On Poisson transforms for differential forms on real hyperbolic spaces

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ON POISSON TRANSFORMS FOR DIFFERENTIAL FORMS ON REAL HYPERBOLIC SPACES

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ABSTRACT. We study the Poisson transform for differential forms on the real hyperbolic space $\mathbb{H}^n$. For $1 < r < \infty$, we prove that the Poisson transform is a topological isomorphism from the space of L^r differential q-forms on the boundary $\partial\mathbb{H}^n$ onto a Hardy-type subspace of p-eigenforms of the Hodge-de Rham Laplacian on $\mathbb{H}^n$, for $0 \leq p < \frac{n-1}{2}$ and $q \in \{p-1, p\}$.

CONTENTS

1. INTRODUCTION

Let G/K be a Riemannian symmetric space of non-compact type. For each parabolic subgroup P of G there exists a natural Poisson transform from the space of C^∞-functions on G/P to space of analytic functions on G/K.

When the parabolic P is minimal, one of the main problem stated by Helgason [12] claims that all eigenfunctions of G-invariant differential operators on G/K are obtained as Poisson transforms of hyperfunctions on the Furstenberg boundary G/P. This conjecture was proved by Helgason when G/K is of rank one, and in full generality by Kashiwara *et al.* [16]. Since then, this problem has received a lot of attention by many people in different settings (see, e.g., [2–6, 14, 18, 19, 23, 26]).

A natural extension of this problem is to investigate the analogous of Helgason's claim for Poisson transforms for homogeneous vector bundles over G/K (see,

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e.g., [8, 9, 11, 15, 20–22, 25, 28, 32]). One of the most interesting vector bundles is the bundle of differential forms on G/K. In this paper we consider the vector bundle of differential forms on the real hyperbolic space.

Let $\mathbb{H}^n = G/K$ be the real hyperbolic space realized as the open unit ball in $\mathbb{R}^n$, where $G = \mathrm{SO_o}(n,1)$ and $K \simeq \mathrm{SO}(n)$. Its boundary $\partial\mathbb{H}^n$ is the unit sphere $\mathbb{S}^{n-1}$. As a homogeneous space, we have $\partial\mathbb{H}^n = G/P$, where $P = MAN$. Here $M \simeq \mathrm{SO}(n-1)$, $A \simeq \mathbb{R}$ and $N \simeq \mathbb{R}^{n-1}$.

For $0 \le p \le n$, let τ_p be the p-th exterior power of the coadjoint representation of K on $V_{\tau_p} = \Lambda^p\mathbb{C}^n$. Then the space $C^\infty(\Lambda^p\mathbb{H}^n)$ of smooth p-forms on $\mathbb{H}^n$ can be identified with the space $C^\infty(G/K;\tau_p)$ of V_{τ_p}-valued smooth functions on G that are right covariant of type τ_p.

Throughout this paper we will assume that $0 \le p < \frac{n-1}{2}$ (for this choice of p see Section 2). Then the decomposition of τ_p restricted to M is $\tau_{p_{|M}} = \sigma_{p-1} \oplus \sigma_p$, where σ_q is q-th exterior power of the coadjoint representation of M on $V_{\sigma_q} = \Lambda^q\mathbb{C}^{n-1}$, with $q \in \{p-1, p\}$.

Let $\mathfrak{a}$ be the Lie algebra of A, and identify its complexified dual $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{C}$. For $\lambda \in \mathbb{C}$, we consider the irreducible representation $\sigma_{q,\lambda}$ of $P = MAN$ given by $\sigma_{q,\lambda}(ma_tn) = \sigma_q(m)\mathrm{e}^{(\rho - i\lambda)t}$, where $\rho = \frac{n-1}{2}$. Let $E_{q,\lambda}$ be the corresponding homogeneous vector bundle over $\partial\mathbb{H}^n$. We identify its space of hyperfunction sections with the space $C^{-\omega}(G/P;\sigma_{q,\lambda})$ of all V_{σ_q}-valued hyperfunctions f on G such that

$$f(gma_tn) = \mathrm{e}^{(i\lambda-\rho)t}\sigma_q(m^{-1})f(g) \quad \forall g \in G,\, \forall m \in M,\, \forall n \in N,\, \forall a_t \in A.$$

For $q \in \{p-1, p\}$, let ι_q^p be the natural embedding of V_{σ_q} into V_{τ_p}. Notice that $\iota_q^p \in \mathrm{Hom}_M(V_{\sigma_q}, V_{\tau_p})$. Then we can define a Poisson transform

$$\mathcal{P}^p_{q,\lambda} \colon \mathcal{C}^{-\omega}(G/P;\sigma_{q,\lambda}) \to C^\infty(\Lambda^p\mathbb{H}^n)$$

by

$$\mathcal{P}^p_{q,\lambda}f(g) = \sqrt{\frac{\dim\tau_p}{\dim\sigma_q}}\int_K \tau_p(k)\iota^p_q(f(gk))\mathrm{d}k, \;\; g \in G.$$

We mention that $E_{p,\lambda}$ can be seen as the vector bundle $G \times_P V_{\sigma_p} \otimes \mathcal{E}[\rho - i\lambda]$, where σ_p is extended to a representation of P, and $\mathcal{E}[\rho - i\lambda]$ is the density line bundle over the character $ma_tn \mapsto e^{(\rho-i\lambda)t}$ of P. Sections of the above bundle are q-hyperforms with value in $\mathcal{E}[\rho - i\lambda]$. In view of this observation, $\mathcal{P}^p_{p,\lambda} = \sqrt{\frac{\dim\tau_p}{\dim\sigma_p}}\,\Phi_p^{\rho-i\lambda}$, where $\Phi_p^{\rho-i\lambda}$ is the Poisson transform considered in [8].

Let $\Delta = d\,d^* + d^*\,d$ be the Hodge-de Rham Laplacian, where $d\colon C^\infty(\Lambda^p\mathbb{H}^n) \to C^\infty(\Lambda^{p+1}\mathbb{H}^n)$ is the differential and d^* is the codifferential (the adjoint of d which is defined by the hyperbolic metric).

For $\lambda \in \mathbb{C}$, denote by $C^\infty_{q,\lambda}(\Lambda^p\mathbb{H}^n)$ the space of all $\omega \in \mathcal{C}^\infty(\Lambda^p\mathbb{H}^n)$ which are closed if $q = p-1$ and co-closed if $q = p$, with the additional condition $\Delta\omega = (\lambda^2 + (\rho - q)^2)\omega$. It was proved in [9], that for $0 \le p < (n-1)/2$, the Poisson transforms $\mathcal{P}^p_{q,\lambda}$, $q = p-1, p$ provide the following isomorphisms:

(*i*) $\mathcal{P}^p_{p,\lambda} : \mathcal{C}^{-\omega}(G/P;\sigma_{p,\lambda}) \to C^\infty_{p,\lambda}(\Lambda^p\mathbb{H}^n)$ iff $i\lambda \notin \{-\rho + p\} \cup (\mathbb{Z}_{\le 0} - \rho)$,

and

(ii) $\mathcal{P}^p_{p-1,\lambda}:\mathcal{C}^{-\omega}(G/P;\sigma_{p-1,\lambda})\to C^\infty_{p-1,\lambda}(\Lambda^p\mathbb{H}^n)$ iff $i\lambda\notin\{\rho-p+1\}\cup(\mathbb{Z}_{\leq 0}-\rho)$.

Now, let $C^{-\omega}(K/M;\sigma_q)$ be the space of V_{σ_q}-valued hyperfunctions f on K satisfying $f(km)=\sigma_q(m^{-1})f(k)$, for all $k\in K, m\in M$. By the Iwasawa decomposition, the restriction map $f\mapsto f_{|K}$ gives an isomorphism from $C^{-\omega}(G/P;\sigma_{q,\lambda})$ onto $C^{-\omega}(K/M;\sigma_q)$. Via this isomorphism we can define the Poisson transform from $C^{-\omega}(K/M;\sigma_q)$ into $C^\infty_{q,\lambda}(\Lambda^p\mathbb{H}^n)$. To state our main result, let us introduce further notation.

For $1<r<\infty$, let $L^r(K/M;\sigma_q)$ be the space of V_{σ_q}-valued functions f on K which are covariant of type σ_q, and such that

$$\|f\|_{L^r(K/M;\sigma_q)}=\left(\int_K \parallel f(k)\parallel^r_{\Lambda^q\mathbb{C}^{n-1}}\,\mathrm{d}k\right)^{\frac{1}{r}}<\infty.$$

The space $L^r(K/M;\sigma_q)$ is identified with the space of L^r differential q-forms on the boundary $\partial\mathbb{H}^n=K/M$. From above, it follows that the Poisson transform $\mathcal{P}^p_{q,\lambda}$ maps $L^r(K/M;\sigma_q)$ into the space $C^\infty_{q,\lambda}(\Lambda^p\mathbb{H}^n)$.

The goal of this paper is to characterize those eigenforms in $C^\infty_{q,\lambda}(\Lambda^p\mathbb{H}^n)$ which are Poisson transforms of elements in $L^r(K/M;\sigma_q)$, for $1<r<\infty$. To this end we introduce the Hardy type space $\mathcal{E}^r_{q,\lambda}(G/K;\tau_p)$ of all F in $C^\infty_{q,\lambda}(\Lambda^p\mathbb{H}^n)$ such that

$$\parallel F\parallel_{\mathcal{E}^r_{q,\lambda}}:=\sup_{t>0}\mathrm{e}^{(\rho-\Re(i\lambda))t}\left(\int_K\parallel F(ka_t)\parallel^r_{\Lambda^p\mathbb{C}^n}\,\mathrm{d}k\right)^{\frac{1}{r}}<\infty,$$

where we have identified $C^\infty(\Lambda^p\mathbb{H}^n)$ with $C^\infty(G/K;\tau_p)$.

We pin down that throughout the paper we will often view p-forms in $C^\infty(\Lambda^p\mathbb{H}^n)$ as functions in $C^\infty(G/K;\tau_p)$ and vice-versa.

Our main result is the following:

Theorem A (see Theorem 6.1)**.** *Let $0\leq p<(n-1)/2$ be an integer and $q\in\{p-1,p\}$. Assume $\lambda\in\mathbb{C}$ such that*

$$\begin{cases}\Re(i\lambda)>0 & \text{if } q=p,\\ \Re(i\lambda)>0 \ \text{ and } \ i\lambda\neq\rho-p+1 & \text{if } q=p-1.\end{cases}$$

The Poisson transform $\mathcal{P}^p_{q,\lambda}$ is a topological isomorphism of the space $L^r(K/M;\sigma_q)$ onto the space $\mathcal{E}^r_{q,\lambda}(G/K;\tau_p)$. Moreover, there exists a positive constant γ_λ such that

$$|c_q(\lambda,p)|\,\|f\|_{L^r(K/M;\sigma_q)}\leq\sqrt{\frac{\dim\sigma_q}{\dim\tau_p}}\,\|\mathcal{P}^p_{q,\lambda}f\|_{\mathcal{E}^r_{q,\lambda}}\leq\gamma_\lambda\|f\|_{L^r(K/M;\sigma_q)},$$

for every $f\in L^r(K/M;\sigma_q)$.

Above, $c_q(\lambda,p)$ $(q=p-1,p)$ denote the scalar components of the vector-valued Harish-Chandra c-function $\mathbf{c}(\lambda,p)$. We refer the reader to (4.2) for the integral representation of $\mathbf{c}(\lambda,p)$. The explicit expressions of $c_q(\lambda,p)$ will be given in Proposition 4.6.

As an immediate consequence of Theorem A we obtain when $q=p$ and $i\lambda=\rho-p$ (the harmonic case) a characterization of co-closed harmonic p-forms, see Corollary 6.2. Furthermore, if in addition $p=0$, we recover the classical fact that the Poisson transform is an isometric isomorphism from $L^r(\partial\mathbb{H}^n)$ onto the Hardy-harmonic space on $\mathbb{H}^n$ (see [27]).

Our strategy in proving Theorem A is to begin with the case $r = 2$. The most difficult part is to prove the sufficiency condition. Let us give a short outline of its proof. Let $F \in \mathcal{E}^2_{q,\lambda}(G/K;\tau_p)$, then we show the existence of a functional T on $C^\infty(G/P;\sigma_{q,\overline{\lambda}})$ such that $F = \widetilde{\mathcal{P}^p_{q,\lambda}}(T)$ [1] (Proposition 5.1). To prove that T is indeed in L^2 we need to establish the asymptotic behavior of certain Eisenstein type integrals (see (5.8), (5.9)). To this end we prove a Fatou-type theorem (Theorem 4.3),

$$\lim_{t\to\infty} \mathrm{e}^{(\rho-i\lambda)t}\mathcal{P}^p_{q,\lambda}f(ka_t) = \sqrt{\frac{\dim\tau_p}{\dim\sigma_q}}\,\mathbf{c}(\lambda,p)\iota^p_q(f(k)),$$

in $L^r(K,\Lambda^p\mathbb{C}^n)$, for every $f \in L^r(K/M;\sigma_q)$.

Let us mention that instead of Proposition 5.1 we might use the result of Gaillard, stated in Proposition 3.2 below, to ensure the existence of a hyperform $f \in C^{-\omega}(G/P;\sigma_{q,\lambda})$ such that $F = \mathcal{P}^p_{q,\lambda}f$. We would prefer to keep our argument because it is potentially useful in studying Poisson transform on vector bundles over symmetric spaces of non-compact type.

