# On Poisson transforms for differential forms on real hyperbolic spaces 

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# ON POISSON TRANSFORMS FOR DIFFERENTIAL FORMS ON REAL HYPERBOLIC SPACES 

SALEM BENSAÏD, ABDELHAMID BOUSSEJRA, AND KHALID KOUFANY


#### Abstract

We study the Poisson transform for differential forms on the real hyperbolic space $\mathbb{H}^{n}$. For $1<r<\infty$, we prove that the Poisson transform is a topological isomorphism from the space of $L^{r}$ differential $q$-forms on the boundary $\partial \mathbb{H}^{n}$ onto a Hardy-type subspace of $p$-eigenforms of the Hodge-de Rham Laplacian on $\mathbb{H}^{n}$, for $0 \leq p<\frac{n-1}{2}$ and $q \in\{p-1, p\}$.


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## 1. Introduction

Let $G / K$ be a Riemannian symmetric space of non-compact type. For each parabolic subgroup $P$ of $G$ there exists a natural Poisson transform from the space of $C^{\infty}$-functions on $G / P$ to space of analytic functions on $G / K$.

When the parabolic $P$ is minimal, one of the main problem stated by Helgason [12] claims that all eigenfunctions of $G$-invariant differential operators on $G / K$ are obtained as Poisson transforms of hyperfunctions on the Furstenberg boundary $G / P$. This conjecture was proved by Helgason when $G / K$ is of rank one, and in full generality by Kashiwara et al. [16]. Since then, this problem has received a lot of attention by many people in different settings (see, e.g., $[2-6,14,18,19,23,26]$ ).

A natural extension of this problem is to investigate the analogous of Helgason's claim for Poisson transforms for homogeneous vector bundles over $G / K$ (see,

[^0]e.g., $[8,9,11,15,20-22,25,28,32])$. One of the most interesting vector bundles is the bundle of differential forms on $G / K$. In this paper we consider the vector bundle of differential forms on the real hyperbolic space.

Let $\mathbb{H}^{n}=G / K$ be the real hyperbolic space realized as the open unit ball in $\mathbb{R}^{n}$, where $G=\mathrm{SO}_{\mathrm{o}}(n, 1)$ and $K \simeq \mathrm{SO}(n)$. Its boundary $\partial \mathbb{H}^{n}$ is the unit sphere $\mathbb{S}^{n-1}$. As a homogeneous space, we have $\partial \mathbb{H}^{n}=G / P$, where $P=M A N$. Here $M \simeq \operatorname{SO}(n-1), A \simeq \mathbb{R}$ and $N \simeq \mathbb{R}^{n-1}$.

For $0 \leq p \leq n$, let $\tau_{p}$ be the $p$-th exterior power of the coadjoint representation of $K$ on $V_{\tau_{p}}=\Lambda^{p} \mathbb{C}^{n}$. Then the space $C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ of smooth $p$-forms on $\mathbb{H}^{n}$ can be identified with the space $C^{\infty}\left(G / K ; \tau_{p}\right)$ of $V_{\tau_{p}}$-valued smooth functions on $G$ that are right covariant of type $\tau_{p}$.

Throughout this paper we will assume that $0 \leq p<\frac{n-1}{2}$ (for this choice of $p$ see Section 2). Then the decomposition of $\tau_{p}$ restricted to $M$ is $\tau_{p_{\mid} M}=\sigma_{p-1} \oplus \sigma_{p}$, where $\sigma_{q}$ is $q$-th exterior power of the coadjoint representation of $M$ on $V_{\sigma_{q}}=$ $\Lambda^{q} \mathbb{C}^{n-1}$, with $q \in\{p-1, p\}$.

