k)})/M}|\geq p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2) - B_{k(1-2^{-j_\tau(k)})/M,k/M}\right)\ \text{with (52)}\\
&\quad\leq P_\lambda\left(\exists k\in[c,d),\ |M^{k/M}_{k(1-2^{-j_\tau(k)})/M}|\geq \varrho^{(2)}_{c,d}\right)\\
&\quad\leq P_\lambda\left(\sup_{s,t\in[\tau,d/M]}|M^t_s|\geq \varrho^{(2)}_{c,d}\right)\ \text{with (44)},
\end{aligned}
$$

and we conclude with the same lines as above using the Theorem 4 in [Le Guével, 2021], the inequality (53) and summing over all the interval of the cover $\mathcal{P}$. □

Lemma 14. *Let $L\geq 3$, M in $\mathbb{N}^*$, α and β be fixed levels in $(0,1)$, $\lambda_0>0$ and $R>\lambda_0$. For λ in $\mathcal{S}[\lambda_0,R]$, assume (45) and that for all interval $[c,d)$ of the cover $\mathcal{P}$ one has*

$$\frac{d/M-\tau}{c/M-\tau}=2, \tag{54}$$

and

$$|\delta|\sqrt{\frac{c}{M}-\tau}\geq 2\max\left(\sqrt{\frac{2R}{3}}\sqrt{\frac{\log(2/u_\alpha)}{L}+\frac{2\log(2\Psi/\beta)}{L}}\ ,\ 4\sqrt{\frac{\lambda_0\log(2/u_\alpha)}{L}}+8\sqrt{\frac{R\log(2\Psi/\beta)}{L}}\right). \tag{55}$$

Then the inequality (46) is satisfied.

Proof of Lemma 14 . Let λ in $\mathcal{S}[\lambda_0,R]$ such that $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,1]}$ where δ in $(-\lambda_0,R-\lambda_0]\setminus\{0\}$ and τ in $(0,1)$. Assume first that $0<\delta\leq R-\lambda_0$ and let $[c,d)$ in $\mathcal{P}$. Notice that (55) entails on the one hand

$$\delta\sqrt{\frac{c}{M}-\tau}\geq 8\left(\sqrt{\frac{\lambda_0\log(2/u_\alpha)}{L}}+2\sqrt{\frac{R\log(2\Psi/\beta)}{L}}\right),$$

and then, since $\lambda_0+\delta\leq R$,

$$\delta\left(\frac{c}{M}-\tau\right)L\geq 8\sqrt{\left(\frac{c}{M}-\tau\right)L}\left(\sqrt{\lambda_0\log\left(\frac{2}{u_\alpha}\right)}+2\sqrt{(\lambda_0+\delta)\log\left(\frac{2\Psi}{\beta}\right)}\right),$$

hence, using (54),

$$\delta\left(\frac{c}{M}-\tau\right)L\geq 4\sqrt{2\frac{d/M-\tau}{c/M-\tau}\left(\frac{c}{M}-\tau\right)L}\left(\sqrt{\lambda_0\log\left(\frac{2}{u_\alpha}\right)}+2\sqrt{(\lambda_0+\delta)\log\left(\frac{2\Psi}{\beta}\right)}\right). \tag{56}$$

On the other hand, (55) yields

$$\delta\sqrt{\frac{c}{M}-\tau} \geq 2\sqrt{\frac{2R}{3}}\sqrt{\frac{\log(2/u_\alpha)}{L}+\frac{2\log(2\Psi/\beta)}{L}},$$

and then, since $\delta < R$,

$$\delta^2\left(\frac{c}{M}-\tau\right) \geq \frac{8\delta}{3}\left(\frac{\log(2/u_\alpha)}{L}+\frac{2\log(2\Psi/\beta)}{L}\right),$$

hence

$$\delta\left(\frac{c}{M}-\tau\right)L \geq \frac{8}{3}\log\left(\frac{2}{u_\alpha}\right)+\frac{16}{3}\log\left(\frac{2\Psi}{\beta}\right). \tag{57}$$

Gathered together, the conditions (56) and (57) give

$$\delta\left(\frac{c}{M}-\tau\right)L \geq 2\max\left(\frac{4}{3}\log\left(\frac{2}{u_\alpha}\right)+\frac{8}{3}\log\left(\frac{2\Psi}{\beta}\right),\right.$$
$$\left.2\sqrt{2\left(\frac{d}{M}-\tau\right)L}\left(\sqrt{\lambda_0\log\left(\frac{2}{u_\alpha}\right)}+2\sqrt{(\lambda_0+\delta)\log\left(\frac{2\Psi}{\beta}\right)}\right)\right),$$

and since $2\max(a,b) \geq a+b$ for all $a, b \geq 0$, we get

$$\frac{1}{2}\delta\left(\frac{c}{M}-\tau\right)L \geq \frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)+\frac{4}{3}\log\left(\frac{2\Psi}{\beta}\right)$$
$$+\sqrt{2\left(\frac{d}{M}-\tau\right)L}\left(\sqrt{\lambda_0\log\left(\frac{2}{u_\alpha}\right)}+2\sqrt{(\lambda_0+\delta)\log\left(\frac{2\Psi}{\beta}\right)}\right),$$

hence

$$\frac{1}{2}\delta\left(\frac{c}{M}-\tau\right)L-\frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)-\sqrt{2\lambda_0L\log\left(\frac{2}{u_\alpha}\right)\left(\frac{d}{M}-\tau\right)}$$
$$\geq \frac{4}{3}\log\left(\frac{2\Psi}{\beta}\right)+2\sqrt{2(\lambda_0+\delta)L\log\left(\frac{2\Psi}{\beta}\right)\left(\frac{d}{M}-\tau\right)}. \tag{58}$$

Moreover, the condition (44) ensures that for all k in $[c, d)$,

$$\frac{1}{2}\left(\frac{k}{M}-\tau\right) < \frac{k}{M2^{j_\tau(k)}} \leq \frac{k}{M}-\tau, \tag{59}$$

and we then obtain

$$\frac{1}{2}\delta\left(\frac{c}{M}-\tau\right)L-\frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)-\sqrt{2\lambda_0L\log\left(\frac{2}{u_\alpha}\right)\left(\frac{d}{M}-\tau\right)}$$
$$\leq \frac{1}{2}\delta\left(\frac{k}{M}-\tau\right)L-\frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)-\sqrt{2\lambda_0L\log\left(\frac{2}{u_\alpha}\right)\left(\frac{k}{M}-\tau\right)}$$
$$\leq \frac{\delta Lk}{M2^{j_\tau(k)}}-\frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)-\sqrt{2\lambda_0\frac{Lk}{M2^{j_\tau(k)}}\log\left(\frac{2}{u_\alpha}\right)} \text{ with (59).}$$

Using a lower bound for g^{-1} recalled in (26), we finally obtain for all k in $[c,d)$,

$$\frac{1}{2}\delta\left(\frac{c}{M}-\tau\right)L-\frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)-\sqrt{2\lambda_0 L\log\left(\frac{2}{u_\alpha}\right)\left(\frac{d}{M}-\tau\right)} \leq \frac{Lk}{M2^{j_\tau(k)}}\left(\delta-\lambda_0 g^{-1}\left(\frac{\log(2/u_\alpha)M2^{j_\tau(k)}}{\lambda_0 kL}\right)\right),$$

hence

$$\frac{1}{2}\delta\left(\frac{c}{M}-\tau\right)L-\frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)-\sqrt{2\lambda_0 L\log\left(\frac{2}{u_\alpha}\right)\left(\frac{d}{M}-\tau\right)} \leq \inf_{k\in[c,d)}\left(\frac{Lk}{M2^{j_\tau(k)}}\left(\delta-\lambda_0 g^{-1}\left(\frac{\log(2/u_\alpha)M2^{j_\tau(k)}}{\lambda_0 kL}\right)\right)\right). \quad (60)$$

Using (26) again, we get

$$\frac{4}{3}\log\left(\frac{2\Psi}{\beta}\right)+2\sqrt{2(\lambda_0+\delta)L\log\left(\frac{2\Psi}{\beta}\right)\left(\frac{d}{M}-\tau\right)} \geq 2\left(\frac{d}{M}-\tau\right)L(\lambda_0+\delta)g^{-1}\left(\frac{\log(2\Psi/\beta)}{(d/M-\tau)L(\lambda_0+\delta)}\right). \quad (61)$$

Finally, (60) and (61) combined with (58) lead to the expected result.
If we assume that $-\lambda_0<\delta<0$, the proof follows the same lines as above just replacing δ by $|\delta|$ except when it is involved in $\lambda_0+\delta$. □

Let us turn back now to the proof of Theorem 2 and first recall that for all λ in $\mathcal{S}[\lambda_0,R]$, $\mathcal{T}(\lambda)=\{H_k\in\mathcal{H}_{M,R}\ ,\ \lambda\in H_k\}$ is the set of true hypotheses.

Proof of Theorem 2. Recall that H_k stands for $H_k[\lambda_0,R]$. We begin with the control of $\mathrm{FWER}(\mathcal{R}_2^{(1)})$ over $\overline{\mathcal{S}}[\lambda_0,R]$. To this end, we compute for all λ in $\overline{\mathcal{S}}[\lambda_0,R]$

$$\begin{aligned} P_\lambda\left(\mathcal{R}_2^{(1)}\cap\mathcal{T}(\lambda)\neq\varnothing\right) &= P_\lambda\left(\exists k\in\{1,\dots,M\},\ H_k\in\mathcal{R}_2^{(1)},\ H_k\in\mathcal{T}(\lambda)\right)\\ &= P_\lambda\left(\exists k\in\{1,\dots,\lfloor\tau M\rfloor\},\ k\geq\hat{k}_2^{(1)}+1,\ \lambda\in H_k\right)\end{aligned}$$

because λ belongs to $H_{\lfloor\tau M\rfloor}$ and not to $H_{\lfloor\tau M\rfloor+1}$. If $\tau<1/M$ then $P_\lambda\left(\mathcal{R}_2^{(1)}\cap\mathcal{T}(\lambda)\neq\varnothing\right)=$

$0 \leq \alpha$, and if $\tau \geq 1/M$ one has

$$
\begin{aligned}
&P_\lambda\left(\mathcal{R}_2^{(1)} \cap \mathcal{T}(\lambda) \neq \varnothing\right) \\
&= P_\lambda\left(\hat{k}_2^{(1)} + 1 \leq \lfloor \tau M \rfloor\right) \\
&= P_\lambda\left(\phi^{(1)}_{2,\lfloor \tau M \rfloor} = 1\right) \\
&\leq P_\lambda\left(\exists j \in \{1, \ldots, \lfloor \log_2 L \rfloor\}, N(\lfloor \tau M \rfloor(1-2^{-j})/M, \lfloor \tau M \rfloor/M] > p_{\lambda_0 \lfloor \tau M \rfloor L 2^{-j}/M}(1-u_\alpha/2)\right) \\
&\quad + P_\lambda\left(\exists j \in \{1, \ldots, \lfloor \log_2 L \rfloor\}, N(\lfloor \tau M \rfloor(1-2^{-j})/M, \lfloor \tau M \rfloor/M] < p_{\lambda_0 \lfloor \tau M \rfloor L 2^{-j}/M}(u_\alpha/2)\right) \\
&\leq \sum_{j=1}^{\lfloor \log_2 L \rfloor} P_\lambda\left(N(\lfloor \tau M \rfloor(1-2^{-j})/M, \lfloor \tau M \rfloor/M] > p_{\lambda_0 \lfloor \tau M \rfloor L 2^{-j}/M}(1-u_\alpha/2)\right) \\
&\quad + \sum_{j=1}^{\lfloor \log_2 L \rfloor} P_\lambda\left(N(\lfloor \tau M \rfloor(1-2^{-j})/M, \lfloor \tau M \rfloor/M] < p_{\lambda_0 \lfloor \tau M \rfloor L 2^{-j}/M}(u_\alpha/2)\right) \\
&\leq \sum_{j=1}^{\lfloor \log_2 L \rfloor} u_\alpha/2 + \sum_{j=1}^{\lfloor \log_2 L \rfloor} u_\alpha/2 \leq \alpha,
\end{aligned}
$$

which proves the control of $\text{FWER}(\mathcal{R}_2^{(1)})$ by α.
Let us compute an upper bound for $\text{FWSR}_\beta(\mathcal{R}_2^{(1)}, \mathcal{S}[\lambda_0, R])$. Recall that (10) leads to $\text{FWSR}_\beta(\mathcal{R}_2^{(1)}, \mathcal{S}[\lambda_0, R]) \leq \lambda_0 \vee (R - \lambda_0)$. Now, assume that

$$L > \left(\frac{C(\alpha, \beta, \lambda_0, R, L)}{\lambda_0 \vee (R - \lambda_0)}\right)^2,$$

where

$$
\begin{aligned}
C(\alpha, \beta, \lambda_0, R, L) = 2 \max\Bigg(&\sqrt{\frac{2R}{3}} \sqrt{\log\left(\frac{2\lfloor \log_2 L \rfloor}{\alpha}\right) + 2\log\left(\frac{2\lceil \log_2 L \rceil}{\beta}\right)}, \\
&4\sqrt{\lambda_0 \log\left(\frac{2\lfloor \log_2 L \rfloor}{\alpha}\right)} + 8\sqrt{R \log\left(\frac{2\lceil \log_2 L \rceil}{\beta}\right)}, \\
&2\sqrt{\lambda_0 \log\left(\frac{2\lfloor \log_2 L \rfloor}{\alpha}\right)} + \sqrt{\frac{2R}{\beta}}, \ \frac{R}{\sqrt{2}}, \ \frac{7R}{2}\sqrt{\log\log L}\Bigg),
\end{aligned}
$$

and let $r > 0$ be such that

$$\lambda_0 \vee (R - \lambda_0) > r \geq \frac{C(\alpha, \beta, \lambda_0, R, L)}{\sqrt{L}}. \tag{62}$$

Recall that for λ in $\mathcal{S}[\lambda_0, R]$, $\mathcal{F}_r(\lambda) = \{H_k \in \mathcal{H}_{M,R},\ d_2(\lambda, H_k) \geq r\}$ with $d_2(\lambda, H_k) = |\delta|\sqrt{k/M - \tau}\mathbb{1}_{\tau \leq k/M}$ for all k in $\{1, \ldots, M\}$. To bound $\text{FWSR}_\beta(\mathcal{R}_2^{(1)}, \mathcal{S}[\lambda_0, R])$ by r, it is sufficient to prove that for all λ in $\mathcal{S}[\lambda_0, R]$, $P_\lambda\left(\mathcal{F}_r(\lambda) \subset \mathcal{R}_2^{(1)}\right) \geq 1 - \beta$.
Let us consider then λ in $\mathcal{S}[\lambda_0, R]$ such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau,1]}$ where δ in $(-\lambda_0, R - \lambda_0] \setminus \{0\}$ and τ in $(0, 1)$.

If $\mathcal{F}_r(\lambda) = \varnothing$, we get that $P_\lambda\left(\mathcal{F}_r(\lambda) \subset \mathcal{R}_2^{(1)}\right) = 1 \geq 1-\beta$, so by now we assume that λ is such that $\mathcal{F}_r(\lambda) \neq \varnothing$. We define

$$k_r = \min\{\tau M < k' \leq M,\ \delta^2\left(k'/M - \tau\right) \geq r^2\}, \tag{63}$$

and with the relationship $\{\mathcal{F}_r(\lambda) \subset \mathcal{R}_2^{(1)}\} = \{k_r \geq \hat{k}_2^{(1)}+1\}$, we have to prove the following inequality

$$P_\lambda\left(\hat{k}_2^{(1)} \geq k_r\right) \leq \beta \tag{64}$$

to obtain the expected result.