To establish Theorem A for every $1 < r < \infty$, we prove that any $F \in \mathcal{E}^r_{q,\lambda}(G/K;\tau_p)$ can be approximated by a sequence $(F_m)_m$ in $\mathcal{E}^2_{q,\lambda}(G/K;\tau_p)$. Using the first part of our result which corresponds to $r = 2$, we can deduce that there exists $f_m \in L^2(K/M;\sigma_q)$ such that $F_m = \mathcal{P}^p_{q,\lambda}(f_m)$. By an L^2-inversion formula of the Poisson transform (Theorem 5.5) we conclude that f_m is indeed in $L^r(K/M;\sigma_q)$. Henceforth the linear form

$$T_m(\varphi) = \int_K \langle f_m(k),\varphi(k)\rangle_{\Lambda^q\mathbb{C}^{n-1}}\mathrm{d}k,$$

is uniformly bounded on $L^s(K/M;\sigma_q)$, with $\frac{1}{r}+\frac{1}{s} = 1$. Thanks to Banach-Alaouglu-Bourbaki theorem, there exists a subsequence of bounded operators $(T_{m_j})_j$ which converges to a bounded operator T under the weak-$\star$ topology. Thus by Riesz representation theorem, we conclude that there exists $f \in L^r(K/M;\sigma_q)$ such that $F = \mathcal{P}^p_{q,\lambda}f$.

The paper is organized as follows. Section 2 contains notations and background material for later use. In particular we recall some materials on differential forms on $\mathbb{H}^n$ and $\partial\mathbb{H}^n = \mathbb{S}^{n-1}$ as sections of specific vector bundles. Section 3 is devoted to the definition of the Poisson transform $\mathcal{P}^p_{q,\lambda}$ on the space of differential forms on $\mathbb{S}^{n-1}$. In Section 4 we prove a Fatou type theorem for $\mathcal{P}^p_{q,\lambda}$, which will be of particular use to find the explicit expression of the Harish-Chandra c-function appearing in Theorem A. The Fatou type theorem will essentially play a crucial role in Section 5 where we prove Theorem A for the case $r = 2$. Section 5 contains also an L^2-inversion formula for the Poisson transform. These results will allow us in Section 6 to prove Theorem A for every $1 < r < \infty$.

[1]The parameter λ in Yang [32] corresponds in our notation to $i\lambda$.

2. Background

2.1. **The real hyperbolic space.** Let $\mathbb{H}^n = \mathbb{H}^n(\mathbb{R})$ be the real hyperbolic space of dimension $n \geq 2$ realized as the open unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. Let $G = \mathrm{SO}_{\mathrm{o}}(n,1)$ be the connected component of the identity of the group of all linear transforms of $\mathbb{R}^{n+1}$ with determinant 1 keeping invariant the Lorentzian quadratic form

$$[\mathbf{x}, \mathbf{x}] = x_1^2 + \cdots + x_n^2 - x_{n+1}^2, \quad \mathbf{x} = (x_1, \cdots, x_n, x_{n+1}).$$

Then the group G acts transitively on $\overline{\mathbb{B}^n}$ by fractional transformations and as a homogeneous space we have the identification $\mathbb{H}^n = G/K$, where $K = \mathrm{SO}(n)$, the isotropy subgroup of $\mathbf{0} \in \mathbb{B}^n$, is a maximal compact subgroup of G.

Let $\mathfrak{g} = \mathfrak{so}(n,1)$ and $\mathfrak{k} = \mathfrak{so}(n)$ be the Lie algebras of G and K, respectively. Let as usual $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. The subspace $\mathfrak{p}$ is identified with the tangent space $T_{\mathbf{o}}(G/K) \simeq \mathbb{R}^n$ of $G/K = \mathbb{H}^n$ at the origin $\mathbf{o} = eK$.

Put

$$H_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathfrak{p},$$

then $\mathfrak{a} = \mathbb{R}H_0$ is a maximal abelian subspace of $\mathfrak{p}$, and the corresponding analytic Lie subgroup A of G is parametrized by

$$a_t = \exp(tH_0) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}, \quad t \in \mathbb{R}.$$

Let

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & y & 0 \\ -y^T & 0_{n-1} & y^T \\ 0 & y & 0 \end{pmatrix}, \ y \in \mathbb{R}^{n-1} \right\} \simeq \mathbb{R}^{n-1},$$

so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the Iwasawa decomposition of $\mathfrak{g}$. Here y^T stands for the transpose of a vector $y \in \mathbb{R}^{n-1}$.

Let $N = \exp(\mathfrak{n})$ be the connected Lie subgroup of G having $\mathfrak{n}$ as Lie algebra. According to the Iwasawa decomposition $G = KAN$, every element $g \in G$ can be uniquely written as

$$g = \kappa(g)\mathrm{e}^{H(g)}n,$$

where $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$ and $n \in N$.

Let ρ be the half sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$. Then $\rho(H_0) = \frac{n-1}{2}$ and we will write $\rho = \rho(H_0)$.

Let $P = MAN$ be the standard minimal parabolic subgroup of G, where M is the centralizer of A in K given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix} : m \in \mathrm{SO}(n-1) \right\} \simeq \mathrm{SO}(n-1).$$

Then $G/P = K/M$ may be identified with the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

2.2. **Differential forms on $\mathbb{H}^n$ and $\mathbb{S}^{n-1}$.** Let $\langle\cdot,\cdot\rangle$ be the standard Euclidean scalar product in $\mathbb{R}^n$. Let $(e_1, e_2, \dots, e_n)$ be the standard orthonormal basis of $\mathbb{R}^n$ and denote $(e_1^*, e_2^*, \dots, e_n^*)$ its dual basis.

For an integer p with $0 \le p \le n$, let $\Lambda^p(\mathbb{C}^n)^* = \Lambda^p(\mathbb{R}^n)^* \otimes \mathbb{C}$ be the space of complex-valued alternating multilinear p-forms on $\mathbb{R}^n$. A basis of $\Lambda^p(\mathbb{C}^n)^*$ is given by set of

$$e_I^* := e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \text{ where } \begin{cases} I = \{i_1, \cdots, i_p\}, \\ 1 \le i_1 < \cdots < i_p \le n. \end{cases}$$

The interior product $\iota_v\omega$ of a p-form ω with a vector $v \in \mathbb{R}^n$ is the $(p-1)$-form defined on the given basis by

$$\iota_{e_j}(e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*) = \begin{cases} 0 & \text{if } j \neq \text{any } i_r \\ (-1)^{r-1} e_{i_1}^* \wedge \cdots \wedge \widehat{e_{i_r}^*} \wedge \cdots \wedge e_{i_p}^* & \text{if } j = i_r \end{cases}$$

where $\widehat{\ }$ over $e_{i_r}^*$ means that it is deleted from the exterior product.

For a given $v \in \mathbb{R}^n$, the exterior product $\varepsilon_v\omega$ of a p-form ω with the linear form v^* is the $(p+1)$-form defined by

$$\varepsilon_v\omega = v^* \wedge \omega.$$

For the reader's convenience and to keep the notations simple, we will identify $(\mathbb{C}^n)^*$ with $\mathbb{C}^n$ and $\Lambda^p(\mathbb{C}^n)^*$ with $\Lambda^p\mathbb{C}^n$.

We define an inner product $\langle\cdot,\cdot\rangle_{\Lambda^p\mathbb{C}^n}$ on $\Lambda^p\mathbb{C}^n$ as an extension of the one on $\mathbb{C}^n$ by setting

$$\langle v_1 \wedge \cdots v_p, w_1 \wedge \cdots w_p\rangle_{\Lambda^p\mathbb{C}^n} = \mathbf{det}(\langle v_i, w_j\rangle)_{i,j}. \tag{2.1}$$

It is easy to show that the basis of $\Lambda^p\mathbb{C}^n$ consisting of the p-vectors $e_I := e_{i_1} \wedge \cdots \wedge e_{i_p}$ (where $I = \{i_1, \cdots, i_p\}$, with $1 \le i_1 < \cdots < i_p \le n$) is an orthonormal basis of $\Lambda^p\mathbb{C}^n$ with respect to (2.1). We have further the useful identity

$$\langle \iota_v\omega, \xi\rangle_{\Lambda^{p-1}\mathbb{C}^n} = \langle \omega, \varepsilon_v\xi\rangle_{\Lambda^p\mathbb{C}^n},\ v \in \mathbb{R}^n, \omega \in \Lambda^p\mathbb{C}^n, \xi \in \Lambda^{p-1}\mathbb{C}^n. \tag{2.2}$$

For $0 \le p \le n$, we let τ_p to be the p-exterior product $\Lambda^p\mathrm{Ad}^*$ of the coadjoint representation of $K = \mathrm{SO}(n)$ on $\mathfrak{p}_{\mathbb{C}}^*$. Its representation space being $V_{\tau_p} := \Lambda^p(\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^* \simeq \Lambda^p\mathbb{C}^n$. Notice that τ_p is unitary with respect to the inner product (2.1), and is equivalent to the standard representation of K on $\Lambda^p\mathbb{C}^n$. By [7] or [13], the representation τ_p is irreducible for $p \neq \frac{n}{2}$ (n even), while $\tau_{\frac{n}{2}} = \tau_{\frac{n}{2}}^+ \oplus \tau_{\frac{n}{2}}^-$. The two factors $\tau_{\frac{n}{2}}^{\pm}$ being irreducible, inequivalent and act on the following eigenspaces of the Hodge star operator $\star$,

$$\Lambda_{\frac{n}{2}}^{\pm}\mathbb{C}^n = \{w \in \Lambda^{\frac{n}{2}}\mathbb{C}^n \ :\ \star w = \mu_{\pm} w\},$$

where $\mu_{\pm} = \pm 1$ if $\frac{n}{2}$ is even and $\mu_{\pm} = \pm i$ if $\frac{n}{2}$ is odd. Since the Hodge operator $\star$ induces the equivalence $\tau_p \simeq \tau_{n-p}$, we will restrict our attention to the case $0 \le p < \frac{n}{2}$, without loss of generality.

For $0 \le q \le n-1$, let σ_q be the standard representation of $M \simeq \mathrm{SO}(n-1)$ on $V_{\sigma_q} = \Lambda^q\mathbb{C}^{n-1}$. It is an irreducible representation for $q \neq \frac{n-1}{2}$, and as before $\sigma_{\frac{n-1}{2}} = \sigma_{\frac{n-1}{2}}^+ \oplus \sigma_{\frac{n-1}{2}}^-$.

Lemma 2.1 (See, e.g. [1,13]). *Let $\tau_{p_{|M}}$ be the restriction of τ_p to $M \simeq \mathrm{SO}(n-1)$. Then $\tau_{p_{|M}}$ decomposes into inequivalent factors as follow :*

1) *For $p = 0$, $\tau_{p_{|M}} = \sigma_p$.*
2) *For $0 < p < \frac{n-1}{2}$,*
   $\tau_{p_{|M}} = \sigma_{p-1} \oplus \sigma_p$ *with*

$$\Lambda^p\mathbb{C}^n = e_1 \wedge \Lambda^{p-1}\mathbb{C}^{n-1} \oplus \Lambda^p\mathbb{C}^{n-1} \simeq \Lambda^{p-1}\mathbb{C}^{n-1} \oplus \Lambda^p\mathbb{C}^{n-1}. \tag{2.3}$$

3) *For $p = \frac{n-1}{2}$, $\tau_{p_{|M}} = \sigma_{p-1} \oplus \sigma_p^+ \oplus \sigma_p^-$.*
4) *For $p = \frac{n}{2}$, $\tau_{p_{|M}} = 2\sigma_{p-1} \sim 2\sigma_p$.*

Henceforth, we will assume along this paper that $0 \leq p < \frac{n-1}{2}$ (we say p generic).

Remark 2.2. (1) In the decomposition (2.3) we have identified $\mathbb{C}^{n-1}$ with $\mathrm{span}\{e_2, \cdots, e_n\}$. The isomorphism (2.3) follows from the $\mathrm{SO}(n-1)$-equivariance of the decomposition

$$\omega = e_1 \wedge \omega' + \omega'' \quad \text{with } \ \omega' \in \Lambda^{p-1}\mathbb{C}^{n-1} \text{ and } \omega'' \in \Lambda^p\mathbb{C}^{n-1}$$

for any $\omega \in \Lambda^p\mathbb{C}^n$.

(2) The scalar products on $\Lambda^q\mathbb{C}^{n-1}$, $q \in \{p-1, p\}$, are induced from the one on $\Lambda^p\mathbb{C}^n$ defined in (2.1).

(3) For $q \in \{p-1, p\}$, we will consider the following natural isometric embedding

$$\iota_q^p\colon V_{\sigma_q} = \Lambda^q\mathbb{C}^{n-1} \to V_{\tau_p} = \Lambda^p\mathbb{C}^n. \tag{2.4}$$

Notice that $\iota_q^p \in \mathrm{Hom}_M(V_{\sigma_q}, V_{\tau_p})$ and it is given by

$$\begin{array}{rcl} \iota_{p-1}^p\colon \Lambda^{p-1}\mathbb{C}^{n-1} & \to & e_1 \wedge \Lambda^{p-1}\mathbb{C}^{n-1} \oplus \Lambda^p\mathbb{C}^{n-1} \\ \xi & \mapsto & e_1 \wedge \xi + 0 \end{array}$$

and

$$\begin{array}{rcl} \iota_p^p\colon \Lambda^p\mathbb{C}^{n-1} & \to & e_1 \wedge \Lambda^{p-1}\mathbb{C}^{n-1} \oplus \Lambda^p\mathbb{C}^{n-1} \\ \xi & \mapsto & 0 + \xi \end{array}$$

In particular, for any $\omega, \omega' \in \Lambda^{p-1}\mathbb{C}^{n-1}$,

$$\langle \omega, \omega' \rangle_{\Lambda^{p-1}\mathbb{C}^{n-1}} = \langle e_1 \wedge \omega, e_1 \wedge \omega' \rangle_{\Lambda^p\mathbb{C}^n}$$

(4) For $q \in \{p-1, p\}$, let π_p^q denotes the natural projection

$$\pi_p^q\colon V_{\tau_p} \to V_{\sigma_q}.$$

Then one can see from (2.2) that $(\pi_p^q)^* = \iota_q^p$.