Let $\mathfrak{a}$ be the Lie algebra of $A$, and identify its complexified dual $\mathfrak{a}_{\mathbb{C}}^{*}$ with $\mathbb{C}$. For $\lambda \in \mathbb{C}$, we consider the irreducible representation $\sigma_{q, \lambda}$ of $P=M A N$ given by $\sigma_{q, \lambda}\left(m a_{t} n\right)=\sigma_{q}(m) \mathrm{e}^{(\rho-i \lambda) t}$, where $\rho=\frac{n-1}{2}$. Let $E_{q, \lambda}$ be the corresponding homogeneous vector bundle over $\partial \mathbb{H}^{n}$. We identify its space of hyperfunction sections with the space $C^{-\omega}\left(G / P ; \sigma_{q, \lambda}\right)$ of all $V_{\sigma_{q}}$-valued hyperfunctions $f$ on $G$ such that

$$
f\left(g m a_{t} n\right)=\mathrm{e}^{(i \lambda-\rho) t} \sigma_{q}\left(m^{-1}\right) f(g) \quad \forall g \in G, \forall m \in M, \forall n \in N, \forall a_{t} \in A .
$$

For $q \in\{p-1, p\}$, let $\iota_{q}^{p}$ be the natural embedding of $V_{\sigma_{q}}$ into $V_{\tau_{p}}$. Notice that $\iota_{q}^{p} \in \operatorname{Hom}_{M}\left(V_{\sigma_{q}}, V_{\tau_{p}}\right)$. Then we can define a Poisson transform

$$
\mathcal{P}_{q, \lambda}^{p}: \mathcal{C}^{-\omega}\left(G / P ; \sigma_{q, \lambda}\right) \rightarrow C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)
$$

by

$$
\mathcal{P}_{q, \lambda}^{p} f(g)=\sqrt{\frac{\operatorname{dim} \tau_{p}}{\operatorname{dim} \sigma_{q}}} \int_{K} \tau_{p}(k) \iota_{q}^{p}(f(g k)) \mathrm{d} k, \quad g \in G .
$$

We mention that $E_{p, \lambda}$ can be seen as the vector bundle $G \times{ }_{P} V_{\sigma_{p}} \otimes \mathcal{E}[\rho-i \lambda]$, where $\sigma_{p}$ is extended to a representation of $P$, and $\mathcal{E}[\rho-i \lambda]$ is the density line bundle over the character $m a_{t} n \mapsto e^{(\rho-i \lambda) t}$ of $P$. Sections of the above bundle are $q$-hyperforms with value in $\mathcal{E}[\rho-i \lambda]$. In view of this observation, $\mathcal{P}_{p, \lambda}^{p}=\sqrt{\frac{\operatorname{dim} \tau_{p}}{\operatorname{dim} \sigma_{p}}} \Phi_{p}^{\rho-i \lambda}$, where $\Phi_{p}^{\rho-i \lambda}$ is the Poisson transform considered in [8].

Let $\Delta=d d^{*}+d^{*} d$ be the Hodge-de Rham Laplacian, where $d: C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right) \rightarrow$ $C^{\infty}\left(\Lambda^{p+1} \mathbb{H}^{n}\right)$ is the differential and $d^{*}$ is the codifferential (the adjoint of $d$ which is defined by the hyperbolic metric).

For $\lambda \in \mathbb{C}$, denote by $C_{q, \lambda}^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ the space of all $\omega \in \mathcal{C}^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ which are closed if $q=p-1$ and co-closed if $q=p$, with the additional condition $\Delta \omega=$ $\left(\lambda^{2}+(\rho-q)^{2}\right) \omega$. It was proved in [9], that for $0 \leq p<(n-1) / 2$, the Poisson transforms $\mathcal{P}_{q, \lambda}^{p}, q=p-1, p$ provide the following isomorphisms:
(i) $\mathcal{P}_{p, \lambda}^{p}: \mathcal{C}^{-\omega}\left(G / P ; \sigma_{p, \lambda}\right) \rightarrow C_{p, \lambda}^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ iff $i \lambda \notin\{-\rho+p\} \cup\left(\mathbb{Z}_{\leq 0}-\rho\right)$,
and
(ii) $\mathcal{P}_{p-1, \lambda}^{p}: \mathcal{C}^{-\omega}\left(G / P ; \sigma_{p-1, \lambda}\right) \rightarrow C_{p-1, \lambda}^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ iff $i \lambda \notin\{\rho-p+1\} \cup\left(\mathbb{Z}_{\leq 0}-\rho\right)$.