First, if $k_r = M$ then

$$\begin{aligned}
&P_\lambda\left(\hat{k}_2^{(1)} \geq k_r\right)\\
&= P_\lambda\left(\hat{k}_2^{(1)} = M\right)\\
&= P_\lambda\left(\phi_{2,M}^{(1)} = 0\right)\\
&= P_\lambda\left(\forall j \in \{1,\dots,\lfloor \log_2 L\rfloor\},\ N((1-2^{-j}),1] \leq p_{\lambda_0 L2^{-j}}(1-u_\alpha/2),\ N((1-2^{-j}),1] \geq p_{\lambda_0 L2^{-j}}(u_\alpha/2)\right).
\end{aligned}$$

By definition of k_r, since the condition (62) ensures in particular that

$$|\delta|\sqrt{1-\tau} \geq 2\max\left(\sqrt{\frac{2R\log(2/u_\alpha)}{3L}}\ ,\ 2\sqrt{\frac{\lambda_0\log(2/u_\alpha)}{L}} + \sqrt{\frac{2R}{\beta L}}\ ,\ \frac{R}{\sqrt{2L}}\right),$$

we immediately obtain that

$$P_\lambda\left(\forall j \in \{1,\dots,\lfloor \log_2 L\rfloor\},\ N((1-2^{-j}),1] \leq p_{\lambda_0 L2^{-j}}(1-u_\alpha/2),\ N((1-2^{-j}),1] \geq p_{\lambda_0 L2^{-j}}(u_\alpha/2)\right) \leq \beta,$$

according to the minimax study of the change-point detection done in [Fromont et al., 2022, Proposition 18] which involves the statistic $\max_{j\in\{1,\dots,\lfloor\log_2 L\rfloor\}}((N((1-2^{-j}),1]-p_{\lambda_0 L2^{-j}}(1-u_\alpha/2)) \vee (p_{\lambda_0 L2^{-j}}(u_\alpha/2) - N((1-2^{-j}),1]))$.

Assume by now that $k_r \leq M-1$, and we compute then

$$\begin{aligned}
P_\lambda\left(\hat{k}_2^{(1)} \geq k_r\right) &= P_\lambda\left(\exists k \geq k_r, \phi_{2,k}^{(1)} = 0\right)\\
&= P_\lambda\left(\exists k \geq k_r,\ \forall j \in \{1,\dots,\lfloor \log_2 L\rfloor\};\ N(k(1-2^{-j})/M, k/M] \leq p_{\lambda_0 kL2^{-j}/M}(1-u_\alpha/2),\right.\\
&\quad \left. N(k(1-2^{-j})/M, k/M] \geq p_{\lambda_0 kL2^{-j}/M}(u_\alpha/2)\right).
\end{aligned} \tag{65}$$

Assume first that $0 < \delta \leq R - \lambda_0$. Then (65) ensures that

$$\begin{aligned}
P_\lambda\left(\hat{k}_2^{(1)} \geq k_r\right) &\leq P_\lambda\left(\exists k \geq k_r,\ \forall j \in \{1,\dots,\lfloor \log_2 L\rfloor\};\ N(k(1-2^{-j})/M, k/M] \leq p_{\lambda_0 kL2^{-j}/M}(1-u_\alpha/2)\right)\\
&\leq P_\lambda\left(\exists k \geq k_r,\ N(k(1-2^{-j_\tau(k)})/M, k/M] \leq p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2)\right),
\end{aligned}$$

where $j_\tau(k)$ is defined by (43). The assumption (62) entails that $r \geq 7R\sqrt{\log\log L/L}$ and then for all $k \geq k_r$

$$\delta^2\left(\frac{k}{M} - \tau\right) \geq 49R^2\frac{\log\log L}{L}. \tag{66}$$

Since $\delta < R$, we therefore get $k/M - \tau \geq 49 \log\log L/L$, which leads to the condition (45) for $L \geq 3$. Indeed, for all $L \geq 3$, we have $49 \log\log L/L \geq 2^{-\lfloor \log_2 L \rfloor + 1}$ which entails $k(1 - 2^{-\lfloor \log_2 L \rfloor + 1})/M \geq \tau$. As a consequence, we have $-\log_2(1 - \tau M/k) + 1 \leq \lfloor \log_2 L \rfloor$ which implies (45). Then for all $k \geq k_r$, $j_\tau(k)$ defined by (43) is such that $j_\tau(k)$ belongs to $\{1, \ldots, \lfloor \log_2 L \rfloor)\}$ and satisfies (44).
Consider now the following partition of cardinality $\Psi = \lceil \log_2(L) \rceil$:

$$\bigcup_{i=0}^{\lceil \log_2(L) \rceil - 1} I_{\tau,M,i,k_r} = \bigcup_{i=0}^{\lceil \log_2(L) \rceil - 1} [\tau M + 2^i(k_r - \tau M) \ ; \ \tau M + 2^{i+1}(k_r - \tau M)).$$

The condition (66) yields $1/(k_r/M - \tau) \leq L \leq 2^{\lceil \log_2(L) \rceil}$ and $\tau M + 2^{(\lceil \log_2(L) \rceil - 1) + 1}(k_r - \tau M) > M$, so that

$$[k_r; M] \subset \bigcup_{i=0}^{\lceil \log_2(L) \rceil - 1} I_{\tau,M,i,k_r}.$$

As a consequence

$$\begin{aligned} P_\lambda\left(\hat{k}_2^{(1)} \geq k_r\right) &\leq \sum_{i=0}^{\lceil \log_2(L) \rceil - 1} P_\lambda\left(\exists k \in I_{\tau,M,i,k_r},\ N(k(1 - 2^{-j_\tau(k)})/M, k/M] - p_{\lambda_0 k L 2^{-j_\tau(k)}/M}(1 - u_\alpha/2) \leq 0\right) \\ &= \sum_{i=0}^{\lceil \log_2(L) \rceil - 1} P_\lambda\left(\sup_{k \in I_{\tau,M,i,k_r}} \left(-N(k(1 - 2^{-j_\tau(k)})/M, k/M] + p_{\lambda_0 k L 2^{-j_\tau(k)}/M}(1 - u_\alpha/2)\right) \geq 0\right). \end{aligned}$$

The proof is then completed applying Lemma 13. Recall that (45) is satisfied and notice that

$$\frac{\frac{\tau M + 2^{i+1}(k_r - \tau M)}{M} - \tau}{\frac{\tau M + 2^i(k_r - \tau M)}{M} - \tau} = 2, \tag{67}$$

that is (54). Then, by definition of k_r (see (63)), the condition (62) gives

$$\begin{aligned} |\delta|\sqrt{\frac{k_r}{M} - \tau} \geq 2\max\Bigg(&\sqrt{\frac{2R}{3}}\sqrt{\frac{\log(2\lfloor \log_2 L \rfloor/\alpha)}{L} + \frac{2\log(2\lceil \log_2 L \rceil/\beta)}{L}}, \\ &4\sqrt{\frac{\lambda_0 \log(2\lfloor \log_2 L \rfloor/\alpha)}{L}} + 8\sqrt{\frac{R\log(2\lceil \log_2 L \rceil/\beta)}{L}}\Bigg), \end{aligned}$$

and since $\tau M + 2^i(k_r - \tau M) \geq k_r$ for all i in $\{0, \ldots, \lceil \log_2(L) \rceil - 1\}$, the inequality (55) is satisfied. As a consequence of Lemma 14, the condition (46) is satisfied by the cover $\bigcup_{i=0}^{\lceil \log_2(L) \rceil - 1} I_{\tau,M,i,k_r}$ and we may apply Lemma 13 to conclude the proof.
Now, if we assume that $-\lambda_0 < \delta < 0$, then (65) ensures that

$$P_\lambda\left(\hat{k}_2^{(1)} \geq k_r\right) \leq P_\lambda\left(\exists k \geq k_r,\ \forall j \in \{1, \ldots, \lfloor \log_2 L \rfloor\},\ N(k(1 - 2^{-j})/M, k/M] \geq p_{\lambda_0 k L 2^{-j}/M}(u_\alpha/2)\right),$$

where $j_\tau(k)$ is defined by (43) and the proof essentially follows the same line as above. □

5.3 Proof of Theorem 3

For $\lambda_0 > 0$ and $R > \lambda_0$, recall that the simple hypotheses $H_k[\lambda_0, R]$ are denoted H_k for short and that u_α stands for $\alpha/\lfloor \log_2 L \rfloor$. For k in $\{1, \dots, M\}$ and j in $\{1, \dots, \lfloor \log_2 L \rfloor\}$, we set for all $x > 0$

$$\varphi_{j,k}(x) = \sqrt{\frac{2^j M}{k}} \mathbb{1}_{\{(\frac{k}{M}(1-\frac{1}{2^j}), \frac{k}{M}]\}}(x).$$

$T_{j,k}$ which is defined by (14) can then be written

$$T_{j,k} = U_{j,k} + 2V_{j,k} + C_{j,k}, \tag{68}$$

where

$$U_{j,k} = \frac{1}{L^2} \left(\left(\int_0^1 \varphi_{j,k}(x)(dN_x - \lambda(x)Ldx) \right)^2 - \int_0^1 \varphi_{j,k}^2(x) dN_x \right), \tag{69}$$

$$V_{j,k} = \frac{1}{L} \int_0^1 \varphi_{j,k}(x)(dN_x - \lambda(x)Ldx) \left(\int_0^1 \varphi_{j,k}(x)\lambda(x)dx - \lambda_0 \sqrt{\frac{k}{2^j M}} \right), \tag{70}$$

and $C_{j,k}$ is the squared bias of $T_{j,k}$

$$C_{j,k} = \left(\int_0^1 \varphi_{j,k}(x)\lambda(x)dx - \lambda_0 \sqrt{\frac{k}{2^j M}} \right)^2. \tag{71}$$

We begin by giving an upper bound for the quantile $t_{j,k}$ of $T_{j,k}$ under (H_k). The key argument to obtain the following upper bound is the use of an exponential inequality established in [Le Guével, 2021, Theorem 2] for the square martingale $\left(\int_0^s (dN_x - \lambda_0 Ldx)\right)^2 - \int_0^s dN_x$.

Lemma 15 (Control of the quantiles)**.** *Let $L \geq 3$, α in $(0,1)$, $\lambda_0 > 0$ and $R > \lambda_0$. For all M in $\mathbb{N}^*$, for all j in $\{1, \dots, \lfloor \log_2 L \rfloor\}$ and for all k in $\{1, \dots, M\}$, we have the following inequality*

$$t_{j,k}(1 - u_\alpha) \leq \frac{2k\lambda_0^2}{2^j M} \left(g^{-1} \left(\frac{M 2^j}{\lambda_0 L k} \log \left(\frac{3\lfloor \log_2(L) \rfloor}{\alpha} \right) \right) \right)^2, \tag{72}$$

where the function g is defined by (25)*.*

Proof of Lemma 15. Let k in $\{1, \dots, M\}$ and j in $\{1, \dots, \lfloor \log_2(L) \rfloor\}$.
Under the hypothesis (H_k), N is a simple Poisson process on the interval $[0, k/M]$, of intensity λ_0 with respect to the measure Ldt, so that the equality (68) reduces to $T_{j,k} = U_{j,k}$ where

$$\begin{aligned} U_{j,k} &= \frac{M2^j}{kL^2} \left(\left(\int_{\frac{k}{M}(1-\frac{1}{2^j})}^{\frac{k}{M}} (dN_x - \lambda_0 Ldx) \right)^2 - \int_{\frac{k}{M}(1-\frac{1}{2^j})}^{\frac{k}{M}} dN_x \right) \\ &\overset{d}{=} \frac{M2^j}{kL^2} \left(\left(\int_0^{\frac{k}{2^j M}} (dN_x - \lambda_0 Ldx) \right)^2 - \int_0^{\frac{k}{2^j M}} dN_x \right). \end{aligned}$$

We then apply inequality (8) of Theorem 2 in [Le Guével, 2021] in order to get for all $x > 0$ and for all λ in H_k,

$$P_\lambda\left(\sup_{0\le s\le \frac{k}{2^j M}}\left(\left(\int_0^s (dN_x - \lambda_0 L dx)\right)^2 - \int_0^s dN_x\right) > \frac{xkL^2}{M2^j}\right) \le 3\exp\left(\frac{-\lambda_0 Lk}{M2^j} g\left(\frac{M2^j}{\lambda_0 Lk}\sqrt{\frac{xkL^2}{M2^{j+1}}}\right)\right),$$

where g is defined by (25). To conclude, we obtain for

$$x \ge \frac{2k\lambda_0^2}{2^j M}\left(g^{-1}\left(\frac{M2^j}{\lambda_0 Lk}\log\left(\frac{3}{u_\alpha}\right)\right)\right)^2,$$

that for all λ in H_k,

$$\begin{aligned}
P_\lambda(T_{j,k} > x) &= P_\lambda(U_{j,k} > x)\\
&\le P_\lambda\left(\sup_{0\le s\le \frac{k}{2^j M}}\left(\left(\int_0^s (dN_x - \lambda_0 L dx)\right)^2 - \int_0^s dN_x\right) > \frac{xkL^2}{M2^j}\right)\\
&\le 3\exp\left(\frac{-\lambda_0 Lk}{M2^j} g\left(\frac{M2^j}{\lambda_0 Lk}\sqrt{\frac{xkL^2}{M2^{j+1}}}\right)\right)\\
&\le u_\alpha,
\end{aligned}$$

and (72) holds by definition of the quantile. □

As for the proof of Theorem 2, let us define for τ in $(0,1)$, $k_\tau = \min\{k' \in \{1,\dots,M\},\ k' > \tau M\}$ and

$$j_\tau(k) = \left\lceil -\log_2\left(1 - \frac{\tau M}{k}\right)\right\rceil \wedge \lfloor \log_2(L)\rfloor \tag{73}$$

for $k \ge k_\tau$, in order to get $j_\tau(k)$ in $\{1,\dots,\lfloor\log_2(L)\rfloor\}$ as well as the inequalities

$$\frac{k}{M}\left(1 - \frac{1}{2^{j_\tau(k)-1}}\right) < \tau \le \frac{k}{M}\left(1 - \frac{1}{2^{j_\tau(k)}}\right) \tag{74}$$

under the following condition

$$\left\lceil -\log_2\left(1 - \frac{\tau M}{k}\right)\right\rceil \le \lfloor \log_2(L)\rfloor. \tag{75}$$

Following the same idea of the proof of Theorem 2, we shall consider also a cover of the interval $[k_\tau, M]$, denoted $\mathcal{P} = \cup_{i=0}^{\Psi-1}[x_i, x_{i+1})$ where Ψ in $\mathbb{N}^*$ is the cardinal of the cover $\mathcal{P}$ and the real $x_0 < \ldots < x_\Psi$ satisfy $x_0 = k_\tau$ and $x_\Psi > M$. Note that the two real x_i and x_{i+1} may depend on λ_0, τ, L, M and k_τ. We will write $\sum_{[c,d)\in\mathcal{P}}$ for the sum over each disjoint interval $[c,d)$ of the cover $\mathcal{P}$. Assuming (75), the key argument in the proof of Theorem 3 will consist on giving an upper bound for the probability $P_\lambda(\sup_{k\in\mathcal{P}}(t_{j_\tau(k),k}(1-u_\alpha) - T_{j_\tau(k),k}) \ge 0)$ instead of the probability $P_\lambda(\sup_{k\ge k_\tau}(t_{j_\tau(k),k}(1-u_\alpha) - T_{j_\tau(k),k}) \ge 0)$, in order to get a more refined bound. The following technical lemmas allow us to bound $P_\lambda(\sup_{k\in[c,d)}(t_{j_\tau(k),k}(1-u_\alpha) - T_{j_\tau(k),k}) \ge 0)$ for each interval $[c,d)$ of the cover $\mathcal{P}$, using exponential inequalities related to the oscillation modulus of martingales or square martingales developed in [Le Guével, 2021].