3. Poisson transform on differential forms

In this section we shall define the Poisson transform for differential forms on $\partial\mathbb{H}^n$. We will follow the definition of Okamoto [21], see also Minemura [20], Yang [32], Juhl [15], Van der ven [28], Olbrich [22] and Pedon [24, 25]. There is also another approach to define the differential forms-valued Poisson transforms initiated by Gaillard [8] and generalized by Harrach [11].

Let $G\times_K V_{\tau_p}$ be the homogeneous vector bundle over G/K associated with τ_p. The space of its smooth sections is identified with

$$C^\infty(G/K;\tau_p)=\left\{f\colon G\to V_{\tau_p}\text{smooth}\mid f(gk)=\tau_p(k^{-1})f(g)\;\forall g\in G,\;\forall k\in K\right\}.$$

As a homogeneous vector bundle, we have $\Lambda^p\mathbb{H}^n:=\Lambda^p\,T^*_{\mathbb{C}}\mathbb{H}^n=G\times_K V_{\tau_p}$ and therfore we identify the space $C^\infty(\Lambda^p\mathbb{H}^n)$ of its smooth sections (*i.e.*, smooth differential p-forms on $\mathbb{H}^n$) with the space $C^\infty(G/K;\tau_p)$.

Consider the exterior differentiation operator $d\colon C^\infty(\Lambda^p\mathbb{H}^n)\to C^\infty(\Lambda^{p+1}\mathbb{H}^n)$ and the co-differentiation $d^*=(-1)^{n(p+1)+1}\star d\star\colon C^\infty(\Lambda^p\mathbb{H}^n)\to C^\infty(\Lambda^{p-1}\mathbb{H}^n)$. Let $\Delta=dd^*+d^*d$ be the Hodge-de Rham Laplacian on $C^\infty(\Lambda\mathbb{H}^n)$. Let $\mathbf{D}(\Lambda^p\mathbb{H}^n)$ be the algebra of G-invariant differential operators acting on $C^\infty(\Lambda^p\mathbb{H}^n)$. Its known by [10] that for generic p, $\mathbf{D}(\Lambda^p\mathbb{H}^n)$ is a commutative algebra generated by dd^* and d^*d.

Next, we shall describe the eigenforms for differential operators in $\mathbf{D}(\Lambda^p\mathbb{H}^n)$ by means of Poisson transforms.

For $q\in\{p-1,p\}$ and $\lambda\in\mathbb{C}$, we consider the following irreducible representation of $P=MAN$,

$$\sigma_{q,\lambda}:ma_tn\mapsto\sigma_q(m)\mathrm{e}^{(\rho-i\lambda)t}$$

Let $E_{q,\lambda}$ be the homogeneous vector bundle over $\partial\mathbb{H}^n$ corresponding to $\sigma_{q,\lambda}$. We denote by $C^{-\omega}(\partial\mathbb{H}^n;E_{q,\lambda})$ the space of its hyperfunction sections and we identify it with the space $C^{-\omega}(G/P;\sigma_{q,\lambda})$ of V_q-valued hyperfunctions ϕ on G such that

$$f(gma_tn)=\mathrm{e}^{(i\lambda-\rho)t}\sigma_q(m^{-1})f(g)$$

for all $g\in G,m\in M,n\in N,a_t\in A$. Then, define the Poisson transform

$$\mathcal{P}^p_{q,\lambda}\colon C^{-\omega}(G/P;\sigma_{q,\lambda})\to C^\infty(\Lambda^p\mathbb{H}^n)$$

by

$$\mathcal{P}^p_{q,\lambda}f(g)=c_{p,q}\int_K\tau_p(k)\iota^p_q(f(gk))\mathrm{d}k,\quad g\in G,$$

where ι^p_q is the embedding given by (2.4), $\mathrm{d}k$ denotes the normalized Haar measure on K, and where the constant factor $c_{p,q}$ is given by

$$c_{p,q}=\sqrt{\frac{\dim\tau_p}{\dim\sigma_q}}=\begin{cases}\sqrt{\frac{n}{n-p}} & \text{if } q=p,\\ \sqrt{\frac{n}{p}} & \text{if } q=p-1.\end{cases}\tag{3.1}$$

Let us mention that for $q=p$, $E_{p,\lambda}$ can be seen as the vector bundle $G\times_P V_{\sigma_p}\otimes\mathcal{E}[\rho-i\lambda]$, where σ_p is extended to a representation of P and $\mathcal{E}[\rho-i\lambda]$ is the density line bundle associated to the character $ma_tn\mapsto e^{(\rho-i\lambda)t}$ of P. Then $C^{-\omega}(\partial\mathbb{H}^n;E_{p,\lambda})$ can be viewed as the space of p-hyperforms on $\partial\mathbb{H}^n$ with value in $\mathcal{E}[\rho-i\lambda]$. In view of this observation, $\mathcal{P}^p_{p,\lambda}=c_{p,p}\Phi^{\rho-i\lambda}_p$, where $\Phi^{\rho-i\lambda}_p$ is the Poisson transform considered in [8]. When $i\lambda=\rho-p$ (which corresponds to the harmonic case, see below) the space $C^{-\omega}(\partial\mathbb{H}^n;E_{p,-i(\rho-p)})$ consists of p-hyeprforms with value in $\mathcal{E}[p]$.

By the Iwasawa decomposition, the restriction map of $f\mapsto f_{|K}$ gives an isomorphism from $C^{-\omega}(G/P;\sigma_{q,\lambda})$ onto the space $C^{-\omega}(K/M;\sigma_q)$ of V_q-valued hyperfunctions f on K satisfying $f(km)=\sigma_q(m^{-1})f(k)$, for all $k\in K,m\in M$.

In this compact model, the Poisson transform

$$\mathcal{P}^p_{q,\lambda}\colon C^{-\omega}(K/M;\sigma_q)\to C^\infty(\Lambda^p\mathbb{H}^n)$$

takes the form

$$\mathcal{P}^p_{q,\lambda}f(g)=c_{p,q}\int_K \mathrm{e}^{-(i\lambda+\rho)H(g^{-1}k)}\tau_p(\kappa(g^{-1}k))\iota^p_q(f(gk))\mathrm{d}k,\quad g\in G.$$

Below, we shall give the explicit action of the algebra $\mathbf{D}(\Lambda^p\mathbb{H}^n)$ on the Poisson transform of elements in $C^{-\omega}(K/M;\sigma_q)$. The following result is due to Gaillard [8,9], see also Pedon [25].

Proposition 3.1. *For $f\in C^{-\omega}(K/M;\sigma_q)$ with $q\in\{p-1,p\}$, we have*

$$\begin{array}{ll} d^*\mathcal{P}^p_{p,\lambda}(f)=0, & d\mathcal{P}^p_{p-1,\lambda}(f)=0,\\ d^*d\mathcal{P}^p_{p,\lambda}(f)=(\lambda^2+(\rho-p)^2)\mathcal{P}^p_{p,\lambda}(f), & dd^*\mathcal{P}^p_{p-1,\lambda}(f)=(\lambda^2+(\rho-p+1)^2)\mathcal{P}^p_{p-1,\lambda}(f).\end{array}$$

For a character $\chi\colon \mathbf{D}(\Lambda^p\mathbb{H}^n)\to\mathbb{C}$, let $\mathcal{E}_\chi(\Lambda^p\mathbb{H}^n)$ be the corresponding eigenspace,

$$\mathcal{E}_\chi(\Lambda^p\mathbb{H}^n):=\{f\in C^\infty(\Lambda^p\mathbb{H}^n)\mid Df=\chi(D)f,\ \forall D\in\mathbf{D}(\Lambda^p\mathbb{H}^n)\}.$$

Put $\chi(\Delta)=\gamma$ and suppose $\gamma\neq 0$. Similarly, denote $\chi(dd^*)=\gamma_1$ and $\chi(d^*d)=\gamma_2$. Consider the eigenspace

$$\mathcal{E}_\gamma(\Lambda^p\mathbb{H}^n):=\{f\in C^\infty(\Lambda^p\mathbb{H}^n)\mid \Delta f=\gamma f\}.$$

Since $(d^*d)(dd^*)=0$, we have $\gamma_1\gamma_2=0$. As $\gamma\neq0$ and $\gamma=\gamma_1+\gamma_2$, therefore, we have either ($\gamma_1=0$ and $\gamma_2=\gamma$) or ($\gamma_2=0$ and $\gamma_1=\gamma$). We denote χ by χ_1 in the first case and by χ_2 the second case. Thus,

$$\mathcal{E}_\gamma(\Lambda^p\mathbb{H}^n)=\mathcal{E}_{\chi_1}(\Lambda^p\mathbb{H}^n)\oplus\mathcal{E}_{\chi_2}(\Lambda^p\mathbb{H}^n).$$

In view of Proposition 3.1, we deduce that $\gamma_1=\lambda^2+(\rho-p+1)^2$, $\gamma_2=\lambda^2+(\rho-p)^2$ and

$$\mathcal{E}_{\chi_1}(\Lambda^p\mathbb{H}^n)=\left\{f\in C^\infty(\Lambda^p\mathbb{H}^n)\mid \begin{cases}\Delta f &=(\lambda^2+(\rho-p)^2)f\\ d^*f &=0\end{cases}\right\},$$

$$\mathcal{E}_{\chi_2}(\Lambda^p\mathbb{H}^n)=\left\{f\in C^\infty(\Lambda^p\mathbb{H}^n)\mid \begin{cases}\Delta f &=(\lambda^2+(\rho-p+1)^2)f\\ df &=0\end{cases}\right\}.$$

Under the identification $C^\infty(\Lambda^p\mathbb{H}^n)\simeq C^\infty(G/K;\tau_p)$, we let D, D^* and $-\mathcal{C}$ to be the counterpart of d, d^* and Δ acting on $C^\infty(G/K;\tau_p)$, given by

$$D=\sum_j X_j\varepsilon_{X_j},\quad D^*=-\sum_j X_j\iota_{X_j},\quad \mathcal{C}=\sum_j X_j^2-\sum_j Y_j^2, \tag{3.2}$$

where (X_i) and (Y_i) are orthonormal [2] bases of $\mathfrak{p}$ and $\mathfrak{k}$ respectively. Thus, the spaces $\mathcal{E}_{\chi_1}(\Lambda^p\mathbb{H}^n)$ and $\mathcal{E}_{\chi_2}(\Lambda^p\mathbb{H}^n)$ are identified respectively with

[2] with respect to the normalized Killing form $\frac{1}{2(n-1)}B$

$$\mathcal{E}_{p,\lambda}(G/K;\tau_p) = \left\{ f \in C^\infty(G/K;\tau_p) \mid \begin{cases} \mathcal{C}f & = -(\lambda^2 + (\rho - p)^2) f \\ D^* f & = 0 \end{cases} \right\}, \quad (3.3)$$

$$\mathcal{E}_{p-1,\lambda}(G/K;\tau_p) = \left\{ f \in C^\infty(G/K;\tau_p) \mid \begin{cases} \mathcal{C}f & = -(\lambda^2 + (\rho - p + 1)^2) f \\ Df & = 0 \end{cases} \right\}. \quad (3.4)$$

Notice that $\mathcal{C}$ is the Casimir operator of $\mathfrak{g}$ acting on $C^\infty(G/K;\tau_p)$.

Proposition 3.2 (see [9])**.** *Let $0 \le p < (n-1)/2$, $q \in \{p-1, p\}$ and let $\lambda \in \mathbb{C}$ such that*

$$\begin{cases} i\lambda \notin \{-\rho + p\} \cup (\mathbb{Z}_{\le 0} - \rho) & \text{if } q = p, \\ i\lambda \notin \{\rho - p + 1\} \cup (\mathbb{Z}_{\le 0} - \rho) & \text{if } q = p - 1. \end{cases}$$

The Poisson transform $\mathcal{P}^p_{q,\lambda}$ is a topological isomorphism from the space $C^{-\omega}(K/M;\sigma_q)$ onto the space $\mathcal{E}_{q,\lambda}(G/K;\tau_p)$.

We point out that the above statement was stated in [15] for $q = p$ and n even.

For $1 < r < \infty$, we denote by $L^r(K/M;\sigma_q)$ the space of $\Lambda^q\mathbb{C}^{n-1}$-valued functions on K which are covariant of type σ_q, i.e.,

$$f(km) = \sigma_q(m^{-1}) f(k), \quad \forall k \in K, \ \forall m \in M,$$

and such that

$$\|f\|_{L^r(K/M;\sigma_q)} := \left(\int_K \| f(k) \|^r_{\Lambda^q\mathbb{C}^{n-1}} \, \mathrm{d}k \right)^{\frac{1}{r}} < \infty.$$

Note that, for any $F\colon K \to \Lambda^\kappa\mathbb{C}^N$ we have

$$\left\| \int_K F(k) \mathrm{d}k \right\|_{\Lambda^\kappa\mathbb{C}^N} \le \int_K \|F(k)\|_{\Lambda^\kappa\mathbb{C}^N} \mathrm{d}k. \quad (3.5)$$

From above, it follows that the Poisson transform $\mathcal{P}^p_{q,\lambda}$ maps $L^r(K/M;\sigma_q)$ into $\mathcal{E}_{q,\lambda}(G/K;\tau_p)$. Our aim is to characterize the exact image of the space $L^r(K/M;\sigma_q)$ by the Poisson transform $\mathcal{P}^p_{q,\lambda}$ for generic p and $q \in \{p-1, p\}$.