Now, let $C^{-\omega}\left(K / M ; \sigma_{q}\right)$ be the space of $V_{\sigma_{q}}$-valued hyperfunctions $f$ on $K$ satisfying $f(k m)=\sigma_{q}\left(m^{-1}\right) f(k)$, for all $k \in K, m \in M$. By the Iwasawa decomposition, the restriction map $f \mapsto f_{\left.\right|_{K}}$ gives an isomorphism from $C^{-\omega}\left(G / P ; \sigma_{q, \lambda}\right)$ onto $C^{-\omega}\left(K / M ; \sigma_{q}\right)$. Via this isomorphism we can define the Poisson transform from $C^{-\omega}\left(K / M ; \sigma_{q}\right)$ into $C_{q, \lambda}^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$. To state our main result, let us introduce further notation.

For $1<r<\infty$, let $L^{r}\left(K / M ; \sigma_{q}\right)$ be the space of $V_{\sigma_{q}}$-valued functions $f$ on $K$ which are covariant of type $\sigma_{q}$, and such that

$$
\|f\|_{L^{r}\left(K / M ; \sigma_{q}\right)}=\left(\int_{K}\|f(k)\|_{\Lambda^{q} \mathbb{C}^{n-1}}^{r} \mathrm{~d} k\right)^{\frac{1}{r}}<\infty
$$

The space $L^{r}\left(K / M ; \sigma_{q}\right)$ is identified with the space of $L^{r}$ differential $q$-forms on the boundary $\partial \mathbb{H}^{n}=K / M$. From above, it follows that the Poisson transform $\mathcal{P}_{q, \lambda}^{p}$ maps $L^{r}\left(K / M ; \sigma_{q}\right)$ into the space $C_{q, \lambda}^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$.

The goal of this paper is to characterize those eigenforms in $C_{q, \lambda}^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ which are Poisson transforms of elements in $L^{r}\left(K / M ; \sigma_{q}\right)$, for $1<r<\infty$. To this end we introduce the Hardy type space $\mathcal{E}_{q, \lambda}^{r}\left(G / K ; \tau_{p}\right)$ of all $F$ in $C_{q, \lambda}^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ such that

$$
\|F\| \mathcal{E}_{q, \lambda}^{r}:=\sup _{t>0} \mathrm{e}^{(\rho-\Re(i \lambda)) t}\left(\int_{K}\left\|F\left(k a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{r} \mathrm{~d} k\right)^{\frac{1}{r}}<\infty
$$

where we have identified $C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ with $C^{\infty}\left(G / K ; \tau_{p}\right)$.
We pin down that throughout the paper we will often view $p$-forms in $C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ as functions in $C^{\infty}\left(G / K ; \tau_{p}\right)$ and vice-versa.

Our main result is the following:
Theorem A (see Theorem 6.1). Let $0 \leq p<(n-1) / 2$ be an integer and $q \in$ $\{p-1, p\}$. Assume $\lambda \in \mathbb{C}$ such that

$$
\begin{cases}\Re(i \lambda)>0 & \text { if } q=p \\ \Re(i \lambda)>0 \text { and } i \lambda \neq \rho-p+1 & \text { if } q=p-1 .\end{cases}
$$