Lemma 16 (Control 1)**.** *Let $L \geq 3$, M in $\mathbb{N}^*$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$ and $R > \lambda_0$. For λ in $\mathcal{S}[\lambda_0, R]$, assume (75) and that the following inequality holds for all interval $[c,d)$ of the cover $\mathcal{P}$:*

$$\frac{L^2}{2}\left(\frac{c}{M}-\tau\right)\left(\frac{1}{4}\delta^2\left(\frac{c}{M}-\tau\right)-\sup_{k\in[c,d)}\left(\frac{k\lambda_0^2}{2^{j_\tau(k)}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0 Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2\right)\right) \geq$$
$$8\left(\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg^{-1}\left(\frac{\log\left(20\Psi/\beta\right)}{\left(d/M-\tau\right)(\lambda_0+\delta)L}\right)\right)^2, \tag{76}$$

where the function g is defined by (25). Then

$$\sum_{[c,d)\in\mathcal{P}} P_\lambda\left(\sup_{k\in[c,d)}\left(-U_{j_\tau(k),k}+\frac{t_{j_\tau(k),k}(1-u_\alpha)-C_{j_\tau(k),k}}{2}\right)\geq 0\right)\leq\frac{\beta}{2}.$$

Proof of Lemma 16. Let λ in $\mathcal{S}[\lambda_0, R]$ be such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau,1]}$, where δ in $(-\lambda_0, R-\lambda_0]\setminus\{0\}$ and τ in $(0,1)$. Let $[c,d)$ in $\mathcal{P}$ and notice that for k in $[c,d)$, (71) and (74) ensure the inequality

$$C_{j_\tau(k),k} = \delta^2\frac{k}{M}\frac{1}{2^{j_\tau(k)}} > \frac{1}{2}\delta^2\left(\frac{k}{M}-\tau\right), \tag{77}$$

which combined with (72) in Lemma 15 gives

$$C_{j_\tau(k),k} > t_{j_\tau(k),k}(1-u_\alpha). \tag{78}$$

Let us define now $\tilde{M}_s^t$ by $\tilde{M}_s^t = \left(\int_s^t(dN_x-\lambda(x)Ldx)\right)^2 - \int_s^t dN_x$ for all $0 \leq s < t$, in order to write with (69)

$$U_{j_\tau(k),k} = \frac{M2^{j_\tau(k)}}{kL^2}\tilde{M}^{k/M}_{k/M\left(1-2^{-j_\tau(k)}\right)}.$$

We define also $\xi_{c,d}$ by

$$\xi_{c,d} = \inf_{k\in[c,d)}\frac{L^2k}{M2^{j_\tau(k)}}\frac{C_{j_\tau(k),k}-t_{j_\tau(k),k}(1-u_\alpha)}{2},$$

and we prove now that

$$\xi_{c,d} \geq \frac{L^2}{2}\left(\frac{c}{M}-\tau\right)\left(\frac{1}{4}\delta^2\left(\frac{c}{M}-\tau\right)-\sup_{k\in[c,d)}\left(\frac{k\lambda_0^2}{2^{j_\tau(k)}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0 Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2\right)\right). \tag{79}$$

Notice first that (76) leads to

$$\frac{1}{4}\delta^2\left(\frac{c}{M}-\tau\right)-\sup_{k\in[c,d)}\left(\frac{k\lambda_0^2}{2^{j_\tau(k)}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0 Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2\right)\geq 0.$$

The inequality (74) then ensures that

$$\frac{L^2k}{M2^{j_\tau(k)}} > \frac{L^2}{2}\left(\frac{k}{M}-\tau\right),$$

and with (72) and (77), we get for all k in $[c,d)$

$$\frac{L^2 k}{M 2^{j_\tau(k)}} \frac{C_{j_\tau(k),k} - t_{j_\tau(k),k}(1-u_\alpha)}{2} > \frac{L^2}{2}\left(\frac{c}{M}-\tau\right)\left(\frac{1}{4}\delta^2\left(\frac{c}{M}-\tau\right) - \sup_{k\in[c,d)}\left(\frac{k\lambda_0^2}{2^{j_\tau(k)}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0 Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2\right)\right)$$

which is (79). Using the condition (76) we then obtain

$$\xi_{c,d} \geq 8\left(\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg^{-1}\left(\frac{\log(20\Psi/\beta)}{(d/M-\tau)(\lambda_0+\delta)L}\right)\right)^2$$

which is equivalent to

$$10\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg\left(\frac{1}{(d/M-\tau)L(\lambda_0+\delta)}\sqrt{\frac{\xi_{c,d}}{8}}\right)\right) \leq \frac{\beta}{2\Psi}. \tag{80}$$

Finally, we get the following inequalities:

$$\begin{aligned}
&P_\lambda\left(\sup_{k\in[c,d)}\left(-U_{j_\tau(k),k} + \frac{t_{j_\tau(k),k}(1-u_\alpha) - C_{j_\tau(k),k}}{2}\right) \geq 0\right) \\
&\leq P_\lambda\left(\exists k\in[c,d),\ |U_{j_\tau(k),k}| \geq \frac{C_{j_\tau(k),k} - t_{j_\tau(k),k}(1-u_\alpha)}{2}\right) \quad \text{with (78)} \\
&\leq P_\lambda\left(\exists k\in[c,d),\ \left|\tilde{M}^{k/M}_{k/M(1-2^{-j_\tau(k)})}\right| \geq \xi_{c,d}\right) \\
&\leq P_\lambda\left(\sup_{s,t\in\left[\tau,\frac{d}{M}\right]} |\tilde{M}^t_s| \geq \xi_{c,d}\right) \quad \text{with (74)}\ ,
\end{aligned}$$

and applying inequality (15) of Theorem 4 in [Le Guével, 2021],

$$P_\lambda\left(\sup_{s,t\in\left[\tau,\frac{d}{M}\right]} |\tilde{M}^t_s| \geq \xi_{c,d}\right) \leq 10\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg\left(\frac{1}{(d/M-\tau)L(\lambda_0+\delta)}\sqrt{\frac{\xi_{c,d}}{8}}\right)\right).$$

To conclude, (80) yields

$$P_\lambda\left(\sup_{k\in[c,d)}\left(-U_{j_\tau(k),k} + \frac{t_{j_\tau(k),k}(1-u_\alpha) - C_{j_\tau(k),k}}{2}\right) \geq 0\right) \leq \frac{\beta}{2\Psi},$$

and the result follows summing over all the interval of the cover $\mathcal{P}$. □

Lemma 17 (Control 2)**.** *Let $L \geq 3$, M in $\mathbb{N}^*$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$ and $R > \lambda_0$. For λ in $\mathcal{S}[\lambda_0, R]$, assume (75) and that the following inequality holds for all interval $[c,d)$ of the cover $\mathcal{P}$:*

$$\begin{aligned}
&\frac{L}{|\delta|}\left(\frac{1}{8}\delta^2\left(\frac{c}{M}-\tau\right) - \sup_{k\in[c,d)}\left(\frac{k\lambda_0^2}{2^{j_\tau(k)+1}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0 Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2\right)\right) \\
&\qquad \geq 2\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg^{-1}\left(\frac{\log(4\Psi/\beta)}{(d/M-\tau)(\lambda_0+\delta)L}\right),
\end{aligned} \tag{81}$$

where the function g is defined by (25)*. Then*

$$\sum_{[c,d)\in\mathcal{P}} P_\lambda\left(\sup_{k\in[c,d)}\left(-2V_{j_\tau(k),k}+\frac{t_{j_\tau(k),k}(1-u_\alpha)-C_{j_\tau(k),k}}{2}\right)\geq 0\right)\leq\frac{\beta}{2}.$$

Proof of Lemma 17. We proceed as in the proof of Lemma 16. Let λ in $\mathcal{S}[\lambda_0,R]$ and $[c,d)$ in $\mathcal{P}$. We may fix δ in $(-\lambda_0,R-\lambda_0]\setminus\{0\}$ and τ in $(0,1)$ such that $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,1]}$. For all $0\leq s<t$ we define M_s^t by $M_s^t=\int_s^t(dN_x-\lambda(x)Ldx)$ in order to write with (70)

$$V_{j_\tau(k),k}=\frac{1}{L}\sqrt{\frac{M2^{j_\tau(k)}}{k}}\left(\int_0^1\varphi_{j_\tau(k),k}(x)\lambda(x)dx-\lambda_0\sqrt{\frac{k}{2^{j_\tau(k)}M}}\right)M^{k/M}_{k/M\left(1-2^{-j_\tau(k)}\right)}.$$

We define also $\zeta_{c,d}$ by

$$\zeta_{c,d}=L\inf_{k\in[c,d)}\sqrt{\frac{k}{M2^{j_\tau(k)}}}\left|\int_0^1\varphi_{j_\tau(k),k}(x)\lambda(x)dx-\lambda_0\sqrt{\frac{k}{2^{j_\tau(k)}M}}\right|^{-1}\frac{C_{j_\tau(k),k}-t_{j_\tau(k),k}(1-u_\alpha)}{4},$$

and we prove that

$$\zeta_{c,d}\geq\frac{L}{|\delta|}\left(\frac{1}{8}\delta^2\left(\frac{c}{M}-\tau\right)-\sup_{k\in[c,d)}\left(\frac{k\lambda_0^2}{2^{j_\tau(k)+1}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2\right)\right). \tag{82}$$

A straightforward calculation gives for all k in $[c,d)$

$$\frac{L\sqrt{k}}{\sqrt{M2^{j_\tau(k)}}}\left|\int_0^1\varphi_{j_\tau(k),k}(x)\lambda(x)dx-\lambda_0\sqrt{\frac{k}{2^{j_\tau(k)}M}}\right|^{-1}=\frac{L}{|\delta|},$$

and then with (72) and (77), we get

$$\frac{L\sqrt{k}}{\sqrt{M2^{j_\tau(k)}}}\left|\int_0^1\varphi_{j_\tau(k),k}(x)\lambda(x)dx-\lambda_0\sqrt{\frac{k}{2^{j_\tau(k)}M}}\right|^{-1}\frac{C_{j_\tau(k),k}-t_{j_\tau(k),k}(1-u_\alpha)}{4}>$$
$$\frac{L}{|\delta|}\left(\frac{1}{8}\delta^2\left(\frac{c}{M}-\tau\right)-\sup_{k\in[c,d)}\left(\frac{k\lambda_0^2}{2^{j_\tau(k)+1}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2\right)\right),$$

which entails (82). Using the condition (81) we obtain

$$\zeta_{c,d}\geq 2\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg^{-1}\left(\frac{\log\left(4\Psi/\beta\right)}{\left(d/M-\tau\right)(\lambda_0+\delta)L}\right),$$

which is equivalent to

$$2\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg\left(\frac{\zeta_{c,d}}{2\left(d/M-\tau\right)L(\lambda_0+\delta)}\right)\right)\leq\frac{\beta}{2\Psi}. \tag{83}$$

Finally, we get the following inequalities

$$
\begin{aligned}
&P_\lambda\left(\sup_{k\in[c,d)} -2V_{j_\tau(k),k} + \frac{t_{j_\tau(k),k}(1-u_\alpha) - C_{j_\tau(k),k}}{2} \geq 0\right) \\
&\leq P_\lambda\left(\exists k \in [c,d),\ |V_{j_\tau(k),k}| \geq \frac{C_{j_\tau(k),k} - t_{j_\tau(k),k}(1-u_\alpha)}{4}\right) \quad \text{with (78)} \\
&\leq P_\lambda\left(\exists k \in [c,d),\ \left|M^{k/M}_{k/M(1-2^{-j_\tau(k)})}\right| \geq \zeta_{c,d}\right) \\
&\leq P_\lambda\left(\sup_{s,t\in\left[\tau,\frac{d}{M}\right]} |M^t_s| \geq \zeta_{c,d}\right) \quad \text{with (74)},
\end{aligned}
$$

and Theorem 4 in [Le Guével, 2021] leads to

$$
P_\lambda\left(\sup_{s,t\in\left[\tau,\frac{d}{M}\right]} |M^t_s| \geq \zeta_{c,d}\right) \leq 2\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg\left(\frac{\zeta_{c,d}}{2\,(d/M-\tau)\,L(\lambda_0+\delta)}\right)\right).
$$

Therefore (83) yields

$$
P_\lambda\left(\sup_{k\in[c,d)} -2V_{j_\tau(k),k} + \frac{t_{j_\tau(k),k}(1-u_\alpha) - C_{j_\tau(k),k}}{2} \geq 0\right) \leq \frac{\beta}{2\Psi},
$$

and the result follows summing over all the interval of the cover $\mathcal{P}$.

□

The following lemma gives some conditions on the cover $\mathcal{P}$ to ensure the inequalities (76) and (81) in order to apply Lemma 16 and Lemma 17.

Lemma 18. *Let $L \geq 3$, M in $\mathbb{N}^*$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$ and $R > \lambda_0$. For λ in $\mathcal{S}[\lambda_0, R]$, we assume (75), that for all interval $[c,d)$ of the cover $\mathcal{P}$*

$$
\frac{d/M-\tau}{c/M-\tau} = 2, \tag{84}
$$

and that the following inequality holds

$$
\begin{aligned}
|\delta|\sqrt{\frac{c}{M}-\tau} \geq \max\Bigg(& 32\sqrt{3R}\sqrt{\frac{\log(20\Psi/\beta)}{L}}\ ;\ 8\sqrt{2\lambda_0}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\ ; \\
& 2\cdot 32^{1/3}\cdot\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\ ; \\
& 2\cdot\left(\frac{32}{3}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\ ;\ 128\sqrt{R}\sqrt{\frac{\log(4\Psi/\beta)}{L}}\Bigg).
\end{aligned} \tag{85}
$$

Then the inequalities (76) and (81) are satisfied.