4. Fatou-type theorem and the Harish-Chandra c-function

For $\lambda \in \mathbb{C}$, generic p, and $q \in \{p-1, p\}$, we define for $1 < r < \infty$, the space $\mathcal{E}^r_{q,\lambda}(G/K;\tau_p)$ to be the subspace of all F in $\mathcal{E}_{q,\lambda}(G/K;\tau_p)$ for which

$$\| F \|_{\mathcal{E}^r_{q,\lambda}} := \sup_{t>0} \mathrm{e}^{(\rho - \Re(i\lambda))t} \left(\int_K \| F(ka_t) \|^r_{\Lambda^p\mathbb{C}^n} \, \mathrm{d}k \right)^{\frac{1}{r}}$$

is finite.

Proposition 4.1. *For every $\lambda \in \mathbb{C}$ with $\Re(i\lambda) > 0$, there exists a positive constant γ_λ such that, for any $f \in L^r(K/M;\sigma_q)$ we have*

$$\left(\int_K \|\mathcal{P}^p_{q,\lambda} f(ka_t)\|^r_{\Lambda^p\mathbb{C}^n} \mathrm{d}k \right)^{1/r} \le \gamma_\lambda c_{p,q} \, \mathrm{e}^{(\Re(i\lambda) - \rho)t} \|f\|_{L^r(K/M;\sigma_q)}. \quad (4.1)$$

Proof. By (3.5) we have

$$\begin{aligned}&\parallel \mathcal{P}^p_{q,\lambda} f(ka_t) \parallel_{\Lambda^p\mathbb{C}^n}\\ &\leq c_{p,q}\int_K \mathrm{e}^{-(\Re(i\lambda)+\rho)H(a_t^{-1}k^{-1}h)} \parallel \tau_p(\kappa(a_t^{-1}k^{-1}h)\iota^p_q(f(h)) \parallel_{\Lambda^p\mathbb{C}^n} \mathrm{d}h\\ &\leq c_{p,q}\int_K \mathrm{e}^{-(\Re(i\lambda)+\rho)H(a_t^{-1}k^{-1}h)} \parallel \iota^p_q(f(h)) \parallel_{\Lambda^p\mathbb{C}^n} \mathrm{d}h,\end{aligned}$$

where the last inequality follows from the unitarity of τ_p. Since ι^p_q is an isometric embedding, we can deduce that

$$\begin{aligned}\parallel \mathcal{P}^p_{q,\lambda} f(ka_t) \parallel_{\Lambda^p\mathbb{C}^n} &\leq c_{p,q}\int_K \mathrm{e}^{-(\Re(i\lambda)+\rho)H(a_t^{-1}k^{-1}h)} \parallel f(h) \parallel_{\Lambda^q\mathbb{C}^{n-1}} \mathrm{d}h\\ &= c_{p,q}\, e_{\lambda,t}(\cdot) * \parallel f(\cdot) \parallel_{\Lambda^q\mathbb{C}^{n-1}} (k),\end{aligned}$$

where $e_{\lambda,t}(k) = \mathrm{e}^{-(\Re(i\lambda)+\rho)H(a_t^{-1}k^{-1})}$, and $*$ is the convolution over K. Therefore, by Young's inequality, we obtain

$$\left(\int_K \parallel \mathcal{P}^p_{q,\lambda} f(ka_t) \parallel^r_{\Lambda^p\mathbb{C}^n} \mathrm{d}k\right)^{1/r} \leq c_{p,q} \parallel e_{\lambda,t} \parallel_{L^1(K/M;\sigma_q)}\parallel f \parallel_{L^r(K/M;\,\sigma_q)} .$$

Further,

$$\parallel e_{\lambda,t} \parallel_{L^1(K/M;\,\sigma_q)}= \int_K \mathrm{e}^{-(\Re(i\lambda)+\rho)H(a_t^{-1}k^{-1})}\mathrm{d}k = \phi^{(\rho-\frac{1}{2},-\frac{1}{2})}_{-i\Re(i\lambda)}(t),$$

where $\phi^{(\alpha,\beta)}_\nu$ is the Jacobi function, see (4.7). Since $\Re(i\lambda) > 0$, by (4.8) we have

$$\phi^{(\rho-\frac{1}{2},-\frac{1}{2})}_{-i\Re(i\lambda)}(t) = e^{(\Re(i\lambda)-\rho)t}\left(c_{\rho-\frac{1}{2},-\frac{1}{2}}(-i\Re(i\lambda)) + o(1)\right) \text{ as } t\to\infty,$$

where $c_{\rho-\frac{1}{2},-\frac{1}{2}}(-i\Re(i\lambda))$ is given by (4.9). This proves the estimate (4.1) and consequently that the Poisson transform is continuous from $L^r(K/M;\sigma_q)$ into $\mathcal{E}^r_{q,\lambda}(G/K;\tau_p)$.

□

Let $\bar{N} = \theta(N)$, where θ is the Cartan involution of G. For $\lambda \in \mathbb{C}$ and $0 \leq p < \frac{n-1}{2}$, define the generalized Harish-Chandra c-function by

$$\mathbf{c}(\lambda,p) = \int_{\bar{N}} \mathrm{e}^{-(i\lambda+\rho)H(\overline{n})}\tau_p(\kappa(\overline{n}))\mathrm{d}\overline{n} \in \mathrm{End}(\Lambda^p\mathbb{C}^n). \tag{4.2}$$

Here $\mathrm{d}\bar{n}$ is the Haar measure on $\bar{N}$ with the normalization

$$\int_{\bar{N}} \mathrm{e}^{-2\rho(H(\bar{n}))}\mathrm{d}\bar{n} = 1.$$

The integral (4.2) converges for λ such that $\Re(i\lambda) > 0$ and has a meromorphic continuation to $\mathbb{C}$ (see, e.g. [31]). Since the restriction $\mathbf{c}(\lambda,p)_{|V_{\sigma_q}}$ commutes with σ_q, then by Schur's lemma, there exists a complex scalar $c_q(\lambda,p)$ such that $\mathbf{c}(\lambda,p)_{|V_{\sigma_q}} = c_q(\lambda,p)\mathrm{Id}_{\Lambda^q\mathbb{C}^{n-1}}$. Therefore,

$$\mathbf{c}(\lambda,p) = c_{p-1}(\lambda,p)\mathrm{Id}_{\Lambda^{p-1}\mathbb{C}^{n-1}} + c_p(\lambda,p)\mathrm{Id}_{\Lambda^p\mathbb{C}^{n-1}}. \tag{4.3}$$

In [29], an explicit expression of $c_{p-1}(\lambda,p)$ and $c_p(\lambda,p)$ are given by a direct computation of the integral (4.2). However, below in Proposition 4.6, we will recover their expressions by using a different approach.

The following lemma is needed for later use.

Lemma 4.2. *(1) For every $v\in V_{\sigma_q}$,*

$$\|\mathbf{c}(\lambda,p)\iota_q^p(v)\|_{\Lambda^p\mathbb{C}^n}=|c_q(\lambda,p)|\|v\|_{\Lambda^q\mathbb{C}^{n-1}}. \tag{4.4}$$

(2) For every linear operator L form a vector space V to V_{σ_q},

$$\|\mathbf{c}(\lambda,p)\iota_q^pL\|_{\mathrm{HS}}=|c_q(\lambda,p)|\|L\|_{\mathrm{HS}}, \tag{4.5}$$

Proof. Using Remark 2.2, the first statement follows directly from

$$\mathbf{c}(\lambda,p)\iota_q^p(v)=\begin{cases}c_{p-1}(\lambda,p)e_1\wedge v, & q=p-1\\ c_p(\lambda,p)v, & q=p.\end{cases}$$

On the other hand,

$$\begin{aligned}\|\mathbf{c}(\lambda,p)\iota_q^pL\|_{\mathrm{HS}}^2 &= \mathbf{tr}\left((\mathbf{c}(\lambda,p)\iota_q^pL)^*(\mathbf{c}(\lambda,p)\iota_q^pL)\right)\\ &= \mathbf{tr}\left(L^*(\pi_p^q\mathbf{c}(\lambda,p)^*\mathbf{c}(\lambda,p)\iota_q^p)L\right).\end{aligned}$$

Notice that $\pi_p^q\mathbf{c}(\lambda,p)^*\mathbf{c}(\lambda,p)\iota_q^p\in\mathrm{End_M(V_{\sigma_q})}$, (hence is scalar). By (4.3), we deduce that

$$\mathbf{c}(\lambda,p)^*\mathbf{c}(\lambda,p)=\begin{pmatrix}|c_{p-1}(\lambda,p)|^2\mathrm{Id}_{\Lambda^{p-1}\mathbb{C}^{n-1}} & 0\\ 0 & |c_p(\lambda,p)|^2\mathrm{Id}_{\Lambda^p\mathbb{C}^{n-1}}\end{pmatrix}.$$

Thus $\pi_p^q\mathbf{c}(\lambda,p)^*\mathbf{c}(\lambda,p)\iota_q^p=|c_q(\lambda,p)|^2\mathrm{Id}_{\Lambda^q\mathbb{C}^{n-1}}$, and this proves the second statement. □

Theorem 4.3. *Let $\lambda\in\mathbb{C}$ such that $\Re(i\lambda)>0$. Then*

$$\lim_{t\to\infty}\mathrm{e}^{(\rho-i\lambda)t}\mathcal{P}_{q,\lambda}^pf(ka_t)=c_{p,q}\mathbf{c}(\lambda,p)\iota_q^p(f(k)),$$

(i) uniformly for $f\in C^\infty(K/M;\sigma_q)$,
(ii) in the $L^r(K;\Lambda^p\mathbb{C}^n)$-sens, for every $f\in L^r(K/M;\sigma_q)$.

Proof. The statement (i) has been proved earlier, see for instance [28] and [32].

(ii) Let $f\in L^r(K/M;\sigma_q)$ and $\varepsilon>0$. By density argument, there exists a K-finite vector φ in $C^\infty(K/M;\sigma_q)$ such that $\|f-\varphi\|_{L^r(K/M;\sigma_q)}<\varepsilon$. Put $p_\lambda^t(f)(k)=\mathcal{P}_{q,\lambda}^pf(ka_t)$, then

$$\begin{aligned}\|\mathrm{e}^{-(i\lambda-\rho)t}p_\lambda^t(f)(k)-c_{p,q}\mathbf{c}(\lambda,p)\iota_q^pf(k)\|_{\Lambda^p\mathbb{C}^n}^r \leq\ & \|\mathrm{e}^{-(i\lambda-\rho)t}p_\lambda^t(f-\varphi)(k)\|_{\Lambda^p\mathbb{C}^n}^r\\ &+\|\mathrm{e}^{-(i\lambda-\rho)t}p_\lambda^t(\varphi)(k)-c_{p,q}\mathbf{c}(\lambda,p)\iota_q^p\varphi(k)\|_{\Lambda^p\mathbb{C}^n}^r\\ &+c_{pq}^r\|\mathbf{c}(\lambda,p)\iota_q^p\varphi(k)-\mathbf{c}(\lambda,p)\iota_q^pf(k)\|_{\Lambda^p\mathbb{C}^n}^r.\end{aligned}$$

From Proposition 4.1 we obtain

$$\int_K\|\mathrm{e}^{-(i\lambda-\rho)t}p_\lambda^t(f-\varphi)(k)\|_{\Lambda^p\mathbb{C}^n}^r\mathrm{d}k\leq\gamma_\lambda^rc_{p,q}^r\|f-\varphi\|_{L^r(K/M;\sigma_q)}^r,$$

and form part (i) above it follows that

$$\lim_{t\to\infty}\int_K\|\mathrm{e}^{-(i\lambda-\rho)t}p_\lambda^t(\varphi)(k)-c_{p,q}\mathbf{c}(\lambda,p)\iota_q^p\varphi(k)\|_{\Lambda^p\mathbb{C}^n}\mathrm{d}k=0.$$

Further, according to (4.4) we obtain

$$\int_K \|\mathbf{c}(\lambda,p)\iota_q^p\varphi(k)-\mathbf{c}(\lambda,p)\iota_q^p f(k)\|^r_{\Lambda^p\mathbb{C}^n}\mathrm{d}k \quad\le\quad |c_q(\lambda,p)|^r\|f-\varphi\|^r_{L^r(K/M;\,\sigma_q)}.$$

In conclusion we have

$$\lim_{t\to\infty}\int_K \|\mathrm{e}^{-(i\lambda-\rho)t}p_\lambda^t(f)(k)-c_{p,q}\mathbf{c}(\lambda,p)\iota_q^p f(k)\|^r_{\Lambda^p\mathbb{C}^n}\mathrm{d}k\le\varepsilon^r c^r_{p,q}(\gamma_\lambda^r+|c_q(\lambda,p)|^r),$$

and this proves the desired statement. □

The following inequalities are crucial.