The Poisson transform $\mathcal{P}_{q, \lambda}^{p}$ is a topological isomorphism of the space $L^{r}\left(K / M ; \sigma_{q}\right)$ onto the space $\mathcal{E}_{q, \lambda}^{r}\left(G / K ; \tau_{p}\right)$. Moreover, there exists a positive constant $\gamma_{\lambda}$ such that

$$
\left|c_{q}(\lambda, p)\right|\|f\|_{L^{r}\left(K / M ; \sigma_{q}\right)} \leq \sqrt{\frac{\operatorname{dim} \sigma_{q}}{\operatorname{dim} \tau_{p}}}\left\|\mathcal{P}_{q, \lambda}^{p} f\right\|_{\mathcal{E}_{q, \lambda}^{r}} \leq \gamma_{\lambda}\|f\|_{L^{r}\left(K / M ; \sigma_{q}\right)},
$$

for every $f \in L^{r}\left(K / M ; \sigma_{q}\right)$.
Above, $c_{q}(\lambda, p)(q=p-1, p)$ denote the scalar components of the vector-valued Harish-Chandra $c$-function $\mathbf{c}(\lambda, p)$. We refer the reader to (4.2) for the integral representation of $\mathbf{c}(\lambda, p)$. The explicit expressions of $c_{q}(\lambda, p)$ will be given in Proposition 4.6.

As an immediate consequence of Theorem A we obtain when $q=p$ and $i \lambda=$ $\rho-p$ (the harmonic case) a characterization of co-closed harmonic $p$-forms, see Corollary 6.2. Furthermore, if in addition $p=0$, we recover the classical fact
that the Poisson transform is an isometric isomorphism from $L^{r}\left(\partial \mathbb{H}^{n}\right)$ onto the Hardy-harmonic space on $\mathbb{H}^{n}$ (see [27]).

Our strategy in proving Theorem A is to begin with the case $r=2$. The most difficult part is to prove the sufficiency condition. Let us give a short outline of its proof. Let $F \in \mathcal{E}_{q, \lambda}^{2}\left(G / K ; \tau_{p}\right)$, then we show the existence of a functional $T$ on $C^{\infty}\left(G / P ; \sigma_{q, \bar{\lambda}}\right)$ such that $F=\widetilde{\mathcal{P}_{q, \lambda}^{p}}(T)^{1}$ (Proposition 5.1). To prove that $T$ is indeed in $L^{2}$ we need to establish the asymptotic behavior of certain Eisenstein type integrals (see (5.8), (5.9)). To this end we prove a Fatou-type theorem (Theorem 4.3),

$$
\lim _{t \rightarrow \infty} \mathrm{e}^{(\rho-i \lambda) t} \mathcal{P}_{q, \lambda}^{p} f\left(k a_{t}\right)=\sqrt{\frac{\operatorname{dim} \tau_{p}}{\operatorname{dim} \sigma_{q}}} \mathbf{c}(\lambda, p) \iota_{q}^{p}(f(k)),
$$

in $L^{r}\left(K, \Lambda^{p} \mathbb{C}^{n}\right)$, for every $f \in L^{r}\left(K / M ; \sigma_{q}\right)$.
Let us mention that instead of Proposition 5.1 we might use the result of Gaillard, stated in Proposition 3.2 below, to ensure the existence of a hyperform $f \in C^{-\omega}\left(G / P ; \sigma_{q, \lambda}\right)$ such that $F=\mathcal{P}_{q, \lambda}^{p} f$. We would prefer to keep our argument because it is potentially useful in studying Poisson transform on vector bundles over symmetric spaces of non-compact type.

To establish Theorem A for every $1<r<\infty$, we prove that any $F \in$ $\mathcal{E}_{q, \lambda}^{r}\left(G / K ; \tau_{p}\right)$ can be approximated by a sequence $\left(F_{m}\right)_{m}$ in $\mathcal{E}_{q, \lambda}^{2}\left(G / K ; \tau_{p}\right)$. Using the first part of our result which corresponds to $r=2$, we can deduce that there exists $f_{m} \in L^{2}\left(K / M ; \sigma_{q}\right)$ such that $F_{m}=\mathcal{P}_{q, \lambda}^{p}\left(f_{m}\right)$. By an $L^{2}$-inversion formula of the Poisson transform (Theorem 5.5) we conclude that $f_{m}$ is indeed in $L^{r}\left(K / M ; \sigma_{q}\right)$. Henceforth the linear form