Proof of Lemma 18. Let λ in $\mathcal{S}[\lambda_0, R]$ such that $\lambda = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}$ where δ in $(-\lambda_0, R - \lambda_0] \setminus \{0\}$ and τ in $(0,1)$. Let $[c,d)$ in $\mathcal{P}$, k in $[c,d)$ and notice that (74) ensures that $2^{j_\tau(k)}M/k < 2/\left(c/M - \tau\right)$. Combined with (26), we deduce that

$$\sup_{k\in[c,d)} \frac{k\lambda_0^2}{2^{j_\tau(k)}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0 Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2 \leq \\ 2\lambda_0\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L} + \frac{8}{9}\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2}\frac{1}{c/M-\tau} + \frac{8}{3}\sqrt{\lambda_0}\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}}\frac{1}{\sqrt{c/M-\tau}}. \tag{86}$$

Moreover, applying the inequality (26) again, we obtain for all u in $\mathbb{R}_*^+$

$$g^{-1}\left(\frac{\log(u)}{(d/M-\tau)(\lambda_0+\delta)L}\right) \leq \sqrt{\frac{2\log(u)}{(d/M-\tau)(\lambda_0+\delta)L}} + \frac{2\log(u)}{3\,(d/M-\tau)(\lambda_0+\delta)L}. \tag{87}$$

To get (76), notice first that (87) applied with $u = 20\Psi/\beta$ leads to

$$8\left(\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg^{-1}\left(\frac{\log(20\Psi/\beta)}{\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)L}\right)\right)^2 \\ \leq \frac{32}{9}\log^2\left(\frac{20\Psi}{\beta}\right) + 16\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)L\log\left(\frac{20\Psi}{\beta}\right) + \\ \frac{32\sqrt{2}}{3}\sqrt{\frac{d}{M}-\tau}\sqrt{\lambda_0+\delta}\sqrt{L}\log^{3/2}\left(\frac{20\Psi}{\beta}\right). \tag{88}$$

Using (86) and (88), it is enough to prove that

$$\frac{L^2}{2}\left(\frac{c}{M}-\tau\right)\left(\frac{1}{4}\delta^2\left(\frac{c}{M}-\tau\right) - \left(2\lambda_0\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L} + \frac{8}{9}\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2}\frac{1}{c/M-\tau} + \right.\right. \\ \left.\left.\frac{8}{3}\sqrt{\lambda_0}\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}}\frac{1}{\sqrt{c/M-\tau}}\right)\right) \geq \frac{32}{9}\log^2\left(\frac{20\Psi}{\beta}\right) \\ + 16\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)L\log\left(\frac{20\Psi}{\beta}\right) + \frac{32\sqrt{2}}{3}\sqrt{\frac{d}{M}-\tau}\sqrt{\lambda_0+\delta}\sqrt{L}\log^{3/2}\left(\frac{20\Psi}{\beta}\right) \tag{89}$$

to get (76). Let us prove that (85) implies (89). From (85), we get in particular

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \max\left(32\sqrt{3R}\sqrt{\frac{\log(20\Psi/\beta)}{L}}\ ;\ 4\sqrt{6\lambda_0}\sqrt{\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\ ;\ 16\sqrt{R}\sqrt{\frac{\log(20\Psi/\beta)}{L}}\ ;\right. \\ 2\cdot 32^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\ ;\ 2\cdot\left(\frac{256}{3}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log(20\Psi/\beta)}{L}}\ ; \\ \left. 2\cdot\left(\frac{32}{3}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\right),$$

and using the fact that $a+b \leq 2\max(a,b)$ for all a,b in $\mathbb{R}^+$, one obtains

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \max\left(16\sqrt{3R}\sqrt{\frac{\log\left(20\Psi/\beta\right)}{L}}+2\sqrt{6\lambda_0}\sqrt{\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\ ;\right.$$
$$8\sqrt{R}\sqrt{\frac{\log\left(20\Psi/\beta\right)}{L}}+32^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\ ;$$
$$\left.\left(\frac{256}{3}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log\left(20\Psi/\beta\right)}{L}}+\left(\frac{32}{3}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\right). \tag{90}$$

For all a,b in $\mathbb{R}^+$ and s in $(0,1)$, since $(a+b)^s \leq a^s+b^s$, we deduce from (90) that

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \max\left(\sqrt{768R\frac{\log\left(20\Psi/\beta\right)}{L}+24\lambda_0\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\ ;\right.$$
$$\left(512R^{3/2}\frac{\log^{3/2}\left(20\Psi/\beta\right)}{L^{3/2}}+32\sqrt{\lambda_0}R\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}}\right)^{1/3}\ ;$$
$$\left.\left(\frac{256}{3}R^2\frac{\log^2\left(20\Psi/\beta\right)}{L^2}+\frac{32}{3}R^2\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2}\right)^{1/4}\right). \tag{91}$$

Then (91) entails on the one hand

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \left(512R^{3/2}\frac{\log^{3/2}\left(20\Psi/\beta\right)}{L^{3/2}}+32\sqrt{\lambda_0}R\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}}\right)^{1/3}$$

that is

$$|\delta|^3\left(\frac{c}{M}-\tau\right)^{3/2} \geq 512R^{3/2}\frac{\log^{3/2}\left(20\Psi/\beta\right)}{L^{3/2}}+32\sqrt{\lambda_0}R\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}},$$

which gives

$$\delta^2\left(\frac{c}{M}-\tau\right) \geq \frac{1}{|\delta|\sqrt{c/M-\tau}}\left(512|\delta|\sqrt{\lambda_0+\delta}\frac{\log^{3/2}\left(20\Psi/\beta\right)}{L^{3/2}}+32\sqrt{\lambda_0}|\delta|\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}}\right).$$

On the other hand, we get that

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \left(\frac{256}{3}R^2\frac{\log^2\left(20\Psi/\beta\right)}{L^2}+\frac{32}{3}R^2\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2}\right)^{1/4}$$

that is

$$\delta^4\left(\frac{c}{M}-\tau\right)^2 \geq \frac{256}{3}R^2\frac{\log^2\left(20\Psi/\beta\right)}{L^2}+\frac{32}{3}R^2\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2},$$

and then

$$\delta^2\left(\frac{c}{M}-\tau\right) \geq \frac{1}{\delta^2\left(c/M-\tau\right)}\left(\frac{256}{3}\delta^2\frac{\log^2\left(20\Psi/\beta\right)}{L^2}+\frac{32}{3}\delta^2\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2}\right).$$

We therefore obtain

$$\delta^2\left(\frac{c}{M}-\tau\right) \geq 3\max\Bigg(256(\lambda_0+\delta)\frac{\log\left(20\Psi/\beta\right)}{L}+8\lambda_0\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}\ ;$$
$$\frac{1}{|\delta|\sqrt{c/M-\tau}}\left(\frac{512}{3}|\delta|\sqrt{\lambda_0+\delta}\frac{\log^{3/2}\left(20\Psi/\beta\right)}{L^{3/2}}+\frac{32}{3}|\delta|\sqrt{\lambda_0}\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}}\right)\ ;$$
$$\frac{1}{\delta^2\left(c/M-\tau\right)}\left(\frac{256}{9}\delta^2\frac{\log^2\left(20\Psi/\beta\right)}{L^2}+\frac{32}{9}\delta^2\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2}\right)\Bigg),$$

and using the fact that $a+b+c\leq 3\max(a,b,c)$ for all a,b,c in $\mathbb{R}^+$, it yields

$$\delta^2\left(\frac{c}{M}-\tau\right) \geq 256(\lambda_0+\delta)\frac{\log\left(20\Psi/\beta\right)}{L}+8\lambda_0\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}+$$
$$\frac{1}{|\delta|\sqrt{c/M-\tau}}\left(\frac{512}{3}|\delta|\sqrt{\lambda_0+\delta}\frac{\log^{3/2}\left(20\Psi/\beta\right)}{L^{3/2}}+\frac{32}{3}|\delta|\sqrt{\lambda_0}\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}}\right)+$$
$$\frac{1}{\delta^2\left(c/M-\tau\right)}\left(\frac{256}{9}\delta^2\frac{\log^2\left(20\Psi/\beta\right)}{L^2}+\frac{32}{9}\delta^2\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2}\right). \quad (92)$$

We use now (84) to get

$$\delta^2\left(\frac{c}{M}-\tau\right)$$
$$\geq 128(\lambda_0+\delta)\frac{\log\left(20\Psi/\beta\right)}{L}\frac{d/M-\tau}{c/M-\tau}+8\lambda_0\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}+$$
$$\frac{1}{\sqrt{c/M-\tau}}\left(\frac{256\sqrt{2}}{3}\sqrt{\frac{d/M-\tau}{c/M-\tau}}\sqrt{\lambda_0+\delta}\frac{\log^{3/2}\left(20\Psi/\beta\right)}{L^{3/2}}+\frac{32}{3}\sqrt{\lambda_0}\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}}\right)+$$
$$\frac{1}{c/M-\tau}\left(\frac{256}{9}\frac{\log^2\left(20\Psi/\beta\right)}{L^2}+\frac{32}{9}\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2}\right),$$

which can be rewritten as

$$\frac{1}{4}\delta^2\left(\frac{c}{M}-\tau\right)-\Bigg(2\lambda_0\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}+\frac{8}{9}\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2}\frac{1}{c/M-\tau}$$
$$+\frac{8}{3}\sqrt{\lambda_0}\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}}\frac{1}{\sqrt{c/M-\tau}}\Bigg)$$
$$\geq\frac{64}{9}\frac{\log^2\left(20\Psi/\beta\right)}{L^2}\frac{1}{c/M-\tau}+32\frac{d/M-\tau}{c/M-\tau}(\lambda_0+\delta)\frac{\log\left(20\Psi/\beta\right)}{L}$$
$$+\frac{64\sqrt{2}}{3}\sqrt{\frac{d/M-\tau}{c/M-\tau}}\frac{1}{\sqrt{c/M-\tau}}\sqrt{\lambda_0+\delta}\frac{\log^{3/2}\left(20\Psi/\beta\right)}{L^{3/2}},$$

that is (89). Let us prove now that (81) is satisfied. Using (86) and (87) with $u=4\Psi/\beta$, it is enough to prove that

$$\frac{L}{|\delta|}\left(\frac{1}{8}\delta^2\left(\frac{c}{M}-\tau\right)-\left(\lambda_0\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}+\frac{4}{9}\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2}\frac{1}{c/M-\tau}\right.\right.$$
$$\left.\left.+\frac{4}{3}\sqrt{\lambda_0}\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}}\frac{1}{\sqrt{c/M-\tau}}\right)\right)$$
$$\geq\frac{4}{3}\log\left(\frac{4\Psi}{\beta}\right)+2\sqrt{2(\lambda_0+\delta)}\sqrt{\frac{d}{M}-\tau}\sqrt{L\log\left(\frac{4\Psi}{\beta}\right)}. \quad (93)$$

Let's prove that (85) implies (93). From (85), we get in particular

$$|\delta|\sqrt{\frac{c}{M}-\tau}\geq\max\left(16\sqrt{\frac{2R}{3}}\sqrt{\frac{\log\left(4\Psi/\beta\right)}{L}}\ ;\ 8\sqrt{2\lambda_0}\sqrt{\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\ ;\ 128\sqrt{R}\sqrt{\frac{\log\left(4\Psi/\beta\right)}{L}}\ ;\right.$$
$$\left.\left(\frac{128}{3}\right)^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\ ;\ \left(\frac{128}{9}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\right),$$

which leads, using the fact that $a+b\leq 2\max(a,b)$ for all a,b in $\mathbb{R}^+$, to

$$|\delta|\sqrt{\frac{c}{M}-\tau}\geq\max\left(8\sqrt{\frac{2R}{3}}\sqrt{\frac{\log\left(4\Psi/\beta\right)}{L}}+4\sqrt{2\lambda_0}\sqrt{\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\ ;\ 128\sqrt{R}\sqrt{\frac{\log\left(4\Psi/\beta\right)}{L}}\ ;\right.$$
$$\left.\left(\frac{128}{3}\right)^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\ ;\ \left(\frac{128}{9}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}}\right). \quad (94)$$

With the same computations as before, (94) yields

$$\delta^2\left(\frac{c}{M}-\tau\right)\geq 4\max\left(\frac{32}{3}|\delta|\frac{\log\left(4\Psi/\beta\right)}{L}+8\lambda_0\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}\ ;\ 32\sqrt{\lambda_0+\delta}\sqrt{\frac{\log\left(4\Psi/\beta\right)}{L}}|\delta|\sqrt{\frac{c}{M}-\tau}\ ;\right.$$
$$\left.\frac{32}{3}|\delta|\sqrt{\lambda_0}\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}}\frac{1}{|\delta|\sqrt{c/M-\tau}}\ ;\ \frac{32}{9}\delta^2\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2}\frac{1}{\delta^2\left(c/M-\tau\right)}\right). \quad (95)$$

Since $a+b+c+d\leq 4\max(a,b,c,d)$ for all a,b,c,d in $\mathbb{R}^+$, (95) ensures using (84)

$$\delta^2\left(\frac{c}{M}-\tau\right)\geq\frac{32}{3}|\delta|\frac{\log\left(4\Psi/\beta\right)}{L}+32\sqrt{\lambda_0+\delta}\sqrt{\frac{\log\left(4\Psi/\beta\right)}{L}}|\delta|\sqrt{\frac{c}{M}-\tau}+8\lambda_0\frac{\log\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L}$$
$$+\frac{32}{9}\delta^2\frac{\log^2\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^2}\frac{1}{\delta^2\left(c/M-\tau\right)}+\frac{32}{3}|\delta|\sqrt{\lambda_0}\frac{\log^{3/2}\left(3\lfloor\log_2 L\rfloor/\alpha\right)}{L^{3/2}}\frac{1}{|\delta|\sqrt{c/M-\tau}},$$

that is (93) . □

Let us turn back now to the proof of Theorem 3 and first recall that for all λ in $\mathcal{S}[\lambda_0,R]$, $\mathcal{T}(\lambda)=\{H_k\in\mathcal{H}_{M,R}\ ,\ \lambda\in H_k\}$ is the set of true hypotheses.

Proof of Theorem 3. Recall that H_k stands for $H_k[\lambda_0, R]$. We begin with the control of $\mathrm{FWER}(\mathcal{R}_2^{(2)})$ over $\overline{\mathcal{S}}[\lambda_0, R]$. To this end, we compute for all λ in $\overline{\mathcal{S}}[\lambda_0, R]$

$$\begin{aligned} P_\lambda\left(\mathcal{R}_2^{(2)} \cap \mathcal{T}(\lambda) \neq \varnothing\right) &= P_\lambda\left(\exists k \in \{1, \ldots, M\},\ H_k \in \mathcal{R}_2^{(2)},\ H_k \in \mathcal{T}(\lambda)\right) \\ &= P_\lambda\left(\exists k \in \{1, \ldots, \lfloor \tau M \rfloor\},\ k \geq \hat{k}_2^{(2)} + 1,\ \lambda \in H_k\right), \end{aligned}$$

because λ belongs to $H_{\lfloor \tau M \rfloor}$ and not to $H_{\lfloor \tau M \rfloor + 1}$. If $\tau < 1/M$ then $P_\lambda\left(\mathcal{R}_2^{(2)} \cap \mathcal{T}(\lambda) \neq \varnothing\right) = 0 \leq \alpha$, and if $\tau \geq 1/M$ one has

$$\begin{aligned} P_\lambda\left(\mathcal{R}_2^{(2)} \cap \mathcal{T}(\lambda) \neq \varnothing\right) &= P_\lambda(\hat{k}_2^{(2)} + 1 \leq \lfloor \tau M \rfloor) \\ &= P_\lambda\left(\phi^{(2)}_{2, \lfloor \tau M \rfloor} = 1\right) \\ &= P_\lambda\left(\exists j \in \{1, \ldots, \lfloor \log_2 L \rfloor\}, T_{j, \lfloor \tau M \rfloor} > t_{j, \lfloor \tau M \rfloor}(1 - u_\alpha)\right) \\ &\leq \sum_{j=1}^{\lfloor \log_2 L \rfloor} P_\lambda\left(T_{j, \lfloor \tau M \rfloor} > t_{j, \lfloor \tau M \rfloor}(1 - u_\alpha)\right) \\ &\leq \sum_{j=1}^{\lfloor \log_2 L \rfloor} u_\alpha \leq \alpha, \end{aligned}$$

which proves the control of $\mathrm{FWER}(\mathcal{R}_2^{(2)})$ by α.
Let us compute an upper bound for $\mathrm{FWSR}_\beta(\mathcal{R}_2^{(2)}, \mathcal{S}[\lambda_0, R])$. Recall that (10) leads to $\mathrm{FWSR}_\beta(\mathcal{R}_2^{(2)}, \mathcal{S}[\lambda_0, R]) \leq \lambda_0 \vee (R - \lambda_0)$. Now, assume that

$$L > \left(\frac{C(\alpha, \beta, \lambda_0, R, L)}{\lambda_0 \vee (R - \lambda_0)}\right)^2,$$

where

$$\begin{aligned} C(\alpha, \beta, \lambda_0, R, L) = \max\Bigg(& 128\sqrt{R}\sqrt{\log\left(\frac{20\lceil \log_2 L \rceil}{\beta}\right)},\ 4\sqrt{2\lambda_0 \log\left(\frac{3\lfloor \log_2 L \rfloor}{\alpha}\right)} + 2\sqrt{2R\sqrt{\frac{2}{\beta}}},\ 16\sqrt{\frac{R}{\beta}}, \\ & \max\left(8\sqrt{2\lambda_0},\ 2 \cdot 32^{1/3} \cdot \lambda_0^{1/6} R^{1/3},\ 2 \cdot (32/3)^{1/4}\sqrt{R},\ 2R\right)\sqrt{\log\left(\frac{3\lfloor \log_2 L \rfloor}{\alpha}\right)}\Bigg), \end{aligned}$$

and let $r > 0$ be such that

$$\lambda_0 \vee (R - \lambda_0) > r \geq \frac{C(\alpha, \beta, \lambda_0, R, L)}{\sqrt{L}}. \tag{96}$$