Proposition 4.4. *For every $\lambda\in\mathbb{C}$ such that $\Re(i\lambda)>0$, there exists a positive constant γ_λ such that for all $f\in L^r(K/M;\sigma_q)$, $1<r<\infty$, we have*

$$c_{p,q}|c_q(\lambda,p)|\|f\|_{L^r(K/M;\,\sigma_q)}\le\|\mathcal{P}^p_{q,\lambda}f\|_{\mathcal{E}^r_{q,\lambda}}\le c_{p,q}\,\gamma_\lambda\|f\|_{L^r(K/M;\,\sigma_q)}. \tag{4.6}$$

Proof. The right-hand side inequality is noting but the estimate (4.1). For the left-hand side inequality, by Theorem 4.3[(ii)], there exists a sequence $(t_j)_j$ with $t_j\to\infty$ such that

$$\lim_{j\to\infty}\|\mathrm{e}^{(\rho-i\lambda)t_j}\mathcal{P}^p_{q,\lambda}f(ka_{t_j})\|_{\Lambda^p\mathbb{C}^n}=\|c_{p,q}\,\mathbf{c}(\lambda,p)\iota_q^p(f(k))\|_{\Lambda^p\mathbb{C}^n}$$

almost every where in K. Consequently, by the classical Fatou theorem and (4.4) we get

$$c^r_{p,q}|c_q(\lambda,p)|^r\int_K\|f(k)\|^r_{\Lambda^q\mathbb{C}^{n-1}}\mathrm{d}k\quad\le\quad\sup_j\mathrm{e}^{r\Re(\rho-i\lambda)t_j}\int_K\|p_\lambda^{t_j}(f)(k)\|^r_{\Lambda^p\mathbb{C}^n}\mathrm{d}k,$$

which implies

$$c_{p,q}|c_q(\lambda,p)|\,\|f\|_{L^r(K/M;\sigma_q)}\le\|\mathcal{P}^p_{q,\lambda}f\|_{\mathcal{E}^r_{q,\lambda}}.$$

□

In the rest of this section we will see how the asymptotic behavior formula given in Theorem 4.3 will allows us to give explicitly the Harish-Chandra c-function

$$\mathbf{c}(\lambda,p)=\int_{\overline{N}}\mathrm{e}^{-(i\lambda+\rho)H(\overline{n})}\tau_p(\kappa(\overline{n}))\mathrm{d}\overline{n}.$$

To this aim, recall the Jacobi functions, see, e.g. [17],

$$\phi_\lambda^{(\alpha,\beta)}(t)={}_2F_1\left(\frac{i\lambda+\alpha+\beta+1}{2},\frac{-i\lambda+\alpha+\beta+1}{2};\alpha+1;-\sinh^2 t\right), \tag{4.7}$$

with $\Re(\alpha+1)>0$ and ${}_2F_1$ is the classical hypergeometric function. We shall need the following asymptotic behavior of Jacobi functions,

$$\phi_\lambda^{(\alpha,\beta)}(t)=\mathrm{e}^{(i\lambda-\alpha-\beta-1)t}(c_{\alpha,\beta}(\lambda)+o(1))\ \text{ as } t\to\infty \tag{4.8}$$

for $\Re(i\lambda)>0$, where

$$c_{\alpha,\beta}(\lambda)=\frac{2^{\alpha+\beta+1-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma\left(\frac{i\lambda+\alpha+\beta+1}{2}\right)\Gamma\left(\frac{i\lambda+\alpha-\beta+1}{2}\right)}. \tag{4.9}$$

A continuous function $F\colon G \to \mathrm{End}(V_{\tau_p})$ is called elementary τ_p-spherical if F satisfies

(*i*) (τ_p-radial function) $F(k_1 g k_2) = \tau_p(k_2)^{-1} F(g) \tau_p(k_1^{-1}), \quad \forall g \in G, \forall k_1, k_2 \in K$,

(*ii*) F is a joint-eigenfunction of all $D \in \mathbf{D}(G/K; \tau_p)$ with $F(e) = Id$.

A τ_p-radial function $F\colon G \to \mathrm{End}(V_{\tau_p})$ (*i.e.* satisfying (*i*)) is determined by its restriction $F_{|A}$ to the subgroup A of G. Since A and M commute, $F_{|A}$ becomes an M-morphism of $V_{\tau_p} = \Lambda^p \mathbb{C}^n$. Now, in the generic case, $\tau_{p_{|M}}$ is multiplicity free, therefore by Schur's lemma, $F_{|A}$ is scalar on each M-irreducible component $V_{\sigma_p} = \Lambda^p \mathbb{C}^{n-1}$ and $V_{\sigma_{p-1}} = \Lambda^{p-1}\mathbb{C}^{n-1}$. Thus

$$F_{|A}(a_t) = f_{p-1}(t)\mathrm{Id}_{\Lambda^{p-1}\mathbb{C}^{n-1}} + f_p(t)\mathrm{Id}_{\Lambda^p\mathbb{C}^{n-1}},$$

the coefficients f_{p-1} and f_p are called the scalar components of F.

For $\lambda \in \mathbb{C}$, we define the Eisenstein integral $\Phi^p_q(\lambda, g) \in \mathrm{End}(V_{\tau_p})$ by

$$\Phi^p_q(\lambda, g) = c^2_{p,q} \int_K \mathrm{e}^{-(i\lambda+\rho)H(g^{-1}k)} \tau_p(\kappa(g^{-1}k)) \iota^p_q(\pi^q_p(\tau_p(k)^{-1}))\mathrm{d}k. \tag{4.10}$$

Proposition 4.5 (see [25, Theorem 5.4])**.** *Assume that* $0 \le p < \frac{n-1}{2}$.

(1) *The set* $\{\Phi^p_q(\lambda,\cdot), q = p-1, p; \lambda \in \mathbb{C} \setminus \{\pm 1\}\}$ *exhausts the class of* τ_p-*elementary spherical functions.*

(2) *The scalar components* $\varphi_{q,p-1}(\lambda,t)$, $\varphi_{q,p}(\lambda,t)$ *of* $\Phi^p_q(\lambda,a_t)$ *are given by*

$$\Phi^p_p(\lambda, a_t): \begin{cases} \varphi_{p,p-1}(\lambda,t) = \phi_\lambda^{(\frac{n}{2},-\frac12)}(t), \\ \varphi_{p,p}(\lambda,t) = \frac{n}{n-p}\phi_\lambda^{(\frac{n}{2}-1,-\frac12)}(t) - \frac{p}{n-p}(\cosh t)\phi_\lambda^{(\frac{n}{2},-\frac12)}(t), \end{cases} \tag{4.11}$$

and

$$\Phi^p_{p-1}(\lambda, a_t): \begin{cases} \varphi_{p-1,p-1}(\lambda,t) = \frac{n}{p}\phi_\lambda^{(\frac{n}{2}-1,-\frac12)}(t) - \frac{n-p}{p}(\cosh t)\phi_\lambda^{(\frac{n}{2},-\frac12)}(t), \\ \varphi_{p-1,p}(\lambda,t) = \phi_\lambda^{(\frac{n}{2},-\frac12)}(t). \end{cases} \tag{4.12}$$

For $q \in \{p-1, p\}$, let us introduce the notation $\rho_q = \rho - q = \frac{n-1}{2} - q$.

Proposition 4.6. *Let* $\lambda \in \mathbb{C}$ *such that* $\Re(i\lambda) > 0$. *The generalized Harish-Chandra c-function is given by*

$$\mathbf{c}(\lambda, p) = c_{p-1}(\lambda,p)\mathrm{Id}_{\Lambda^{p-1}\mathbb{C}^{n-1}} + c_p(\lambda,p)\mathrm{Id}_{\Lambda^p\mathbb{C}^{n-1}},$$

where the scalar coefficients are explicitly given by

$$c_{p-1}(\lambda,p) = \frac{i\lambda - \rho_{p-1}}{i\lambda+\rho} c(\lambda),$$

and

$$c_p(\lambda,p) = \frac{i\lambda + \rho_p}{i\lambda+\rho} c(\lambda),$$

with

$$c(\lambda) = 2^{\rho - i\lambda} \frac{\Gamma(i\lambda)\Gamma\left(\rho+\frac12\right)}{\Gamma\left(\frac{i\lambda+\rho}{2}\right)\Gamma\left(\frac{i\lambda+\rho+1}{2}\right)}.$$

Proof. Let $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$. Since

$$\Phi_q^p(\lambda, ka_t) = \mathcal{P}_{q,\lambda}^p\left(c_{p,q}\pi_p^q(\tau(k^{-1}))\right)(a_t),$$

Theorem 4.3 implies

$$\Phi_q^p(\lambda, a_t) = c_{p,q}^2\mathbf{c}(\lambda, p)\mathrm{e}^{(i\lambda-\rho)t}\left(\pi_p^q + o(1)\right) \quad \text{as } t \to \infty, \tag{4.13}$$

with

$$c_{p,q}^2 = \begin{cases} \frac{n}{n-p} & \text{if } q = p, \\ \frac{n}{p} & \text{if } q = p-1. \end{cases}$$

Let us first consider the case $q = p$. Using the asymptotic behavior of Jacobi functions (4.8) together with the relation

$$c_{\frac{n}{2},-\frac{1}{2}}(\lambda) = \frac{2n}{i\lambda+\rho}c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda),$$

we obtain

$$\begin{aligned}\varphi_{p,p}(\lambda, t) &\underset{t\to\infty}{=} \frac{1}{n-p}\mathrm{e}^{(i\lambda-\rho)t}\left(nc_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda) - \frac{p}{2}c_{\frac{n}{2},-\frac{1}{2}}(\lambda) + o(1)\right), \\ &\underset{t\to\infty}{=} \mathrm{e}^{(i\lambda-\rho)t}\frac{n}{n-p}c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda)\left(\frac{i\lambda+\rho-p}{i\lambda+\rho} + o(1)\right).\end{aligned}$$

Similarly, we get

$$\varphi_{p,p-1}(\lambda, t) \underset{t\to\infty}{=} \mathrm{e}^{(i\lambda-\rho-1)t}\left(c_{\frac{n}{2},-\frac{1}{2}}(\lambda) + o(1)\right).$$

Thus

$$\begin{aligned}\Phi_p^p(\lambda, a_t) &= \mathrm{e}^{(i\lambda-\rho-1)t}\left(c_{\frac{n}{2},-\frac{1}{2}}(\lambda) + o(1)\right)\mathrm{Id}_{\Lambda^{p-1}\mathbb{C}^{n-1}} \\ &\quad + \mathrm{e}^{(i\lambda-\rho)t}\frac{n}{n-p}c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda)\left(\frac{i\lambda+\rho-p}{i\lambda+\rho} + o(1)\right)\mathrm{Id}_{\Lambda^p\mathbb{C}^{n-1}},\end{aligned}$$

from which we deduce that

$$\lim_{t\to\infty}\mathrm{e}^{(\rho-i\lambda)t}\Phi_p^p(\lambda, a_t) = \frac{n}{n-p}\left(\frac{i\lambda+\rho-p}{i\lambda+\rho}\right)c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda)\mathrm{Id}_{\Lambda^p\mathbb{C}^{n-1}}. \tag{4.14}$$

Finally, by identification of (4.13) and (4.14) it follows that

$$c_p(\lambda, p) = \frac{i\lambda+\rho-p}{i\lambda+\rho}c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda) = \frac{i\lambda+\rho-p}{i\lambda+\rho}c(\lambda).$$

Similarly, for $q = p-1$ we can prove that

$$\lim_{t\to\infty}\mathrm{e}^{(i\lambda-\rho)t}\Phi_{p-1}^p(\lambda, a_t) = \frac{n}{p}\left(\frac{i\lambda-\rho+p-1}{i\lambda+\rho}\right)c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda)\mathrm{Id}_{\Lambda^{p-1}\mathbb{C}^{n-1}},$$

from which we deduce that

$$c_{p-1}(\lambda, p) = \frac{i\lambda-\rho+p-1}{i\lambda+\rho}c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda) = \frac{i\lambda-\rho+p-1}{i\lambda+\rho}c(\lambda).$$

□

5. The L^2-range of the Poisson transform

Recall that our main goal is to characterize the image of the space $L^r(K/M;\sigma_q)$ under the Poisson transform $\mathcal{P}^p_{q,\lambda}$, for $1<r<\infty$. To do so, we will start with the case $r=2$.
Fix $\sigma_q\in\widehat{M}$ acting on the space $V_{\sigma_q}=\Lambda^q\mathbb{C}^{n-1}$ of dimension d_{σ_q}. To simplify notations, we will write sometimes (σ,V_σ) instead of (σ_q,V_{σ_q}).

Let (δ,V_δ) be an element in $\widehat{K}(\sigma)$, where $\widehat{K}(\sigma)\subset\widehat{K}$ denotes the subset of those classes containing σ upon restriction to K. It follows from Frobenius reciprocity theorem together with [13] that σ occurs in $\delta_{|M}$ with multiplicity one and therefore $\dim\mathrm{Hom}_M(V_\delta,V_\sigma)=1$. We choose the orthogonal projection $P_\delta:V_\delta\to V_\sigma$ as a generator of $\mathrm{Hom}_M(V_\delta,V_\sigma)$.

let $(v_j)_{j=1}^{d_\delta}$ be an orthonormal basis for V_δ, where $d_\delta=\dim V_\delta$. Then the functions

$$k\mapsto\phi^\delta_j(k)=P_\delta(\delta(k^{-1})v_j),\quad 1\le j\le d_\delta,\ \delta\in\widehat{K}(\sigma)$$

define an orthogonal basis of the space $L^2(K/M;\sigma_q)$, see, e.g. [30]. Thus, the Fourier expansion of every $f\in L^2(K/M;\sigma_q)$ is given by

$$f(k)=\sum_{\delta\in\widehat{K}(\sigma)}\sum_{j=1}^{d_\delta}a^\delta_j\phi^\delta_j(k),$$

with

$$\| f\|^2_{L^2(K/M;\sigma)}=\sum_{\delta\in\widehat{K}(\sigma)}\frac{d_\delta}{d_\sigma}\sum_{j=1}^{d_\delta}|a^\delta_j|^2. \tag{5.1}$$

Next, we will prove a general result giving the Poisson integral representation of a joint eigensections of the algebra $\mathbf{D}(G/K;\tau_p)$ of G-invariant differential operators acting on $C^\infty(G/K;\tau_p)$.