Recall that for λ in $\mathcal{S}[\lambda_0, R]$, $\mathcal{F}_r(\lambda) = \{H_k \in \mathcal{H}_{M,R},\ d_2(\lambda, H_k) \geq r\}$ with $d_2(\lambda, H_k) = |\delta|\sqrt{k/M - \tau}\mathbb{1}_{\tau \leq k/M}$ for all k in $\{1, \ldots, M\}$. To bound $\mathrm{FWSR}_\beta(\mathcal{R}_2^{(2)}, \mathcal{S}[\lambda_0, R])$ by r, it is sufficient to prove that for all λ in $\mathcal{S}[\lambda_0, R]$, $P_\lambda\left(\mathcal{F}_r(\lambda) \subset \mathcal{R}_2^{(2)}\right) \geq 1 - \beta$.
Let us consider then λ in $\mathcal{S}[\lambda_0, R]$ such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau, 1]}$ where δ in $(-\lambda_0, R - \lambda_0] \setminus \{0\}$ and τ in $(0, 1)$.

If $\mathcal{F}_r(\lambda) = \varnothing$, we get that $P_\lambda\left(\mathcal{F}_r(\lambda) \subset \mathcal{R}_2^{(2)}\right) = 1 \geq 1 - \beta$, so by now we assume that λ is such that $\mathcal{F}_r(\lambda) \neq \varnothing$. We define

$$k_r = \min\{\tau M < k' \leq M,\ \delta^2 \left(k'/M - \tau\right) \geq r^2\}, \tag{97}$$

and with the relationship $\{\mathcal{F}_r(\lambda) \subset \mathcal{R}_2^{(2)}\} = \{k_r \geq \hat{k}_2^{(2)} + 1\}$, we have to prove the following inequality

$$P_\lambda\left(\hat{k}_2^{(2)} \geq k_r\right) \leq \beta \tag{98}$$

to obtain the expected result.

First, if $k_r = M$ then

$$\begin{aligned} P_\lambda\left(\hat{k}_2^{(2)} \geq k_r\right) &= P_\lambda\left(\hat{k}_2^{(2)} = M\right) \\ &= P_\lambda\left(\phi_{2,M}^{(2)} = 0\right) \\ &= P_\lambda\left(\forall j \in \{1, \ldots, \lfloor \log_2 L \rfloor\},\ T_{j,M} \leq t_{j,M}\left(1 - u_\alpha\right)\right), \end{aligned}$$

where for all j in $\{1, \ldots, \lfloor \log_2 L \rfloor\}$,

$$T_{j,M}(N) = \frac{2^j}{L^2}\left(N\left(\left(1 - \frac{1}{2^j}\right), 1\right]^2 - N\left(\left(1 - \frac{1}{2^j}\right), 1\right]\right) - \frac{2\lambda_0}{L} N\left(\left(1 - \frac{1}{2^j}\right), 1\right] + \frac{\lambda_0^2}{2^j}.$$

By definition of k_r, since the condition (96) ensures in particular that

$$|\delta|\sqrt{1-\tau} \geq \max\left(4\sqrt{\frac{2\lambda_0 \log\left(3/u_\alpha\right)}{L}} + 2\sqrt{2\sqrt{\frac{2}{\beta}\frac{R}{L}}}\ ,\ 2\sqrt{\frac{2\sqrt{2}R\log\left(3/u_\alpha\right)}{3L}}\ ,\right.$$
$$\left. 4\left(\frac{2}{3}\right)^{1/3} \lambda_0^{1/6} R^{1/3} \sqrt{\frac{\log\left(3/u_\alpha\right)}{L}}\ , 16\sqrt{\frac{R}{\beta L}}\ ,\ \frac{\sqrt{2}R}{\sqrt{L}}\right),$$

we immediately conclude that

$$P_\lambda\left(\forall j \in \{1, \ldots, \lfloor \log_2 L \rfloor\},\ T_{j,M} \leq t_{j,M}\left(1 - u_\alpha\right)\right) \leq \beta,$$

according to the minimax study of the change-point detection done in [Fromont et al., 2022, Proposition 18] which involves the statistic $\max_{j \in \{1, \ldots, \lfloor \log_2 L \rfloor\}}(T_{j,M} - t_{j,M})$.

Assume by now that $k_r \leq M - 1$, and we compute then

$$\begin{aligned} P_\lambda\left(\hat{k}_2^{(2)} \geq k_r\right) &= P_\lambda\left(\exists k \geq k_r, \phi_{2,k}^{(2)} = 0\right) \\ &= P_\lambda\left(\exists k \geq k_r,\ \forall j \in \{1, \ldots, \lfloor \log_2 L \rfloor\},\ T_{j,k} \leq t_{j,k}\left(1 - u_\alpha\right)\right) \\ &\leq P_\lambda\left(\exists k \geq k_r,\ T_{j_\tau(k),k} \leq t_{j_\tau(k),k}\left(1 - u_\alpha\right)\right) \\ &= P_\lambda\left(\exists k \geq k_r;\ U_{j_\tau(k),k} + 2V_{j_\tau(k),k} + C_{j_\tau(k),k} - t_{j_\tau(k),k}(1 - u_\alpha) \leq 0\right), \end{aligned}$$

where $j_\tau(k)$ is defined by (73). The assumption (96) entails that $r \geq 2R\sqrt{\log\left(3\lfloor \log_2 L \rfloor/\alpha\right)/L}$ and then for all $k \geq k_r$

$$\delta^2\left(\frac{k}{M} - \tau\right) \geq 4R^2 \frac{\log\left(3\lfloor \log_2 L \rfloor/\alpha\right)}{L}. \tag{99}$$

We therefore get $k/M - \tau \geq 4\log\left(3\lfloor \log_2 L\rfloor/\alpha\right)/L$, which leads to the condition (75) for $L \geq 3$. Indeed, for all $L \geq 3$, we have $2\log\left(3\lfloor \log_2 L\rfloor/\alpha\right)/L \geq 2^{-\lfloor \log_2 L\rfloor}$ which entails $k/M(1-2^{-\lfloor \log_2 L\rfloor+1}) \geq \tau$. As a consequence, we have $-\log_2(1-\tau M/k)+1 \leq \lfloor \log_2 L\rfloor$ which implies (75). Then for all $k \geq k_r$, $j_\tau(k)$ defined by (73) is such that $j_\tau(k)$ belongs to $\{1,\ldots,\lfloor \log_2 L\rfloor)\}$ and satisfies (74).
Consider now the following partition of cardinality $\Psi = \lceil \log_2(L)\rceil$:

$$\bigcup_{i=0}^{\lceil \log_2(L)\rceil-1} I_{\tau,M,i,k_r} = \bigcup_{i=0}^{\lceil \log_2(L)\rceil-1} [\tau M + 2^i(k_r-\tau M)\ ;\ \tau M + 2^{i+1}(k_r-\tau M)).$$

The condition (99) implies $1/(k_r/M-\tau) \leq L \leq 2^{\lceil \log_2(L)\rceil}$ and $\tau M + 2^{(\lceil \log_2(L)\rceil-1)+1}(k_r - \tau M) > M$, so that

$$[k_r; M] \subset \bigcup_{i=0}^{\lceil \log_2(L)\rceil-1} I_{\tau,M,i,k_r}.$$

As a consequence

$$\begin{aligned}
P_\lambda\left(\hat{k}_2^{(2)} \geq k_r\right) &\leq \sum_{i=0}^{\lceil \log_2(L)\rceil-1} P_\lambda\left(\exists k \in I_{\tau,M,i,k_r}, U_{j_\tau(k),k} + 2V_{j_\tau(k),k} + C_{j_\tau(k),k} - t_{j_\tau(k),k}(1-u_\alpha) \leq 0\right)\\
&= \sum_{i=0}^{\lceil \log_2(L)\rceil-1} P_\lambda\left(\sup_{k\in I_{\tau,M,i,k_r}} \left(-U_{j_\tau(k),k} - 2V_{j_\tau(k),k} - C_{j_\tau(k),k} + t_{j_\tau(k),k}(1-u_\alpha)\right) \geq 0\right)\\
&\leq \sum_{i=0}^{\lceil \log_2(L)\rceil-1} P_\lambda\left(\sup_{k\in I_{\tau,M,i,k_r}} \left(-U_{j_\tau(k),k} + \frac{t_{j_\tau(k),k}(1-u_\alpha) - C_{j_\tau(k),k}}{2}\right) \geq 0\right)\\
&\quad + \sum_{i=0}^{\lceil \log_2(L)\rceil-1} P_\lambda\left(\sup_{k\in I_{\tau,M,i,k_r}} \left(-2V_{j_\tau(k),k} + \frac{t_{j_\tau(k),k}(1-u_\alpha) - C_{j_\tau(k),k}}{2}\right) \geq 0\right).
\end{aligned}$$

The proof is then completed applying Lemma 16 and Lemma 17. Recall that (74) is satisfied and notice that

$$\frac{\frac{\tau M + 2^{i+1}(k_r-\tau M)}{M} - \tau}{\frac{\tau M + 2^{i}(k_r-\tau M)}{M} - \tau} = 2, \tag{100}$$

that is (84). Then, by definition of k_r (97), the condition (96) gives

$$|\delta|\sqrt{\frac{k_r}{M}-\tau} \geq \max\left(128\sqrt{R}\sqrt{\frac{\log\left(20\lceil \log_2 L\rceil/\beta\right)}{L}}\ ,\ C(\lambda_0,R)\sqrt{\frac{\log\left(3\lfloor \log_2 L\rfloor/\alpha\right)}{L}}\right),$$

where $C(\lambda_0,R) = \max\left(8\sqrt{2\lambda_0}\ ,\ 2\cdot 32^{1/3}\cdot \lambda_0^{1/6}R^{1/3}\ ,\ 2\cdot(32/3)^{1/4}\sqrt{R}\ ,\ 2R\right)$, and since $\tau M + 2^i(k_r - \tau M) \geq k_r$ for all i in $\{0,\ldots,\lceil \log_2(L)\rceil - 1\}$, the inequality (85) is satisfied. As a consequence of Lemma 18, the two conditions (76) and (81) are satisfied by the cover $\bigcup_{i=0}^{\lceil \log_2(L)\rceil-1} I_{\tau,M,i,k_r}$ and we may apply Lemma 16 and Lemma 17 to conclude the proof. □

5.4 Proofs of Section 4

5.4.1 Proof of Lemma 5

Let $\mathcal{R}$ be a multiple testing procedure on $\mathcal{H}$ such that $\text{FWER}(\mathcal{R}) \leq \alpha$ and let λ in $\overline{\mathcal{S}}$. Recall that we define I_α by $I_\alpha = \{x \in [0,1] :\ x \leq ((\sup\{k \in \{1,\dots,M\},\ H_k \notin \mathcal{R}\} + 1)/M) \wedge 1\}$. If $\tau < 1/M$, one obtains immediately that $P_\lambda(\tau \in I_\alpha) = 1 \geq 1-\alpha$, and if $\tau \geq 1/M$ we get

$$\begin{aligned}\alpha \geq P_\lambda(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing) &= P_\lambda(\exists H_k \in \mathcal{H},\ H_k \in \mathcal{R},\ \lambda \in H_k)\\ &= P_\lambda(\exists k \in \{1,\dots,\lfloor \tau M \rfloor\},\ H_k \in \mathcal{R},\ \lambda \in H_k)\end{aligned}$$

because λ belongs to $H_{\lfloor \tau M \rfloor}$ and not to $H_{\lfloor \tau M \rfloor + 1}$. Then

$$P_\lambda(\exists k \in \{1,\dots,\lfloor \tau M \rfloor\},\ H_k \in \mathcal{R},\ \lambda \in H_k) \geq P_\lambda(\tau \notin I_\alpha)$$

because on the event $\{\tau \notin I_\alpha\}$, one has $H_{\lfloor \tau M \rfloor}$ in $\mathcal{R}$. Therefore $P_\lambda(\tau \in I_\alpha) \geq 1-\alpha$ for all λ in $\overline{\mathcal{S}}$ which proves the first part of the lemma.
Conversely, let λ in $\overline{\mathcal{S}}$ and let I_α a $(1-\alpha)$-confidence region for τ on $\overline{\mathcal{S}}$. Set $\mathcal{R} = \{H_k \in \mathcal{H},\ k/M > \sup\{x \in [0,1],\ x \in I_\alpha\}\}$ and notice that if $\tau < 1/M$ then $P_\lambda(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing) = 0 \leq \alpha$. If $\tau \geq 1/M$, we get

$$\begin{aligned}P_\lambda(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing) &= P_\lambda(\exists H_k \in \mathcal{H},\ H_k \in \mathcal{R},\ \lambda \in H_k)\\ &= P_\lambda(\exists k \in \{1,\dots,\lfloor \tau M \rfloor\},\ k/M > \sup\{x \in [0,1],\ x \in I_\alpha\},\ \lambda \in H_k)\\ &= P_\lambda\left(\sup\{x \in [0,1],\ x \in I_\alpha\} < \frac{\lfloor \tau M \rfloor}{M}\right)\\ &= 1 - P_\lambda\left(\sup\{x \in [0,1],\ x \in I_\alpha\} \geq \frac{\lfloor \tau M \rfloor}{M}\right).\end{aligned}$$

On the event $\{\tau \in I_\alpha\}$, $\tau \leq \sup\{x \in [0,1],\ x \in I_\alpha\}$ and since $\lfloor \tau M \rfloor / M \leq \tau$, we obtain that $\lfloor \tau M \rfloor / M \leq \sup\{x \in [0,1],\ x \in I_\alpha\}$. Then

$$P_\lambda(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing) \leq 1 - P_\lambda(\tau \in I_\alpha),$$

and $P_\lambda(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing) \leq \alpha$ for all λ in $\overline{\mathcal{S}}$ which is the expected result.

5.4.2 Proof of Lemma 6

The control of $\text{FWER}(\mathcal{R})$ over $\overline{\mathcal{S}}[\lambda_0, \delta^*]$ by α is a consequence of the second part of Lemma 5.
Now, let λ in $\mathcal{S}[\lambda_0, \delta^*]$ and notice that on the event $\{\tau \in (\phi(N)-a, \phi(N)+b]\}$, if H_k belongs to $\{H_k \in \mathcal{H}_{M,\delta^*},\ k/M - \tau \geq a+b\}$, then $k/M \geq \tau + a + b$ and since $\phi(N) + b < \tau + a + b$, we obtain $k/M > \phi(N) + b$ that is H_k belongs to $\mathcal{R}$. Then (16) leads to

$$1-\alpha \leq P_\lambda(\tau \in (\phi(N)-a, \phi(N)+b]) \leq P_\lambda\left(\{H_k \in \mathcal{H}_{M,\delta^*},\ k/M - \tau \geq a+b\} \subset \mathcal{R}\right).$$

Therefore, for all λ in $\mathcal{S}[\lambda_0, \delta^*]$,

$$\begin{aligned}1-\alpha &\leq P_\lambda\left(\{H_k \in \mathcal{H}_{M,\delta^*},\ \delta^{*2}\left(k/M - \tau\right) \geq \delta^{*2}(a+b)\} \subset \mathcal{R}\right)\\ &= P_\lambda\left(\mathcal{F}_{|\delta^*|\sqrt{a+b}}(\lambda) \subset \mathcal{R}\right),\end{aligned}$$

and then $\mathrm{FWSR}_\alpha(\mathcal{R}, \mathcal{S}[\lambda_0, \delta^*]) \leq |\delta^*|\sqrt{a+b}$. In particular one has $\mathrm{mFWSR}_{\alpha,\alpha}(\mathcal{S}[\lambda_0, \delta^*]) \leq |\delta^*|\sqrt{a+b}$ hence

$$\mathcal{L}_\alpha(\overline{\mathcal{S}}[\lambda_0, \delta^*]) \geq \frac{\mathrm{mFWSR}_{\alpha,\alpha}(\mathcal{S}[\lambda_0, \delta^*])^2}{\delta^{*2}}$$

which proves the first part of the Lemma.
Assume now that $\mathcal{R}$ is a multiple procedure on $\mathcal{H}_{M,\delta^*}$ satisfying $\mathrm{FWER}(\mathcal{R}) \leq \alpha$ and $\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}[\lambda_0, \delta^*]) \leq r$. Recall that we define an estimator of τ from $\mathcal{R}$ by $\hat{\tau} = \hat{k}/M$ where $\hat{k} = \sup\{k \in \{1, \ldots, M\},\ H_k \notin \mathcal{R}\}$. If $\tau < 1/M$ then $P_\lambda(\tau > \hat{\tau} + 1/M) = 0 \leq \alpha$ and if $\tau \geq 1/M$, we get for all λ in $\overline{\mathcal{S}}[\lambda_0, \delta^*]$,

$$\begin{aligned}\alpha \geq P_\lambda(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing) &= P_\lambda(\exists H_k \in \mathcal{H}_{M,\delta^*},\ H_k \in \mathcal{R},\ H_k \in \mathcal{T}(\lambda)) \\ &= P_\lambda(\exists k \in \{1, \ldots, \lfloor \tau M \rfloor\},\ H_k \in \mathcal{R},\ \lambda \in H_k)\end{aligned}$$

because λ belongs to $H_{\lfloor \tau M \rfloor}$ and not in $H_{\lfloor \tau M \rfloor + 1}$. On the event $\{\tau > \hat{\tau} + 1/M\}$, $\hat{k} < \lfloor \tau M \rfloor$ and then $H_{\lfloor \tau M \rfloor}$ belongs to $\mathcal{R}$. This leads to $P_\lambda(\exists k \in \{1, \ldots, \lfloor \tau M \rfloor\},\ H_k \in \mathcal{R},\ \lambda \in H_k) \geq P_\lambda(\tau > \hat{\tau} + 1/M)$ and

$$P_\lambda(\tau > \hat{\tau} + 1/M) \leq \alpha$$

for all λ in $\overline{\mathcal{S}}[\lambda_0, \delta^*]$. Moreover, since $\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}[\lambda_0, \delta^*]) \leq r$, one has $P_\lambda(\mathcal{F}_r(\lambda) \subset \mathcal{R}) \geq 1 - \beta$ for all λ in $\mathcal{S}[\lambda_0, \delta^*]$. We get then

$$\begin{aligned}\beta \geq P_\lambda(\mathcal{F}_r(\lambda) \cap (\mathcal{H}_{M,\delta^*} \setminus \mathcal{R}) \neq \varnothing) &= P_\lambda(\exists k \in \{1, \ldots, M\},\ d_2(\lambda, H_k) \geq r,\ H_k \notin \mathcal{R}) \\ &= P_\lambda(\exists k \in \{\lceil \tau M \rceil, \ldots, \hat{k}\},\ \delta^{*2}(k/M - \tau) \geq r^2,\ H_k \notin \mathcal{R}) \\ &\geq P_\lambda(\hat{k}/M - \tau \geq r^2/\delta^{*2}),\end{aligned}$$

and one obtains $P_\lambda\left(\tau \leq \hat{\tau} - r^2/\delta^{*2}\right) \leq \beta$ for all λ in $\mathcal{S}[\lambda_0, \delta^*]$. This inequality remains true for $\lambda = \lambda_0$ (and $\tau = 1$). Finally,

$$\inf_{\lambda \in \overline{\mathcal{S}}[\lambda_0, \delta^*]} P_\lambda\left(\tau \in \left(\hat{\tau} - \frac{r^2}{\delta^{*2}},\ \hat{\tau} + \frac{1}{M}\right]\right) \geq 1 - \alpha - \beta.$$

5.4.3 Proof of Lemma 7

Let α in $(0, 1/2)$, $\lambda_0 > 0$, $R > \lambda_0$ and $L \geq 1$. Taking $\phi(N) = a = b = 1/2$, we obtain $\mathcal{L}_\alpha(\overline{\mathcal{S}}[\lambda_0, R]) \leq 1$. Assume now that $\mathcal{L}_\alpha(\overline{\mathcal{S}}[\lambda_0, R]) < 1$. We may fix ϕ in $\mathcal{MD}$ such that $\mathcal{L}_\alpha(\phi, \overline{\mathcal{S}}[\lambda_0, R]) < 1$, and then there exists $a > 0$ and $b > 0$ such that $a + b < 1$ and

$$\inf_{\lambda \in \overline{\mathcal{S}}[\lambda_0, R]} P_\lambda(\tau \in (\phi(N) - a, \phi(N) + b]) \geq 1 - \alpha.$$

On the one hand, for $\lambda = \lambda_0$ we get

$$P_{\lambda_0}\left(1 \in (\phi(N) - a, \phi(N) + b]\right) \geq 1 - \alpha,$$

hence $P_{\lambda_0}(\phi(N) \geq 1 - b) \geq 1 - \alpha$. Since $1 - b > a$, this yields $P_{\lambda_0}(\phi(N) > a) \geq 1 - \alpha$ and

$$P_{\lambda_0}(\phi(N) \leq a) \leq \alpha. \tag{101}$$

On the other hand, for every δ in $(0, R - \lambda_0]$ and every τ in $(0, 1)$,

$$P_\lambda(\tau \in (\phi(N) - a, \phi(N) + b]) \geq 1 - \alpha$$

with $\lambda = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}$. Using the Girsanov theorem (see [Brémaud, 1981]), one computes

$$P_\lambda(\tau \in (\phi(N) - a, \phi(N) + b]) = E_{\lambda_0}\left[\exp\left(\int_\tau^1 \log\left(\frac{\lambda_0 + \delta}{\lambda_0}\right) dN_t - \delta(1-\tau)L\right) \mathbb{1}_{\tau \in (\phi(N)-a, \phi(N)+b]}\right],$$

so letting δ tends to 0, we obtain by dominated convergence that for every τ in $(0, 1)$:

$$P_{\lambda_0}(\tau \in (\phi(N) - a, \phi(N) + b]) \geq 1 - \alpha.$$

In particular, when τ tends to 0, this leads to

$$P_{\lambda_0}(\phi(N) \leq a) \geq 1 - \alpha. \tag{102}$$

Therefore, (101) and (102) leads to

$$1 - \alpha \leq P_{\lambda_0}(\phi(N) \leq a) \leq \alpha.$$

This entails $\alpha \geq 1/2$ which is a contradiction.

5.4.4 Proof of Lemma 8

Let $\lambda_0 > 0$ and ϕ_α a level-α test of (H_0) "$\lambda = \lambda_0$" versus (H_1) "$\lambda \in \mathcal{S}_{\geq\Delta}[\lambda_0]$", with $\mathcal{S}_{\geq\Delta}[\lambda_0]$ defined by (19).
Let $r > 0$ and λ in $\mathcal{S}_{\geq\Delta}[\lambda_0]$ such that $d_2(\lambda, \{\lambda_0\}) \geq r$. We have

$$\begin{aligned} P_\lambda(\phi_\alpha(N) = 0) &= 1 - P_\lambda(\phi_\alpha(N) = 1) + P_{\lambda_0}(\phi_\alpha(N) = 1) - P_{\lambda_0}(\phi_\alpha(N) = 1) \\ &\geq 1 - \alpha - |P_{\lambda_0}(\phi_\alpha(N) = 1) - P_\lambda(\phi_\alpha(N) = 1)| \\ &\geq 1 - \alpha - V(P_\lambda, P_{\lambda_0}), \end{aligned}$$

where $V(P_\lambda, P_{\lambda_0})$ is the total variation distance between the probability measures P_λ and P_{λ_0}. Then, using the Pinsker inequality (see for example Lemma 2.5 in [Tsybakov, 2008]),

$$P_\lambda(\phi_\alpha(N) = 0) \geq 1 - \alpha - \sqrt{\frac{K(P_\lambda, P_{\lambda_0})}{2}},$$

where $K(P_\lambda, P_{\lambda_0})$ is the Kullback divergence between the probability measures P_λ and P_{λ_0}. From Lemma 42 recalled in [Fromont et al., 2022], we thus deduce that if there exists λ in $\mathcal{S}_{\geq\Delta}[\lambda_0]$ such that $d_2(\lambda, \{\lambda_0\}) \geq r$ satisfying $1 - \alpha - \sqrt{K(P_\lambda, P_{\lambda_0})/2} \geq \beta$, then $\mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}_{\geq\Delta}[\lambda_0]) \geq r$.
For $r \geq \Delta$, let us introduce for all τ in $(0, 1)$, $\lambda_r = \lambda_0 + r(1-\tau)^{-1/2}\mathbb{1}_{(\tau,1]}$ in $\mathcal{S}_{\geq\Delta}[\lambda_0]$, and such that $d_2(\lambda_r, \{\lambda_0\}) = r$. Then, the Girsanov theorem (see [Brémaud, 1981]) entails

$$K(P_{\lambda_r}, P_{\lambda_0}) = \int \log\left(\frac{dP_{\lambda_r}}{dP_{\lambda_0}}\right) dP_{\lambda_r} = \log\left(1 + \frac{r}{\lambda_0\sqrt{1-\tau}}\right)\left(\lambda_0 + \frac{r}{\sqrt{1-\tau}}\right)(1-\tau)L - Lr\sqrt{1-\tau}.$$

Hence choosing τ close enough to 1 – which is allowed as long as λ_r is not constrained to be upper bounded by some given constant, $K(P_{\lambda_r}, P_{\lambda_0}) \leq 2(1 - \alpha - \beta)^2$. This entails

$$\mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}_{\geq\Delta}[\lambda_0]) \geq r, \quad \text{for all } r \geq \Delta,$$

which allows to conclude that $\mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}_{\geq\Delta}[\lambda_0]) = +\infty$.

5.4.5 Proof of Theorem 4

By now, for $\lambda_0 > 0$, $R > 0$ and Δ in $(0, \lambda_0 \wedge (R - \lambda_0))$, the simple hypothesis $H_k[\lambda_0, \Delta, R]$ is simply written H_k for short. The following lemma gives an upper bound for the quantiles $s_{\Delta,k}(1-\alpha/2)$ (respectively $s_{-\Delta,k}(1-\alpha/2)$) of $S_{\Delta,k}$ (respectively of $S_{-\Delta,k}$) under H_k, and its proof follows the same lines as the one of Lemma 11, just replacing δ^* by Δ when $\delta^* > 0$ and by $-\Delta$ when $\delta^* < 0$. It highlights in particular that we can bound these quantiles by some constants which do not depend on k, L and M.

Lemma 19 (Control of the quantiles)**.** *Let α in $(0,1)$, $\lambda_0 > 0$, $R > 0$ and Δ in $(0, \lambda_0 \wedge (R - \lambda_0))$. For all M in $\mathbb{N}^*$, for all $L \geq 1$ and for all $k \in \{1, \dots, M\}$, one has*

$$\begin{cases} s_{\Delta,k}\left(1-\frac{\alpha}{2}\right) \leq q_{\lambda_0}\left(1-\frac{\alpha}{2}, \frac{\Delta}{2}\right), \\ s_{-\Delta,k}\left(1-\frac{\alpha}{2}\right) \leq \frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0 - \Delta/2}\right)}, \end{cases} \tag{103}$$

where $q_{\lambda_0}\left(1-\alpha/2, \Delta/2\right)$ is defined in Lemma 3.

Let us prove the Theorem 4 and first recall that for λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$, $\mathcal{T}(\lambda) = \{H_k \in \mathcal{H}_{M,\Delta,R}\ ,\ \lambda \in H_k\}$ is the set of true hypotheses.

Proof of Theorem 4. For all k in $\{1, \dots, M\}$, recall that H_k stands for $H_k[\lambda_0, \Delta, R]$. We start with the control of FWER($\mathcal{R}_3$) over $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0, R]$, and for λ in $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0, R]$ we compute to this end

$$P_\lambda\left(\mathcal{R}_3 \cap \mathcal{T}(\lambda) \neq \varnothing\right) = P_\lambda\left(\exists k \in \{1, \dots, \lfloor \tau M \rfloor\},\ k \geq \hat{k}_3 + 1,\ \lambda \in H_k\right)$$

because λ belongs to $H_{\lfloor \tau M \rfloor}$ and not to $H_{\lfloor \tau M \rfloor + 1}$. If $\tau < 1/M$ then $P_\lambda\left(\mathcal{R}_3 \cap \mathcal{T}(\lambda) \neq \varnothing\right) = 0$, and if $\tau \geq 1/M$ one has

$$\begin{aligned} P_\lambda\left(\mathcal{R}_3 \cap \mathcal{T}(\lambda) \neq \varnothing\right) &= P_\lambda\left(\hat{k}_3 + 1 \leq \lfloor \tau M \rfloor\right) \\ &= P_\lambda\left(\phi_{3,\lfloor \tau M \rfloor} = 1\right) \\ &\leq P_\lambda\left(S_{\Delta,\lfloor \tau M \rfloor}(N) > s_{\Delta,\lfloor \tau M \rfloor}(1-\alpha/2)\right) + P_\lambda\left(S_{-\Delta,\lfloor \tau M \rfloor}(N) > s_{-\Delta,\lfloor \tau M \rfloor}(1-\alpha/2)\right) \\ &\leq \alpha, \end{aligned}$$

that is FWER($\mathcal{R}_3$) is bounded by α.