By a functional on $E_{q,\lambda}=G\times_P V_{\sigma_q}$ we shall mean a linear form T on $C^\infty(G/P;\sigma_{q,\bar\lambda})$. For a such functional T, we define $\widetilde{\mathcal{P}^p_{q,\lambda}}(T)$ by

$$\langle v,\widetilde{\mathcal{P}^p_{q,\lambda}}T(g)\rangle_{\Lambda^p\mathbb{C}^n}=c_{p,q}(T,\pi^q_pL_g\Phi_\lambda v),\quad\forall v\in\Lambda^p\mathbb{C}^n \tag{5.2}$$

where L_g is the left regular action, and $\Phi_\lambda\colon G\to\mathrm{End}(V_{\tau_p})$ is given by

$$\Phi_\lambda(g)=\mathrm{e}^{(i\bar\lambda-\rho)H(g)}\tau_p^{-1}(\kappa(g)). \tag{5.3}$$

Notice that $\Phi_\lambda(g^{-1}k)^*=P^p_{q,\lambda}(g,k)$, where $P^p_{q,\lambda}\colon G\times K\to\mathrm{End}(V_{\tau_p})$ is the Poisson kernel given by

$$P^p_{q,\lambda}(g,k)=\mathrm{e}^{-(i\lambda+\rho)H(g^{-1}k)}\tau_p(\kappa(g^{-1}k)). \tag{5.4}$$

If $T=T_f$ is a functional given by $f\in C^\infty(G/P;\sigma_{q,\lambda})$, then

$$\widetilde{\mathcal{P}^p_{q,\lambda}}(T_f)=\mathcal{P}^p_{q,\lambda}(f). \tag{5.5}$$

Indeed,

$$\begin{aligned}
\langle v, \widetilde{\mathcal{P}^p_{q,\lambda}} T_f(g)\rangle_{\Lambda^p\mathbb{C}^n} &= c_{p,q}(T, \pi_p^q L_g \Phi_\lambda v),\\
&= c_{p,q}\int_K \langle f(k), \pi_p^q L_g\Phi_\lambda(k)v\rangle_{\Lambda^q\mathbb{C}^{n-1}}\mathrm{d}k,\\
&= c_{p,q}\int_K \langle f(k), \pi_p^q \Phi_\lambda(g^{-1}k)v\rangle_{\Lambda^q\mathbb{C}^{n-1}}\mathrm{d}k,\\
&= c_{p,q}\int_K \langle \Phi_\lambda^*(g^{-1}k)\iota_q^p f(k), v\rangle_{\Lambda^p\mathbb{C}^n}\mathrm{d}k,\\
&= c_{p,q}\int_K \langle P^p_{q,\lambda}(g,k)\iota_q^p f(k), v\rangle_{\Lambda^p\mathbb{C}^n}\mathrm{d}k,\\
&= \langle v, \mathcal{P}^p_{q,\lambda} f(g)\rangle_{\Lambda^p\mathbb{C}^n}.
\end{aligned}$$

Proposition 5.1. *For every eigensection F of $\mathbf{D}(G/K;\tau_p)$, there exists a functional T on $C^\infty(G/P;\sigma_{q,\bar\lambda})$ such that $F = \widetilde{\mathcal{P}^p_{q,\lambda}} T$.*

Proof. Let F be an arbitrary joint eigensection of all $D \in \mathbf{D}(G/K;\tau_p)$. Then F has an expansion

$$F(g) = \sum_{\delta\in\widehat{K}(\sigma)} F_\delta(g)$$

in $C^\infty(G/K;\tau_p)$. Since F_δ is K-finite of type δ, then, by [32, Corollary 10.8], there exists a K-finite vector f_δ in $C^\infty(G/P;\sigma_{q,\lambda})$ such that $F_\delta = \mathcal{P}^p_{q,\lambda} f_\delta$. We have

$$f_\delta(k) = \sum_{j=1}^{d_\delta} a_j^\delta P_\delta(\delta(k^{-1})v_j).$$

Define a functional T by

$$(T,\varphi) = \sum_{\delta\in\widehat{K}(\sigma)}\sum_{j=1}^{d_\delta} \overline{a_j^\delta}\int_K \langle \varphi(k), P_\delta(\delta(k^{-1})v_j)\rangle_{\Lambda^q\mathbb{C}^{n-1}}\,\mathrm{d}k, \tag{5.6}$$

for all $\varphi \in C^\infty(G/P;\sigma_{q,\overline{\lambda}})$ for which the above sum converges. Choose φ in (5.6) to be $\varphi\colon k \mapsto c_{p,q}\pi_p^q(\Phi_\lambda(g^{-1}k)w)$ with $w \in V_{\tau_p}$, then we get

$$\begin{aligned}
(T,\varphi) &= c_{p,q}\sum_{\delta\in\widehat{K}(\sigma)}\sum_{j=1}^{d_\delta}\overline{a_j^\delta}\int_K \langle \pi_p^q\Phi_\lambda(g^{-1}k)w, P_\delta(\delta(k^{-1})v_j)\rangle_{\Lambda^q\mathbb{C}^{n-1}}\,\mathrm{d}k,\\
&= c_{p,q}\sum_\delta\sum_j\overline{a_j^\delta}\int_K \langle w, \Phi_\lambda(g^{-1}k)^*(\pi_p^q)^*P_\delta(\delta(k^{-1})v_j)\rangle_{\Lambda^p\mathbb{C}^n}\,\mathrm{d}k,\\
&= c_{p,q}\sum_\delta\sum_j\overline{a_j^\delta}\int_K \Big\langle w, \mathrm{e}^{-(i\lambda+\rho)H(g^{-1}k)}\tau_p(\kappa(g^{-1}k))\iota_q^pP_\delta(\delta(k^{-1})v_j)\Big\rangle_{\Lambda^p\mathbb{C}^n}\,\mathrm{d}k,\\
&= \big\langle w, \sum_\delta c_{p,q}\int_K \mathrm{e}^{-(i\lambda+\rho)H(g^{-1}k)}\tau_p(\kappa(g^{-1}k))\iota_q^p\sum_j a_j^\delta P_\delta(\delta(k^{-1})v_j)\,\mathrm{d}k\big\rangle_{\Lambda^p\mathbb{C}^n},\\
&= \big\langle w, \sum_{\delta\in\widehat{K}(\sigma)}\mathcal{P}^q_{\lambda,p}f_\delta(g)\big\rangle_{\Lambda^p\mathbb{C}^n},\\
&= \big\langle w, F(g)\big\rangle_{\Lambda^p\mathbb{C}^n}.
\end{aligned}$$

On the other hand, by the definition (5.2) of the Poisson transform on functionals, we have

$$(T, c_{p,q}\pi_p^q(L_g\Phi_\lambda w)) = \langle w, \widetilde{\mathcal{P}^p_{q,\lambda}}T(g)\rangle_{\Lambda^p\mathbb{C}^n},$$

from which we deduce that $F(g) = \widetilde{\mathcal{P}^p_{q,\lambda}}T(g)$, since the vector w is arbitrary. □

Theorem 5.2. *Assume that $\lambda \in \mathbb{C}$ such that*

$$\begin{cases}\Re(i\lambda) > 0 & \text{if } q = p,\\ \Re(i\lambda) > 0 \ \text{ and } \ i\lambda \neq \rho - p + 1 & \text{if } q = p-1.\end{cases} \tag{5.7}$$

The Poisson transform $\mathcal{P}^p_{q,\lambda}$ is a topological isomorphism from the space $L^2(K/M;\sigma_q)$ onto the space $\mathcal{E}^2_{q,\lambda}(G/K;\tau_p)$. Moreover, there exists a positive constant γ_λ such that

$$c_{p,q}|c_q(\lambda,p)|\|f\|_{L^2(K/M;\sigma_q)} \leq \|\mathcal{P}^p_{q,\lambda}f\|_{\mathcal{E}^2_{q,\lambda}} \leq c_{p,q}\,\gamma_\lambda\|f\|_{L^2(K/M;\sigma_q)},$$

for every $f \in L^2(K/M;\sigma_q)$.

Proof. On one hand, by Proposition 3.2 and Proposition 4.4 it follows that $\mathcal{P}^p_{q,\lambda}$ is a continuous map from $L^2(K/M;\sigma_q)$ into $\mathcal{E}^2_{q,\lambda}(G/K;\tau_p)$.

On the other hand, for $F \in \mathcal{E}^2_{q,\lambda}(G/K;\tau_p)$, by Proposition 5.1, there exists a functional T on $C^\infty(G/P;\sigma_{q,\overline{\lambda}})$ defined by (5.6) such that $F = \widetilde{\mathcal{P}^p_{q,\lambda}}T$. From the proof of Proposition 5.1, it follows that

$$F(g) = c_{p,q}\sum_{\delta\in\widehat{K}(\sigma)}\sum_{j=1}^{d_\delta}a_j^\delta\int_K \mathrm{e}^{-(i\lambda+\rho)H(g^{-1}k)}\tau_p(\kappa(g^{-1}k))\iota_q^pP_\delta(\delta(k^{-1})v_j)\mathrm{d}k.$$

Define $\Phi_{\lambda,\delta}$ by

$$\Phi_{\lambda,\delta}(g)(v) = c_{p,q}\int_K \mathrm{e}^{-(i\lambda+\rho)H(g^{-1}k)}\tau_p(\kappa(g^{-1}k))\iota_q^pP_\delta(\delta(k^{-1})v)\mathrm{d}k, \tag{5.8}$$

for $g \in G$ and $v \in V_\delta$. Clearly $\Phi_{\lambda,\delta}(k_1 g k_2) = \tau_p(k_2^{-1})\Phi_{\lambda,\delta}(g)\delta(k_1^{-1})$ for every $g \in G$ and $k_1, k_2 \in K$. Further

$$\int_K \langle F(ka_t), F(ka_t)\rangle_{\Lambda^p\mathbb{C}^n} \mathrm{d}k = \sum_{\delta,\delta'}\sum_{j,\ell} a_j^\delta \overline{a_\ell^{\delta'}} \int_K \langle \Phi_{\lambda,\delta}(ka_t)v_j, \Phi_{\lambda,\delta'}(ka_t)v_\ell\rangle_{\Lambda^p\mathbb{C}^n}\, \mathrm{d}k.$$

By the covariance property and Schur's lemma, we obtain

$$\begin{aligned}\int_k \langle \Phi_{\lambda,\delta}(ka_t)v_j, \Phi_{\lambda,\delta'}(ka)v_\ell\rangle_{\Lambda^p\mathbb{C}^n}\, \mathrm{d}k &= \int_K \langle \Phi_{\lambda,\delta'}(a_t)^*\Phi_{\lambda,\delta}(a_t)\delta(k^{-1})v_j, \delta'(k^{-1})v_\ell\rangle_{V_\delta}\, \mathrm{d}k \\ &= \begin{cases} 0 & \text{if } \delta' \nsim \delta \\ \frac{1}{d_\delta}\mathbf{tr}\left(\Phi_{\lambda,\delta}(a_t)^*\Phi_{\lambda,\delta}(a_t)\right)\langle v_j, v_\ell\rangle_{V_\delta} & \text{otherwise}\end{cases}\end{aligned}$$

Thus

$$\begin{aligned}\int_K \parallel F(ka_t) \parallel^2_{\Lambda^p\mathbb{C}^n}\, \mathrm{d}k &= \sum_{\delta\in\widehat{K}(\sigma)} \frac{1}{d_\delta}\sum_{j=1}^{d_\delta} |a_j^\delta|^2 \mathbf{tr}\left(\Phi_{\lambda,\delta}(a_t)^*\Phi_{\lambda,\delta}(a_t)\right), \\ &= \sum_\delta \frac{1}{d_\delta}\|\Phi_{\lambda,\delta}(a_t)\|^2_{\mathrm{HS}} \sum_j |a_j^\delta|^2,\end{aligned}$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm. Hence, for a finite subset $\Lambda \subset \widehat{K}(\sigma)$ we get

$$\begin{aligned}\sum_{\delta\in\Lambda}\frac{1}{d_\delta}\sum_j \|a_j^\delta \mathrm{e}^{(\rho-i\lambda)t}\Phi_{\lambda,\delta}(a_t)\|^2_{\mathrm{HS}} &\leq \sup_{t>0} \mathrm{e}^{2(\rho-\Re(i\lambda))t}\int_K \parallel F(ka_t) \parallel^2_{\Lambda^p\mathbb{C}^n}\, \mathrm{d}k, \\ &= \|F\|^2_{\mathcal{E}^2_{2,\lambda}}.\end{aligned}$$

Under the assumption (5.7) we may use Theorem 4.3, *i.e.*,

$$\lim_{t\to\infty} \mathrm{e}^{(\rho-i\lambda)t}\Phi_{\lambda,\delta}(a_t) = c_{p,q}\mathbf{c}(\lambda,p)\iota_q^p P_\delta, \tag{5.9}$$

and (4.5) to obtain

$$c_{p,q}^2|c_q(\lambda,p)|^2\sum_{\delta\in\Lambda}\frac{1}{d_\delta}\sum_j \|a_j^\delta P_\delta\|^2_{\mathrm{HS}} \leq \|F\|^2_{\mathcal{E}^2_{2,\lambda}}.$$

That is

$$c_{p,q}^2|c_q(\lambda,p)|^2\sum_{\delta\in\Lambda}\frac{1}{d_\delta}\sum_j d_\sigma|a_j^\delta|^2 \leq \|F\|^2_{\mathcal{E}^2_{2,\lambda}}.$$

Since the subset $\Lambda \subset \widehat{K}(\sigma)$ is arbitrary, it follows that

$$c_{p,q}^2|c_q(\lambda,p)|^2\sum_{\delta\in\widehat{K}(\sigma)}\frac{d_\sigma}{d_\delta}\sum_j \mid a_j^\delta \mid^2 \leq \parallel F \parallel^2_{\mathcal{E}^2_{2,\lambda}} < \infty.$$

This shows that the functional $T(k) \sim \sum_{\delta\in\widehat{K}(\sigma)}\sum_{j=1}^{d_\delta} a_j^\delta P_\delta \delta(k^{-1})v_j$ defines a function $f \in L^2(K/M;\sigma)$ and by (5.5), we deduce that $F = \mathcal{P}^p_{q,\lambda}f$ with

$$c_{p,q}|c_q(\lambda,p)| \parallel f \parallel_{L^2(K/M;\sigma)} \leq \parallel \mathcal{P}^p_{q,\lambda}f \parallel_{\mathcal{E}^2_{2,\lambda}}.$$

□

Lemma 5.3. *We have*

$$\sup_{t>0} \mathrm{e}^{(\rho-\Re(i\lambda))t}\|\Phi_{\lambda,\delta}(a_t)\|_{\mathrm{HS}} \leq \gamma_\lambda c_{p,q}\|P_\delta\|_{\mathrm{HS}} = \gamma_\lambda c_{p,q}\sqrt{d_\sigma}.$$

Proof. By Proposition 4.1 we have

$$\sup_{t>0} \mathrm{e}^{(\rho-\Re(i\lambda))t}\left(\int_K \|\mathcal{P}^p_{q,\lambda}(P_\delta(\delta^{-1}(\cdot)v))(ka_t)\|^2_{\Lambda^p\mathbb{C}^n}\mathrm{d}k\right)^{1/2} \leq \gamma_\lambda c_{p,q}\|P_\delta(\delta^{-1}(\cdot)v)\|_{L^2(K/M;\,\sigma_q)}.$$

Since $\mathcal{P}^p_{q,\lambda}(P_\delta(\delta^{-1}(\cdot)v))(ka_t) = \Phi_{\lambda,\delta}(ka_t)(v)$, we get

$$\begin{aligned}\int_K \|\mathcal{P}^p_{q,\lambda}(P_\delta(\delta^{-1}(\cdot)v)(ka_t))\|^2_{\Lambda^p\mathbb{C}^n}\mathrm{d}k &= \int_K \langle\Phi_{\lambda,\delta}(a_t)\delta(k^{-1})v, \Phi_{\lambda,\delta}(a_t)\delta(k^{-1})v\rangle_{\Lambda^p\mathbb{C}^n}\,\mathrm{d}k,\\ &= \frac{1}{d_\delta}\mathbf{tr}\left(\Phi_{\lambda,\delta}(a_t)^*\Phi_{\lambda,\delta}(a_t)\right)\|v\|^2_{V_\delta},\\ &= \frac{1}{d_\delta}\|\Phi_{\lambda,\delta}(a_t)\|^2_{\mathrm{HS}}\|v\|^2_{V_\delta}.\end{aligned}$$

Now the desired inequality follows from

$$\|P_\delta(\delta^{-1}(\cdot)v)\|^2_{L^2(K/M;\,\sigma_q)} = \frac{d_\sigma}{d_\delta}\,\|v\|^2_{V_\delta}.$$

□

Lemma 5.4. *We have*

$$\lim_{t\to\infty} \mathrm{e}^{2(\rho-\Re(i\lambda))t}\|\Phi_{\lambda,\delta}(a_t)\|^2_{\mathrm{HS}} = c^2_{p,q}|c_q(\lambda,p)|^2 d_{\sigma_q},$$

where $c_q(\lambda,p)$ is the scalar component of $\mathbf{c}(\lambda,p)$ on $V_{\sigma_q} = \Lambda^q\mathbb{C}^{n-1}$.

Proof. Recall that $\Phi_{\lambda,\delta}(a_t) = \mathcal{P}^p_{q,\lambda}(P_\delta(\delta^{-1}(\cdot)))(a_t)$. Then

$$\begin{aligned}e^{2(\rho-\Re(i\lambda))t}\|\Phi_{\lambda,\delta}(a_t)\|^2_{\mathrm{HS}} &= \sum_{j=1}^{d_\delta}\|e^{(\rho-\Re(i\lambda))t}\Phi_{\lambda,\delta}(a_t)v_j\|^2_{\Lambda^p\mathbb{C}^n},\\ &= \sum_{j=1}^{d_\delta}\|e^{(\rho-\Re(i\lambda))t}\mathcal{P}^p_{q,\lambda}(P_\delta(\delta^{-1}(\cdot)v_j))(a_t)\|^2_{\Lambda^p\mathbb{C}^n}.\end{aligned}$$

Using Theorem 4.3 and (4.5), we obtain

$$\begin{aligned}\lim_{t\to\infty} e^{2(\rho-\Re(i\lambda))t}\|\Phi_{\lambda,\delta}(a_t)\|^2_{\mathrm{HS}} &= c^2_{p,q}\sum_{j=1}^{d_\delta}\langle\mathbf{c}(\lambda,p)i^p_qP_\delta v_j, \mathbf{c}(\lambda,p)\iota^p_qP_\delta v_j\rangle_{\Lambda^p\mathbb{C}^n}.\\ &= c^2_{p,q}\|\mathbf{c}(\lambda,p)\iota^p_qP_\delta\|_{\mathrm{HS}}\\ &= c^2_{p,q}|c_q(\lambda,p)|^2 d_{\sigma_q}.\end{aligned}$$

□

Theorem 5.5 (Inversion formula)**.** *Assume $\lambda\in\mathbb{C}$ such that*

$$\begin{cases}\Re(i\lambda) > 0 & \textit{if } q = p,\\ \Re(i\lambda) > 0 \ \textit{ and } \ i\lambda \neq \rho - p + 1 & \textit{if } q = p-1.\end{cases}$$

Let $F \in \mathcal{E}^2_{q,\lambda}(G/K;\tau_p)$ and let $f \in L^2(K/M;\sigma_q)$ be its boundary value. Then the following inversion formula holds in $L^2(K/M;\sigma_q)$

$$f(k) = c_{p,q}^{-1}|c_q(\lambda,p)|^{-2} \lim_{t\to\infty} e^{2(\rho-\Re(i\lambda))t}\pi_p^q \left(\int_K P^p_{q,\lambda}(ha_t,k)^* F(ha_t)\,\mathrm{d}h\right),$$

where $P^p_{q,\lambda}(\cdot,\cdot)$ is the Poisson kernel given in (5.4).

Proof. Let $F \in \mathcal{E}^2_{q,\lambda}(G/K;\tau_p)$. By Theorem 5.2, there exists a unique $f \in L^2(K/M;\sigma_q)$ such that $F = \mathcal{P}^p_{q,\lambda} f$. Write

$$f(k) = \sum_{\delta\in\widehat{K}(\sigma_q)} \sum_{j=1}^{d_\delta} a_j^\delta P_\delta(\delta(k^{-1}))v_j.$$

Then

$$F(ka_t) = \sum_\delta \sum_j a_j^\delta \Phi_{\lambda,\delta}(a_t)\delta(k^{-1})v_j,$$

and therefore

$$\int_K \|F(ka_t)\|^2_{\Lambda^p\mathbb{C}^n}\,\mathrm{d}k = \sum_\delta\sum_j \frac{|a_j^\delta|^2}{d_\delta}\|\Phi_{\lambda,\delta}(a_t)\|^2_{\mathrm{HS}}.$$

From Lemma 5.4 we deduce

$$\lim_{t\to\infty} \mathrm{e}^{2(\rho-\mathbb{R}(i\lambda))t}\int_K \|\mathcal{P}^p_{q,\lambda}f(ka_t)\|^2_{\Lambda^p\mathbb{C}^n}\,\mathrm{d}k = c_{p,q}^2|c_q(\lambda,p)|^2\|f\|^2_{L^2(K/M;\sigma_q)},$$

which implies

$$\lim_{t\to\infty}(g_t,\varphi)_{L^2(K/M;\,\sigma_q)} = (f,\varphi)_{L^2(K/M;\,\sigma_q)}, \quad \forall\varphi\in L^2(K/M;\sigma_q),$$

where g_t is the V_{σ_q}-valued function defined by

$$g_t(k) = c_{p,q}^{-1}|c_q(\lambda,p)|^{-2}\mathrm{e}^{2(\rho-\Re(i\lambda))t}\pi_p^q\int_K P^p_{q,\lambda}(ha_t,k)^*F(ha_t)\,\mathrm{d}h.$$

To obtain the inversion formula, it is only required to show that

$$\lim_{t\to\infty}\|g_t\|_{L^2(K/M;\,\sigma_q)} = \|f\|_{L^2(K/M;\,\sigma_q)}.$$

To do so, let us first compute the Fourier coefficients $c_j^\delta(g_t)$ of g_t:

$$\begin{aligned} c_j^\delta(g_t) &= \frac{d_\delta}{d_\sigma}\int_K \langle g_t(k), P_\delta\delta(k^{-1})v_j\rangle_{\Lambda^q\mathbb{C}^{n-1}}\,\mathrm{d}k \\ &= c_{p,q}^{-1}|c_q(\lambda,p)|^{-2}\mathrm{e}^{2(\rho-\Re(i\lambda))t} \\ &\times \frac{d_\delta}{d_\sigma}\sum_{\delta',\ell} a_\ell^{\delta'}\int_K \left\langle \pi_p^q\int_K P^p_{q,\lambda}(ha_t,k)^*\Phi_{\lambda,\delta'}(a_t)\delta'(h^{-1})v_\ell\mathrm{d}h, P_\delta\delta(k^{-1})v_j\right\rangle_{\Lambda^q\mathbb{C}^{n-1}}\mathrm{d}k. \end{aligned}$$

Since $(\pi_p^q)^* = \iota_q^p$, we get

$$\begin{aligned} c_j^\delta(g_t) &= c_{p,q}^{-1}|c_q(\lambda,p)|^{-2}\mathrm{e}^{2(\rho-\Re(i\lambda))t} \\ &\times \frac{d_\delta}{d_\sigma}\sum_{\delta',\ell} a_\ell^{\delta'}\int_K\int_K \left\langle \Phi_{\lambda,\delta'}(a_t)\delta'(h^{-1})v_\ell, P_{q,\lambda}^p(ha_t,k)\iota_q^p P_\delta\delta(k^{-1})v_j\right\rangle_{\Lambda^p\mathbb{C}^n}\mathrm{d}h\mathrm{d}k, \end{aligned}$$

As $\int_K P_{q,\lambda}^p(ha_t,k)\iota_q^p P_\delta\delta(k^{-1})\mathrm{d}k = c_{p,q}^{-1}\Phi_{\lambda,\delta}(ha_t)$, we obtain

$$\begin{aligned} c_j^\delta(g_t) &= c_{p,q}^{-2}|c_q(\lambda,p)|^{-2}\mathrm{e}^{2(\rho-\Re(i\lambda))t} \\ &\times \frac{d_\delta}{d_\sigma}\sum_{\delta',\ell} a_\ell^{\delta'}\int_K \langle \Phi_{\lambda,\delta'}(a_t)\delta'(h^{-1})v_\ell, \Phi_{\lambda,\delta}(a_t)\delta(h^{-1})v_j\rangle_{\Lambda^p\mathbb{C}^n}\,\mathrm{d}h, \\ &= c_{p,q}^{-2}|c_q(\lambda,p)|^{-2}\mathrm{e}^{2(\rho-\Re(i\lambda))t} \\ &\times \frac{d_\delta}{d_\sigma}\sum_{\delta',\ell} a_\ell^{\delta'}\int_K \langle \delta(h)\Phi_{\lambda,\delta}(a_t)^*\Phi_{\lambda,\delta'}(a_t)\delta'(h^{-1})v_\ell, v_j\rangle_{\Lambda^p\mathbb{C}^n}\,\mathrm{d}h. \end{aligned}$$

By the Schur lemma, we get

$$\begin{aligned} c_j^\delta(g_t) &= c_{p,q}^{-2}|c_q(\lambda,p)|^{-2}\mathrm{e}^{2(\rho-\Re(i\lambda))t} \\ &\times \frac{d_\delta}{d_\sigma}\sum_{\ell} a_\ell^{\delta}\int_K \frac{1}{d_\delta}\mathbf{tr}\left(\Phi_{\lambda,\delta}(a_t)^*\Phi_{\lambda,\delta}(a_t)\right)\langle v_\ell, v_j\rangle_{V_\delta}\,\mathrm{d}h, \\ &= c_{p,q}^{-2}|c_q(\lambda,p)|^{-2}\mathrm{e}^{2(\rho-\Re(i\lambda))t}\frac{1}{d_\sigma}a_j^\delta\|\Phi_{\lambda,\delta}(a_t)\|_{\mathrm{HS}}^2. \end{aligned}$$

From all the above computations, we conclude that,

$$\|g_t\|_{L^2(K/M,\sigma)}^2 = \left(\mathrm{e}^{2(\rho-\Re(i\lambda))t}|c_{p,q}c_q(\lambda,p)|^{-2}\right)^2\sum_\delta\frac{d_\sigma}{d_\delta}\sum_j\frac{1}{d_\sigma^2}|a_j^\delta|^2\|\Phi_{\lambda,\delta}(a_t)\|_{\mathrm{HS}}^4,$$

and by Lemma 5.4 we get

$$\lim_{t\to\infty}\|g_t\|_{L^2(K/M;\sigma)}^2 = \sum_\delta\frac{d_\sigma}{d_\delta}\sum_j|a_j^\delta|^2 = \|f\|_{L^2(K/M;\sigma)}^2.$$

To finish the proof, we have to justify that we can reverse $\lim_{t\to\infty}$ and $\sum_\delta$ by proving that the serie

$$\sum_{\delta\in\widehat{K}(\sigma)}\frac{1}{d_\delta}\sum_{j=1}^{d_\delta}|a_j^\delta|^2\|\Phi_{\lambda,\delta}(a_t)\|_{\mathrm{HS}}^4,$$

is uniformly convergent. This follows easily from Lemma 5.3. □

6. The L^r-range of the Poisson transform

In this section we shall generalize Theorem 5.2 to $L^r(K/M;\sigma_q)$ with $1<r<\infty$.