Let us compute now an upper bound for $\mathrm{FWSR}_\beta(\mathcal{R}_3, \mathcal{S}_{\geq\Delta}[\lambda_0, R])$.
Let $r > 0$ and M in $\mathbb{N}^*$. Recall that for λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$,

$$\mathcal{F}_r(\lambda) = \{H_k[\lambda_0, \Delta, R] \in \mathcal{H}_{M,\Delta,R},\ d_2(\lambda, H_k[\lambda_0, \Delta, R]) \geq r\}$$

with $d_2(\lambda, H_k[\lambda_0, \Delta, R]) = |\delta|\sqrt{k/M - \tau}\,\mathbb{1}_{\tau \leq k/M}$ for all k in $\{1, \dots, M\}$. One has the following straightforward assertion

$$\forall \lambda \in \mathcal{S}_{\geq\Delta}[\lambda_0, R],\ \mathcal{F}_r(\lambda) = \varnothing \iff r \geq \lambda_0 \vee (R - \lambda_0),$$

and in particular, for all multiple testing procedure $\mathcal{R}$,

$$\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}_{\geq\Delta}[\lambda_0, R]) \leq \lambda_0 \vee (R - \lambda_0).$$

Indeed, let $r > 0$ be such that $r \geq \lambda_0 \vee (R - \lambda_0)$. Then for all λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$, $\mathcal{F}_r(\lambda) = \varnothing$ and then $P_\lambda(\mathcal{F}_r(\lambda) \subset \mathcal{R}) = 1 \geq 1 - \beta$ for all multiple testing procedure $\mathcal{R}$ and for all λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$.

Now, assume that $L > (C(\alpha, \beta, \lambda_0, \Delta, R)/(\lambda_0 \vee (R - \lambda_0)))^2$ and let $r > 0$ be such that

$$\lambda_0 \vee (R - \lambda_0) > r \geq \frac{C(\alpha, \beta, \lambda_0, \Delta, R)}{\sqrt{L}}, \tag{104}$$

where $C(\alpha, \beta, \lambda_0, \Delta, R)$ is defined by

$$C(\alpha, \beta, \lambda_0, \Delta, R) = 2 \max \left(\sqrt{R - \lambda_0} \sqrt{q_{\lambda_0}\left(1 - \frac{\alpha}{2}, \frac{\Delta}{2}\right) + \frac{\log(3/\beta)}{\log\left(\frac{\lambda_0 + \Delta}{\lambda_0 + \Delta/2}\right)}}\ , \right.$$
$$\left. \sqrt{\lambda_0} \sqrt{q_{\lambda_0 - \Delta}\left(1 - \frac{\beta}{3}, \frac{\Delta}{2}\right) + \frac{\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0 - \Delta/2}\right)}}\ ,\ 2\sqrt{\frac{3R}{\beta}} \right), \tag{105}$$

where $q_{\lambda_0}(1 - \alpha/2, \Delta/2)$ and $q_{\lambda_0 - \Delta}(1 - \beta/3, \Delta/2)$ are two positive constants defined by Lemma 3.
To bound $\mathrm{FWSR}_\beta(\mathcal{R}_3, \mathcal{S}_{\geq\Delta}[\lambda_0, R])$ by r, it is sufficient to obtain $P_\lambda(\mathcal{F}_r(\lambda) \subset \mathcal{R}_3) \geq 1 - \beta$ for all λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$.
We consider λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$ of the form $\lambda = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}$ with τ in $(0,1)$ and δ in $\{(-\lambda_0, -\Delta] \cup [\Delta, R - \lambda_0]\}$. If $\mathcal{F}_r(\lambda) = \varnothing$, we easily get $P_\lambda(\mathcal{F}_r(\lambda) \subset \mathcal{R}_3) = 1 \geq 1 - \beta$. We therefore assume by now that λ is satisfying $\mathcal{F}_r(\lambda) \neq \varnothing$, and we define

$$k_r = \min\{\tau M < k' \leq M,\ \delta^2 (k'/M - \tau) \geq r^2\}. \tag{106}$$

By virtue of $\{\mathcal{F}_r(\lambda) \subset \mathcal{R}_3\} = \{k_r \geq \hat{k}_3 + 1\}$, we want to prove the following inequality

$$P_\lambda\left(\hat{k}_3 \geq k_r\right) \leq \beta \tag{107}$$

to obtain the expected result.

First, if $k_r = M$ then

$$\begin{aligned} P_\lambda\left(\hat{k}_3 \geq k_r\right) &= P_\lambda\left(\hat{k}_3 = M\right) \\ &= P_\lambda\left(\phi_{3,M} = 0\right) \\ &= P_\lambda\left(S_{\Delta,M}(N) \leq s_{\Delta,M}(1 - \alpha/2),\ S_{-\Delta,M} \leq s_{-\Delta,M}(1 - \alpha/2)\right). \end{aligned} \tag{108}$$

Assume that $\Delta \leq \delta \leq R - \lambda_0$ and notice that (108) ensures

$$\begin{aligned} P_\lambda\left(\hat{k}_3 \geq k_r\right) &\leq P_\lambda\left(S_{\Delta,M}(N) \leq s_{\Delta,M}(1 - \alpha/2)\right) \\ &\leq P_\lambda\left(S_{\Delta,M}(N) \leq q_{\lambda_0}(1 - \alpha/2, \Delta/2)\right) \quad \text{using (103)} \\ &\leq P_\lambda\left(N(\tau, 1] - (\lambda_0 + \Delta/2)(1 - \tau)L \leq q_{\lambda_0}(1 - \alpha/2, \Delta/2)\right). \end{aligned} \tag{109}$$

By definition of k_r, the condition (104) entails in particular that

$$\delta\sqrt{1 - \tau} \geq \frac{2}{\sqrt{L}} \max\left(\sqrt{(R - \lambda_0) q_{\lambda_0}\left(1 - \frac{\alpha}{2}, \frac{\Delta}{2}\right)}\ ,\ 2\sqrt{\frac{R}{\beta}} \right),$$

and then

$$\delta\sqrt{1-\tau} \geq \frac{2}{\sqrt{L}} \max\left(\sqrt{\delta q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)}\ ,\ 2\sqrt{\frac{\lambda_0+\delta}{\beta}}\right). \tag{110}$$

Therefore, we get on the one hand $\delta\sqrt{1-\tau} \geq 2\sqrt{\delta q_{\lambda_0}(1-\alpha/2,\Delta/2)}/\sqrt{L}$ and then $\delta(1-\tau) \geq 4q_{\lambda_0}(1-\alpha/2,\Delta/2)/L$, and on the other hand $\delta\sqrt{1-\tau} \geq 4\sqrt{(\lambda_0+\delta)/(\beta L)}$, hence $\delta(1-\tau) \geq 4\sqrt{(\lambda_0+\delta)(1-\tau)/(\beta L)}$. Thus, using $2\max(a+b) \geq a+b$ for all $a, b \geq 0$, (110) leads to

$$\delta\,(1-\tau) \geq 2\left(\frac{q_{\lambda_0}(1-\alpha/2,\Delta/2)}{L} + \sqrt{\frac{(\lambda_0+\delta)(1-\tau)}{\beta L}}\right),$$

and since $\delta/2 \leq \delta - \Delta/2$, we get

$$\left(\delta - \frac{\Delta}{2}\right)(1-\tau)\,L \geq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right) + \sqrt{\frac{(\lambda_0+\delta)(1-\tau)L}{\beta}}. \tag{111}$$

Moreover, one has $E_\lambda[N(\tau,1]-(\lambda_0+\Delta/2)(1-\tau)L] = (\delta-\Delta/2)(1-\tau)L$, $\mathrm{Var}_\lambda[N(\tau,1]-(\lambda_0+\Delta/2)(1-\tau)L] = (\lambda_0+\delta)(1-\tau)L$, and then (109) gives

$$\begin{aligned}
P_\lambda\left(\hat{k}_3 \geq k_r\right) &\leq P_\lambda\left(N(\tau,1]-(\lambda_0+\delta)(1-\tau)L \leq q_{\lambda_0}(1-\alpha/2,\Delta/2)-(\delta-\Delta/2)(1-\tau)L\right)\\
&\leq P_\lambda\left(N(\tau,1]-(\lambda_0+\delta)(1-\tau)L \leq -\sqrt{(\lambda_0+\delta)(1-\tau)L/\beta}\right) \quad \text{with (111)}\\
&\leq \beta \quad \text{with the Bienayme-Chebyshev inequality.}
\end{aligned}$$

Secondly, assume that $-\lambda_0 < \delta \leq -\Delta$ and notice that (108) yields

$$\begin{aligned}
P_\lambda\left(\hat{k}_3 \geq k_r\right) &\leq P_\lambda\left(S_{-\Delta,M}(N) \leq s_{-\Delta,M}(1-\alpha/2)\right)\\
&\leq P_\lambda\left(S_{-\Delta,M}(N) \leq \frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}\right) \quad \text{using (103)}\\
&\leq P_\lambda\left(-N(\tau,1]+\left(\lambda_0-\frac{\Delta}{2}\right)(1-\tau)L \leq \frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}\right).
\end{aligned} \tag{112}$$

By definition of k_r, the condition (104) also entails in particular that

$$|\delta|\sqrt{1-\tau} \geq \frac{2}{\sqrt{L}} \max\left(\sqrt{\frac{\lambda_0\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}}\ ,\ 2\sqrt{\frac{R}{\beta}}\right),$$

hence

$$|\delta|\sqrt{1-\tau} \geq \frac{2}{\sqrt{L}} \max\left(\sqrt{\frac{|\delta|\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}}\ ,\ 2\sqrt{\frac{\lambda_0+\delta}{\beta}}\right),$$

and then, using the inequality $2\max(a+b) \geq a+b$ for all $a, b \geq 0$ again,

$$\frac{|\delta|}{2}\,(1-\tau)\,L \geq \frac{\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)} + \sqrt{\frac{(\lambda_0+\delta)(1-\tau)L}{\beta}}.$$

Since $|\delta|/2 \leq |\delta| - \Delta/2$, we finally get

$$\left(|\delta| - \frac{\Delta}{2}\right)(1-\tau)\,L \geq \frac{\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)} + \sqrt{\frac{(\lambda_0+\delta)(1-\tau)L}{\beta}}, \tag{113}$$

and since $E_\lambda[-N(\tau,1] + (\lambda_0 - \Delta/2)(1-\tau)L] = (|\delta| - \Delta/2)(1-\tau)L$ and $\mathrm{Var}_\lambda[-N(\tau,1] + (\lambda_0 - \Delta/2)(1-\tau)L] = (\lambda_0+\delta)(1-\tau)L$, we conclude with the following inequalities

$$\begin{aligned}
P_\lambda\left(\hat{k}_3 \geq k_r\right) &\leq P_\lambda\left(-N(\tau,1] + (\lambda_0 - |\delta|)(1-\tau)L \leq \frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)} - \left(|\delta| - \frac{\Delta}{2}\right)(1-\tau)L\right) \\
&\leq P_\lambda\left(-N(\tau,1] + (\lambda_0 - |\delta|)(1-\tau)L \leq -\sqrt{\frac{(\lambda_0+\delta)(1-\tau)L}{\beta}}\right) \quad \text{with (113)} \\
&\leq \beta \quad \text{with the Bienayme-Chebyshev inequality,}
\end{aligned}$$

which concludes the first part of the proof when $k_r = M$.

Assume by now that $k_r \leq M-1$, and we compute

$$\begin{aligned}
P_\lambda\left(\hat{k}_3 \geq k_r\right) &= P_\lambda\left(\exists k \geq k_r, \phi_{3,k} = 0\right) \\
&= P_\lambda\left(\exists k \geq k_r,\ S_{\Delta,k}(N) \leq s_{\Delta,k}(1-\alpha/2),\ S_{-\Delta,k} \leq s_{-\Delta,k}(1-\alpha/2)\right).
\end{aligned} \tag{114}$$

Assume first that $\Delta \leq \delta \leq R - \lambda_0$.
The equality (114) leads to

$$P_\lambda\left(\hat{k}_3 \geq k_r\right) \leq P_\lambda\left(\exists k \geq k_r,\ S_{\Delta,k} \leq s_{\Delta,k}(1-\alpha/2)\right),$$

and we use (103) in Lemma 19 to get

$$\begin{aligned}
P_\lambda\left(\hat{k}_3 \geq k_r\right) &\leq P_\lambda\left(\exists k \geq k_r,\ \sup_{t\in(0,k/M)}\left(N\left(t,\frac{k}{M}\right] - \left(\lambda_0 + \frac{\Delta}{2}\right)L\left(\frac{k}{M} - t\right)\right) \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)\right) \\
&\leq P_\lambda\left(\exists k \geq k_r,\ N\left(\tau,\frac{k}{M}\right] - \left(\lambda_0 + \frac{\Delta}{2}\right)L\left(\frac{k}{M} - \tau\right) \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)\right) \\
&= P_\lambda\left(\inf_{k\in[k_r,M]}\left(N\left(\tau,\frac{k}{M}\right] - \left(\lambda_0 + \frac{\Delta}{2}\right)L\left(\frac{k}{M} - \tau\right)\right) \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)\right) \\
&\leq P_\lambda\left(\inf_{s\in[k_r/M,1]}\left(N\left(\tau,s\right] - \left(\lambda_0 + \frac{\Delta}{2}\right)L\left(s - \tau\right)\right) \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)\right) \\
&= P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right] - \left(\lambda_0 + \frac{\Delta}{2}\right)L\left(\frac{k_r}{M} - \tau\right) + \inf_{s\in(k_r/M,1]} Z_s \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)\right) \\
&= P_\lambda\left(\inf_{s\in(k_r/M,1]} Z_s \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right) - N\left(\tau,\frac{k_r}{M}\right] + \left(\lambda_0 + \frac{\Delta}{2}\right)\left(\frac{k_r}{M} - \tau\right)L\right),
\end{aligned}$$

where $Z_t = N(k_r/M, t] - (\lambda_0 + \Delta/2)(t - k_r/M) L$ for t in $(k_r/M, 1]$. Let us write J for the interval

$$J = \left[(\lambda_0 + \delta) \left(\frac{k_r}{M} - \tau \right) L \pm \sqrt{\frac{3(\lambda_0 + \delta)(k_r/M - \tau) L}{\beta}} \right]. \tag{115}$$