Theorem 6.1. *Let $0\le p<(n-1)/2$, and $\lambda\in\mathbb{C}$ such that*

$$\begin{cases}\Re(i\lambda)>0 & \text{if } q=p,\\ \Re(i\lambda)>0 \text{ and } i\lambda\neq\rho-p+1 & \text{if } q=p-1.\end{cases}$$

For $1 < r < \infty$, the Poisson transform $\mathcal{P}^p_{q,\lambda}$ is a topological isomorphism from the space $L^r(K/M;\sigma_q)$ onto the space $\mathcal{E}^r_{q,\lambda}(G/K;\tau_p)$. Moreover, there exists a positive constant γ_λ such that

$$c_{p,q}|c_q(\lambda,p)|\|f\|_{L^r(K/M;\sigma_q)} \leq \|\mathcal{P}^p_{q,\lambda}f\|_{\mathcal{E}^r_{q,\lambda}} \leq c_{p,q}\,\gamma_\lambda\|f\|_{L^r(K/M;\sigma_q)},$$

for every $f \in L^r(K/M;\sigma_q)$.

Proof. The necessary condition follows from Proposition 3.2 and Proposition 4.4. For the sufficiency condition, let $F \in \mathcal{E}^r_{q,\lambda}(G/K;\tau_p)$ and write $F(g) = \sum_i F_i(g)u_i$ where $(u_i)_i$ is an orthonormal basis of $\Lambda^p\mathbb{C}^n$. Fix $(\chi_m)_m$ to be an approximation of the identity in $C^\infty(K)$ and let $F_{i,m}(g) = \int_K \chi_m(k)F_i(k^{-1}g)\mathrm{d}k$. Then $(F_{i,m})_m$ converges point-wise to F_i. Define $F_m : G \to \Lambda^p\mathbb{C}^n$ by $F_m(g) = \sum_i F_{i,m}(g)u_i$. Then

$$\begin{aligned}
F_m(g) &= \sum_i \left(\int_K \chi_m(k)F_i(k^{-1}g)\mathrm{d}k\right)u_i,\\
&= \int_K \chi_m(k)\sum_i F_i(k^{-1}g)u_i\mathrm{d}k,\\
&= \int_K \chi_m(k)F(k^{-1}g)\mathrm{d}k.
\end{aligned}$$

We have $\|F_m(g) - F(g)\|^2_{\Lambda^p\mathbb{C}^n} \underset{m\to\infty}{\to} 0$ and since the operators $\mathcal{C}$, D and D^* in (3.2) are K-invariant, then $F_m \in \mathcal{E}_{q,\lambda}(G/K;\tau_p)$ for every m. Further,

$$\begin{aligned}
F_m(ka_t) &= \int_K \chi_m(h)F(h^{-1}ka_t)\mathrm{d}h,\\
&= \left(\chi_m * F^t\right)(k),
\end{aligned}$$

where $F^t : K \to \Lambda^p\mathbb{C}^n$ is defined for any $t > 0$ by $F^t(k) = F(ka_t)$ for every F. By (3.5) we have

$$\|(\chi_m * F^t)(k)\|_{\Lambda^p\mathbb{C}^n} \leq \int_K |\chi_m(h)|\|F^t(h^{-1}k)\|_{\Lambda^p\mathbb{C}^n}\mathrm{d}h,$$

that is

$$\|F^t_m(k)\|_{\Lambda^p\mathbb{C}^n} \leq \left(|\chi_m(\cdot)| * \|F^t(\cdot)\|_{\Lambda^p\mathbb{C}^n}\right)(k).$$

Therefore

$$\|F^t_m\|_{L^r(K;\Lambda^p\mathbb{C}^n)} \leq \left\||\chi_m(\cdot)| * \|F^t(\cdot)\|_{\Lambda^p\mathbb{C}^n}\right\|_{L^r(K)}.$$

By Young's inequalities we obtain

$$\begin{aligned}
\|F^t_m\|_{L^r(K;\Lambda^p\mathbb{C}^n)} &\leq \|\chi_m\|_{L^1(K)}\left\|\|F^t(\cdot)\|_{\Lambda^p\mathbb{C}^n}\right\|_{L^r(K)},\\
&= \|F^t\|_{L^r(K;\Lambda^p\mathbb{C}^n)}, \qquad (6.1)
\end{aligned}$$

and

$$\begin{aligned}
\|F^t_m\|_{L^2(K;\Lambda^p\mathbb{C}^n)} &\leq \|\chi_m\|_{L^2(K)}\left\|\|F^t(\cdot)\|_{\Lambda^p\mathbb{C}^n}\right\|_{L^1(K)},\\
&= \|\chi_m\|_{L^2(K)}\,\|F^t\|_{L^r(K;\Lambda^p\mathbb{C}^n)}. \qquad (6.2)
\end{aligned}$$

The inequality (6.2) implies

$$\sup_{t>0} \mathrm{e}^{(\rho-\Re(i\lambda))t} \left(\int_K \|F_m(ka_t)\|^2_{\Lambda^p\mathbb{C}^n} \right)^{1/2} \leq \|\chi_m\|_{L^2(K)} \|F\|_{\mathcal{E}^r_{q,\lambda}} < \infty.$$

Hence, for each m, $F_m \in \mathcal{E}^2_{q,\lambda}(G/K;\tau_p)$ and from Theorem 5.2 it follows that there exists $f_m \in L^2(K/M;\sigma_q)$ such that $F_m = \mathcal{P}^p_{q,\lambda} f_m$. To prove that $f_m \in L^r(K/M;\sigma_q)$ we will follow the same method as in [5]. According to Theorem 5.5 we have, for any $\varphi \in C^\infty(K/M;\sigma_q)$,

$$\int_K \langle f_m(k), \varphi(k)\rangle_{\Lambda^q\mathbb{C}^{n-1}} \mathrm{d}k = \lim_{t\to\infty} \int_K \langle g^t_m(k), \varphi(k)\rangle_{\Lambda^q\mathbb{C}^{n-1}} \mathrm{d}k,$$

where

$$g^t_m(k) := c^{-2}_{p,q} |c_q(\lambda,p)|^{-2} e^{2(\rho-\Re(i\lambda))t} \pi^q_p \int_K P_\lambda(ha_t,k)^* F_m(ha_t) \mathrm{d}h.$$

Further,

$$\begin{aligned}
&\int_K \langle g^t_m(k), \varphi(k)\rangle_{\Lambda^q\mathbb{C}^{n-1}} \mathrm{d}k \\
&= c^{-2}_{p,q} |c_q(\lambda,p)|^{-2} e^{2(\rho-\Re(i\lambda))t} \int_K \langle \pi^q_p \int_K P_\lambda(ha_t,k)^* F_m(ha_t)\mathrm{d}h, \varphi(k)\rangle_{\Lambda^q\mathbb{C}^{n-1}} \mathrm{d}k, \\
&= c^{-2}_{p,q} |c_q(\lambda,p)|^{-2} e^{2(\rho-\Re(i\lambda))t} \int_K \int_K \langle F_m(ha_t), P_\lambda(ha_t,k) i^p_q \varphi(k)\mathrm{d}k\rangle_{\Lambda^p\mathbb{C}^n} \mathrm{d}h, \\
&= c^{-3}_{p,q} |c_q(\lambda,p)|^{-2} e^{2(\rho-\Re(i\lambda))t} \int_K \langle F_m(ha_t), (\mathcal{P}^p_{q,\lambda}\varphi)(ha_t)\rangle_{\Lambda^p\mathbb{C}^n} \mathrm{d}h.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left| \int_K \langle g^t_m(k), \varphi(k)\rangle_{\Lambda^q\mathbb{C}^{n-1}} \mathrm{d}k \right| \\
&\leq c^{-3}_{p,q} |c_q(\lambda,p)|^{-2} e^{2(\rho-\Re(i\lambda))t} \int_K \|F_m(ha_t)\|_{\Lambda^p\mathbb{C}^n} \|\mathcal{P}^p_{q,\lambda}\varphi(ha_t)\|_{\Lambda^p\mathbb{C}^n} \mathrm{d}h.
\end{aligned}$$

By Hölder's inequality (with $\frac{1}{r} + \frac{1}{s} = 1$), we deduce

$$\begin{aligned}
&\left| \int_K \langle g^t_m(k), \varphi(k)\rangle_{\Lambda^q\mathbb{C}^{n-1}} \mathrm{d}k \right| \\
&\leq c^{-3}_{p,q} |c_q(\lambda,p)|^{-2} e^{2(\rho-\Re(i\lambda))t} \|F^t_m\|_{L^r(K;\Lambda^p\mathbb{C}^n)} \|(\mathcal{P}^p_{q,\lambda}\varphi)^t\|_{L^s(K;\Lambda^p\mathbb{C}^n)},
\end{aligned}$$

where $(\mathcal{P}^p_{q,\lambda}\varphi)^t(k) = (\mathcal{P}^p_{q,\lambda}\varphi)(ka_t)$. Using (6.1) and Proposition 4.1 we get

$$\begin{aligned}
\left| \int_K \langle f_m(k), \varphi(k)\rangle_{\Lambda^q\mathbb{C}^{n-1}} \mathrm{d}k \right| &\leq \gamma_\lambda c^{-2}_{p,q} |c_q(\lambda,p)|^{-2} \|F_m\|_{\mathcal{E}^r_{q,\lambda}} \|\varphi\|_{L^s(K/M;\sigma_q)}, \\
&\leq \gamma_\lambda c^{-2}_{p,q} |c_q(\lambda,p)|^{-2} \|F\|_{\mathcal{E}^r_{q,\lambda}} \|\varphi\|_{L^s(K/M;\sigma_q)}.
\end{aligned}$$

By taking the supremum over all $\varphi \in C^\infty(K/M;\sigma_q)$ with $\|\varphi\|_{L^s(K/M;\sigma_q)} = 1$ we obtain

$$\|f_m\|_{L^r(K/M;\sigma_q)} \leq \gamma_\lambda c^{-2}_{p,q} |c_q(\lambda,p)|^{-2} \|F\|_{\mathcal{E}^r_{q,\lambda}},$$

which implies f_m, initially belongs to $L^2(K/M;\sigma_q)$, is in fact in $L^r(K/M;\sigma_q)$.

For every m, define the linear form T_m on $L^s(K/M;\sigma_q)$ by

$$T_m(\varphi) = \int_K \langle f_m(k), \varphi(k)\rangle_{\Lambda^q\mathbb{C}^{n-1}} \mathrm{d}k.$$

Clearly, T_m is continuous and

$$|T_m(\varphi)| \quad \leq \quad \gamma_\lambda c_{p,q}^{-2} |c_q(\lambda,p)|^{-2} \|F\|_{\mathcal{E}^r_{q,\lambda}} \|\varphi\|_{L^s(K/M;\sigma_q)}.$$

This shows that $(T_m)_m$ is uniformly bounded in $L^s(K/M;\sigma_q)$, with

$$\sup_m \|T_m\|_{\mathrm{op}} \leq \gamma_\lambda c_{p,q}^{-2} |c_q(\lambda,p)|^{-2} \|F\|_{\mathcal{E}^r_{q,\lambda}} .$$

The Banach-Alaouglu-Bourbaki theorem will then ensures the existence of a subsequence of bounded operators (T_{m_j}) which converges to a bounded operator T under the weak-$*$ topology, with

$$\|T\|_{\mathrm{op}} \leq \gamma_\lambda c_{p,q}^{-2} |c_q(\lambda,p)|^{-2} \|F\|_{\mathcal{E}^r_{q,\lambda}} .$$

Thus, Riesz's representation theorem guarantees the existence of a unique $f \in L^r(K/M;\sigma_q)$ such that

$$T(\varphi) = \int_K \langle \varphi(k), f(k)\rangle_{\Lambda^q\mathbb{C}^{n-1}} \mathrm{d}k.$$

By means of the Poisson kernel (5.4), we consider the test function $\varphi_g(k) = P^p_{q,\lambda}(g,k)v$ with $v \in \Lambda^p\mathbb{C}^n$, then

$$T(\varphi_g) = \langle v, \mathcal{P}^p_{q,\lambda} f(g)\rangle_{\Lambda^p\mathbb{C}^n}.$$

On the other hand

$$T_{m_j}(\varphi_g) = \langle v, \mathcal{P}^p_{q,\lambda} f_{m_j}(g)\rangle_{\Lambda^p\mathbb{C}^n} = \langle v, F_{m_j}(g)\rangle_{\Lambda^p\mathbb{C}^n}.$$

Taking the limit of the above identity when $j \to \infty$ we conclude that $F(g) = \mathcal{P}^p_{q,\lambda} f(g)$ for every $g \in G$. □

As an immediate consequence of Theorem 6.1 we obtain the following characterization of co-closed harmonic p-forms on $\mathbb{H}^n$:

Corollary 6.2. *Let p be an integer with $0 \leq p < (n-1)/2$. For $1 < r < \infty$, the Poisson transform $\mathcal{P}^p_{p,i(p-\rho)}$ is a topological isomorphism from the space $L^r(K/M;\sigma_p)$ onto the space $\mathcal{E}^r_{p,i(p-\rho)}(G/K;\tau_p)$. Moreover, for every $f \in L^r(K/M;\sigma_p)$ the following estimates hold,*

$$\frac{2(\rho-p)}{2\rho-p} c_p(\rho)\|f\|_{L^r(K/M;\sigma_p)} \leq \|\mathcal{P}^p_{p,i(p-\rho)} f\|_{\mathcal{E}^r_{p,i(p-\rho)}} \leq c_p(\rho)\|f\|_{L^r(K/M;\sigma_p)},$$

where

$$c_p(\rho) = c_{p,p} \frac{2^p \Gamma(\rho+\frac{1}{2})\Gamma(\rho-p)}{\Gamma(\rho-\frac{p}{2})\Gamma(\rho-\frac{p}{2}+\frac{1}{2})}.$$

In the case where $p=0$, we recover the classical fact that the Poisson transform is an isometric isomorphism from $L^r(\partial\mathbb{H}^n)$ onto the Hardy-harmonic space on $\mathbb{H}^n$ (see [27]).

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