Using the total probability formula, we get

$$\begin{aligned}
& P_\lambda \left(\hat{k}_3 \geq k_r \right) \\
& \leq P_\lambda \left(\inf_{s \in (k_r/M, 1]} Z_s \leq q_{\lambda_0} \left(1 - \frac{\alpha}{2}, \frac{\Delta}{2} \right) - N\left(\tau, \frac{k_r}{M} \right] + \left(\lambda_0 + \frac{\Delta}{2} \right) \left(\frac{k_r}{M} - \tau \right) L \ , \ N\left(\tau, \frac{k_r}{M} \right] \in J \right) \\
& \quad + P_\lambda \left(N\left(\tau, \frac{k_r}{M} \right] < (\lambda_0 + \delta) \left(\frac{k_r}{M} - \tau \right) L - \sqrt{\frac{3(\lambda_0 + \delta)(k_r/M - \tau) L}{\beta}} \right) \\
& \quad + P_\lambda \left(N\left(\tau, \frac{k_r}{M} \right] > (\lambda_0 + \delta) \left(\frac{k_r}{M} - \tau \right) L + \sqrt{\frac{3(\lambda_0 + \delta)(k_r/M - \tau) L}{\beta}} \right) \\
& \leq P_\lambda \left(\inf_{s \in (k_r/M, 1]} Z_s \leq q_{\lambda_0} \left(1 - \frac{\alpha}{2}, \frac{\Delta}{2} \right) - N\left(\tau, \frac{k_r}{M} \right] + \left(\lambda_0 + \frac{\Delta}{2} \right) \left(\frac{k_r}{M} - \tau \right) L \ , \ N\left(\tau, \frac{k_r}{M} \right] \in J \right) \\
& \quad + \frac{2\beta}{3}
\end{aligned} \tag{116}$$

with the Bienayme-Chebyshev inequality. To compute this last probability, we consider a simple Poisson process $(N_t^{\lambda_0+\delta})_{t \geq 0}$ of intensity $(\lambda_0 + \delta)L$ with respect to the Lebesgue measure on $\mathbb{R}^+$, which is the distribution of N_t for t greater than k_r/M. We then obtain

$$\begin{aligned}
& P_\lambda \left(\inf_{s \in (k_r/M, 1]} Z_s \leq q_{\lambda_0} \left(1 - \frac{\alpha}{2}, \frac{\Delta}{2} \right) - N\left(\tau, \frac{k_r}{M} \right] + \left(\lambda_0 + \frac{\Delta}{2} \right) \left(\frac{k_r}{M} - \tau \right) L \ , \ N\left(\tau, \frac{k_r}{M} \right] \in J \right) \\
& \leq P_\lambda \left(\inf_{s \in (k_r/M, 1]} Z_s \leq q_{\lambda_0} \left(1 - \frac{\alpha}{2}, \frac{\Delta}{2} \right) - \left(\delta - \frac{\Delta}{2} \right) \left(\frac{k_r}{M} - \tau \right) L + \sqrt{\frac{3(\lambda_0 + \delta)(k_r/M - \tau) L}{\beta}} \right) \\
& = \mathbb{P} \left(\inf_{t \in (0, 1 - k_r/M]} \left(N^{\lambda_0 + \delta}(0, t] - \left(\lambda_0 + \frac{\Delta}{2} \right) Lt \right) \leq q_{\lambda_0} \left(1 - \frac{\alpha}{2}, \frac{\Delta}{2} \right) - \left(\delta - \frac{\Delta}{2} \right) \left(\frac{k_r}{M} - \tau \right) L \right. \\
& \qquad \left. + \sqrt{\frac{3(\lambda_0 + \delta)(k_r/M - \tau) L}{\beta}} \right).
\end{aligned}$$

By definition of k_r in (106), the condition (104) gives with (105),

$$\delta \sqrt{\frac{k_r}{M} - \tau} \geq \frac{2}{\sqrt{L}} \max \left(\sqrt{\delta} \sqrt{q_{\lambda_0} \left(1 - \frac{\alpha}{2}, \frac{\Delta}{2} \right) + \frac{\log(3/\beta)}{\log \left(\frac{\lambda_0 + \Delta}{\lambda_0 + \Delta/2} \right)}} \ , \ 2\sqrt{3} \sqrt{\frac{\lambda_0 + \delta}{\beta}} \right). \tag{117}$$

On the one hand, (117) leads to

$$\delta \sqrt{k_r/M - \tau} \geq 2\sqrt{\delta/L} \sqrt{q_{\lambda_0}(1 - \alpha/2, \Delta/2) + \log(3/\beta)/\log((\lambda_0 + \Delta)/(\lambda_0 + \Delta/2))},$$

and then $\delta\left(k_r/M-\tau\right)L \geq 4\left(q_{\lambda_0}\left(1-\alpha/2,\Delta/2\right)+\log(3/\beta)/\log\left(\left(\lambda_0+\Delta\right)/\left(\lambda_0+\Delta/2\right)\right)\right)$. On the other hand, (117) yields $\delta\sqrt{k_r/M-\tau} \geq 4\sqrt{3\left(\lambda_0+\delta\right)/(\beta L)}$ and then $\delta\left(k_r/M-\tau\right)L \geq 4\sqrt{3\left(\lambda_0+\delta\right)\left(k_r/M-\tau\right)L/\beta}$. This leads to

$$\delta\left(\frac{k_r}{M}-\tau\right)L \geq 4\max\left(q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)+\frac{\log(3/\beta)}{\log\left(\frac{\lambda_0+\Delta}{\lambda_0+\Delta/2}\right)}\ ,\ \sqrt{\frac{3(\lambda_0+\delta)\left(k_r/M-\tau\right)L}{\beta}}\right),$$

and using the fact that $a+b\leq 2\max(a,b)$ for all $a,b\geq 0$, one obtains

$$q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\frac{\delta}{2}\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)\left(k_r/M-\tau\right)L}{\beta}} \leq \frac{\log(\beta/3)}{\log\left(\frac{\lambda_0+\Delta}{\lambda_0+\Delta/2}\right)},$$

hence using the fact that $\delta\geq\Delta$,

$$q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)\left(k_r/M-\tau\right)L}{\beta}} \leq \frac{\log(\beta/3)}{\log\left(\frac{\lambda_0+\delta}{\lambda_0+\Delta/2}\right)}. \tag{118}$$

The inequality (118) is equivalent to

$$\exp\left(\left(q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)\left(k_r/M-\tau\right)L}{\beta}}\right)\log\left(\frac{\lambda_0+\delta}{\lambda_0+\Delta/2}\right)\right) \leq \frac{\beta}{3}. \tag{119}$$

Notice that since $\beta<1$, (118) ensures

$$q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)\left(k_r/M-\tau\right)L}{\beta}} \leq 0. \tag{120}$$

We therefore may apply Theorem 3 and equation (15) in [Pyke, 1959] to obtain

$$\begin{aligned}\mathbb{P}\Bigg(\inf_{t\in(0,1-k_r/M]}&\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0+\frac{\Delta}{2}\right)Lt\right)\leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\\&+\sqrt{\frac{3(\lambda_0+\delta)\left(k_r/M-\tau\right)L}{\beta}}\Bigg)\\\leq \exp&\left(\left(q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)\left(k_r/M-\tau\right)L}{\beta}}\right)\omega\right),\end{aligned}$$

where ω is the largest real root of the equation $(\lambda_0+\delta)(1-e^{-\omega})=\omega\left(\lambda_0+\Delta/2\right)$. The root ω satisfies $\omega>\log\left((\lambda_0+\delta)/(\lambda_0+\Delta/2)\right)$, and then

$$\begin{aligned}\mathbb{P}\Bigg(\inf_{t\in(0,1-k_r/M]}&\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0+\frac{\Delta}{2}\right)Lt\right)\leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\\&+\sqrt{\frac{3(\lambda_0+\delta)\left(k_r/M-\tau\right)L}{\beta}}\Bigg)\\\leq \exp&\left(\left(q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)\left(k_r/M-\tau\right)L}{\beta}}\right)\log\left(\frac{\lambda_0+\delta}{\lambda_0+\Delta/2}\right)\right),\end{aligned}$$

which entails with (119)

$$\mathbb{P}\left(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0+\frac{\Delta}{2}\right)Lt\right)\le q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\right.$$
$$\left.+\sqrt{\frac{3(\lambda_0+\delta)\,(k_r/M-\tau\,)\,L}{\beta}}\right)\le\frac{\beta}{3}.$$

Gathering this inequality with (116) leads finally to $P_\lambda\left(\hat{k}_1\ge k_r\right)\le\beta$.

Now, assume that $-\lambda_0<\delta<-\Delta$.
We proceed as in the case $\delta>0$. Let us define to this end $X_t=N\,(k_r/M,t]-(\lambda_0-\Delta/2)\,(t-k_r/M\,)\,L$ for all t in $(k_r/M,1]$.
The equality (114) leads to

$$P_\lambda\left(\hat{k}_3\ge k_r\right)\le P_\lambda\left(\exists k\ge k_r,\ S_{-\Delta,k}(N)\le s_{-\Delta,k}(1-\alpha/2)\right),$$

and applying the inequality (103),

$$P_\lambda\left(\hat{k}_3\ge k_r\right)\le P_\lambda\left(\exists k\ge k_r,\ \left(\lambda_0-\frac{\Delta}{2}\right)L\left(\frac{k}{M}-\tau\right)-N\left(\tau,\frac{k}{M}\right]\le\frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}\right)$$
$$\le P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\le\frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\right).$$

Using the interval J defined by (115), we obtain using the total probability formula and the Bienayme-Chebyshev inequality

$$P_\lambda\left(\hat{k}_3\ge k_r\right)$$
$$\le P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\le\frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right)$$
$$+P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right]<(\lambda_0+\delta)\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta)\,(k_r/M-\tau\,)\,L}{\beta}}\right)$$
$$+P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right]>(\lambda_0+\delta)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)\,(k_r/M-\tau\,)\,L}{\beta}}\right)$$
$$\le P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\le\frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right)$$
$$+\frac{2\beta}{3}. \tag{121}$$

We conclude the proof giving an upper bound for this last probability. We compute

$$P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\le \frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right)$$

$$\le P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\le \frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}-\left(|\delta|-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)\,(k_r/M-\tau)\,L}{\beta}}\right)$$

$$= P_\lambda\left(\sup_{s\in(k_r/M,1]}X_s\ge \frac{\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+\left(|\delta|-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta)\,(k_r/M-\tau)\,L}{\beta}}\right)$$

$$\le \mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0-\frac{\Delta}{2}\right)Lt\right)\ge \frac{\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+\left(|\delta|-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L - \sqrt{\frac{3(\lambda_0+\delta)\,(k_r/M-\tau)\,L}{\beta}}\right), \tag{122}$$

By definition of k_r in (106) and the condition (104), we get with (105),

$$|\delta|\sqrt{\frac{k_r}{M}-\tau}\ge\frac{2}{\sqrt{L}}\max\left(\sqrt{\lambda_0}\sqrt{q_{\lambda_0-\Delta}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right)+\frac{\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}}\ ,\ 2\sqrt{\frac{3R}{\beta}}\right),$$

hence

$$|\delta|\sqrt{\frac{k_r}{M}-\tau}>\frac{2}{\sqrt{L}}\max\left(\sqrt{|\delta|}\sqrt{q_{\lambda_0-\Delta}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right)+\frac{\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}}\ ,\ 2\sqrt{\frac{3(\lambda_0+\delta)}{\beta}}\right).$$

Therefore, as in the case $\delta>0$, this leads to

$$\frac{|\delta|}{2}\left(\frac{k_r}{M}-\tau\right)L>q_{\lambda_0-\Delta}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right)+\frac{\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}},$$

and then, since $|\delta|\ge\Delta$,

$$\frac{\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+\left(|\delta|-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta)\,(k_r/M-\tau)\,L}{\beta}}>q_{\lambda_0-\Delta}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right). \tag{123}$$

We then obtain

$$\mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0-\frac{\Delta}{2}\right)Lt\right)\geq\frac{\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+\left(|\delta|-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\right.$$
$$\left.-\sqrt{\frac{3(\lambda_0+\delta)\,(k_r/M-\tau\,)\,L}{\beta}}\right)$$
$$\leq\mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0-\frac{\Delta}{2}\right)Lt\right)>q_{\lambda_0-\Delta}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right)\right)\quad\text{with (123)}$$
$$\leq\mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0-\frac{\Delta}{2}\right)Lt\right)>q_{\lambda_0-|\delta|}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right)\right)\quad\text{with Lemma 9}$$
$$\leq\mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0-|\delta|+\frac{\Delta}{2}\right)Lt\right)>q_{\lambda_0-|\delta|}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right)\right)\quad\text{since } |\delta|\geq\Delta$$
$$\leq\frac{\beta}{3}$$

by definition of $q_{\lambda_0-|\delta|}\,(1-\beta/3,\Delta/2\,)$ in Lemma 3. The proof is then complete using (121).

□

5.4.6 Proof of Lemma 10

The control of FWER($\mathcal{R}$) over $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]$ by α is a consequence of the second part of Lemma 5.
To prove the first part of the lemma, one obtains by the same lines as for Lemma 6 that for all λ in $\mathcal{S}_{\geq\Delta}[\lambda_0,R]$, since $|\delta|\leq\lambda_0\wedge(R-\lambda_0)$,

$$\begin{aligned}1-\alpha&\leq P_\lambda\left(\{H_k\in\mathcal{H}_{M,\Delta,R},\ \delta^2\,(k/M-\tau\,)\geq\delta^2(a+b)\}\subset\mathcal{R}\right)\\&\leq P_\lambda\left(\{H_k\in\mathcal{H}_{M,\Delta,R},\ \delta^2\,(k/M-\tau\,)\geq(\lambda_0^2\wedge(R-\lambda_0)^2)(a+b)\}\subset\mathcal{R}\right)\\&=P_\lambda\left(\mathcal{F}_{(\lambda_0\wedge(R-\lambda_0))\sqrt{a+b}}(\lambda)\subset\mathcal{R}\right).\end{aligned}$$

Then $\mathrm{FWSR}_\alpha(\mathcal{R},\mathcal{S}_{\geq\Delta}[\lambda_0,R])\leq(\lambda_0\wedge(R-\lambda_0))\sqrt{a+b}$ and in particular one has the inequality $\mathrm{mFWSR}_{\alpha,\alpha}(\mathcal{S}_{\geq\Delta}[\lambda_0,R])\leq(\lambda_0\wedge(R-\lambda_0))\sqrt{a+b}$ hence

$$\mathcal{L}_\alpha(\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R])\geq\frac{\mathrm{mFWSR}_{\alpha,\alpha}(\mathcal{S}_{\geq\Delta}[\lambda_0,R])^2}{(\lambda_0\wedge(R-\lambda_0))^2}$$

which proves the first part of the Lemma.
Assume now that $\mathcal{R}$ is a multiple procedure on $\mathcal{H}_{M,\Delta,R}$ satisfying FWER($\mathcal{R}$) $\leq\alpha$ and $\mathrm{FWSR}_\beta(\mathcal{R},\mathcal{S}_{\geq\Delta}[\lambda_0,R])$
$\leq r$. Recall that we define an estimator of τ from $\mathcal{R}$ by $\hat{\tau}=\hat{k}/M$ where $\hat{k}=\sup\{k\in\{1,\ldots,M\},\ H_k\notin\mathcal{R}\}$. Following again the same lines as the proof of Lemma 6, we get that for all λ in $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]$,

$$P_\lambda(\tau>\hat{\tau}+1/M)\leq\alpha.$$

Now, since $\text{FWSR}_\beta(\mathcal{R}, \mathcal{S}_{\geq\Delta}[\lambda_0, R]) \leq r$, one has $P_\lambda(\mathcal{F}_r(\lambda) \subset \mathcal{R}) \geq 1 - \beta$ for all λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$. Since $|\delta| \geq \Delta$, we get then

$$\begin{aligned}
\beta \geq P_\lambda(\mathcal{F}_r(\lambda) \cap (\mathcal{H}_{M,\delta^*} \setminus \mathcal{R}) \neq \varnothing) &= P_\lambda(\exists k \in \{1, \ldots, M\},\ d_2(\lambda, H_k) \geq r,\ H_k \notin \mathcal{R}) \\
&= P_\lambda(\exists k \in \{\lceil \tau M \rceil, \ldots, \hat{k}\},\ \delta^2(k/M - \tau) \geq r^2,\ H_k \notin \mathcal{R}) \\
&\geq P_\lambda(\hat{k}/M - \tau \geq r^2/\delta^2) \\
&\geq P_\lambda(\hat{k}/M - \tau \geq r^2/\Delta^2),
\end{aligned}$$

and one obtains $P_\lambda\left(\tau \leq \hat{\tau} - r^2/\Delta^2\right) \leq \beta$ for all λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$. This last inequality being true for $\lambda = \lambda_0$ (and $\tau = 1$), we finally get

$$\inf_{\lambda \in \overline{\mathcal{S}}_{\geq\Delta}[\lambda_0, R]} P_\lambda\left(\tau \in \left(\hat{\tau} - \frac{r^2}{\Delta^2},\ \hat{\tau} + \frac{1}{M}\right]\right) \geq 1 - \alpha - \beta.$$

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