$$
T_{m}(\varphi)=\int_{K}\left\langle f_{m}(k), \varphi(k)\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k,
$$

is uniformly bounded on $L^{s}\left(K / M ; \sigma_{q}\right)$, with $\frac{1}{r}+\frac{1}{s}=1$. Thanks to Banach-Alaouglu-Bourbaki theorem, there exists a subsequence of bounded operators $\left(T_{m_{j}}\right)_{j}$ which converges to a bounded operator $T$ under the weak-» topology. Thus by Riesz representation theorem, we conclude that there exists $f \in L^{r}\left(K / M ; \sigma_{q}\right)$ such that $F=\mathcal{P}_{q, \lambda}^{p} f$.

The paper is organized as follows. Section 2 contains notations and background material for later use. In particular we recall some materials on differential forms on $\mathbb{H}^{n}$ and $\partial \mathbb{H}^{n}=\mathbb{S}^{n-1}$ as sections of specific vector bundles. Section 3 is devoted to the definition of the Poisson transform $\mathcal{P}_{q, \lambda}^{p}$ on the space of differential forms on $\mathbb{S}^{n-1}$. In Section 4 we prove a Fatou type theorem for $\mathcal{P}_{q, \lambda}^{p}$, which will be of particular use to find the explicit expression of the Harish-Chandra $c$-function appearing in Theorem A. The Fatou type theorem will essentially play a crucial role in Section 5 where we prove Theorem A for the case $r=2$. Section 5 contains also an $L^{2}$-inversion formula for the Poisson transform. These results will allow us in Section 6 to prove Theorem A for every $1<r<\infty$.

[^1]
## 2. Background

2.1. The real hyperbolic space. Let $\mathbb{H}^{n}=\mathbb{H}^{n}(\mathbb{R})$ be the real hyperbolic space of dimension $n \geq 2$ realized as the open unit ball $\mathbb{B}^{n}$ in $\mathbb{R}^{n}$. Let $G=\operatorname{SO}_{\mathrm{o}}(n, 1)$ be the connected component of the identity of the group of all linear transforms of $\mathbb{R}^{n+1}$ with determinant 1 keeping invariant the Lorentzian quadratic form

$$
[\mathbf{x}, \mathbf{x}]=x_{1}^{2}+\cdots+x_{n}^{2}-x_{n+1}^{2}, \quad \mathbf{x}=\left(x_{1}, \cdots, x_{n}, x_{n+1}\right)
$$

Then the group $G$ acts transitively on $\overline{\mathbb{B}^{n}}$ by fractional transformations and as a homogeneous space we have the identification $\mathbb{H}^{n}=G / K$, where $K=\operatorname{SO}(n)$, the isotropy subgroup of $\mathbf{0} \in \mathbb{B}^{n}$, is a maximal compact subgroup of $G$.

Let $\mathfrak{g}=\mathfrak{s o}(n, 1)$ and $\mathfrak{k}=\mathfrak{s o}(n)$ be the Lie algebras of $G$ and $K$, respectively. Let as usual $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. The subspace $\mathfrak{p}$ is identified with the tangent space $T_{\mathbf{o}}(G / K) \simeq \mathbb{R}^{n}$ of $G / K=\mathbb{H}^{n}$ at the origin $\mathbf{o}=e K$.

Put

$$
H_{0}=\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0_{n-1} & 0 \\
1 & 0 & 0
\end{array}\right) \in \mathfrak{p}
$$

then $\mathfrak{a}=\mathbb{R} H_{0}$ is a maximal abelian subspace of $\mathfrak{p}$, and the corresponding analytic Lie subgroup $A$ of $G$ is parametrized by

$$
a_{t}=\exp \left(t H_{0}\right)=\left(\begin{array}{ccc}
\cosh t & 0 & \sinh t \\
0 & I_{n-1} & 0 \\
\sinh t & 0 & \cosh t
\end{array}\right), \quad t \in \mathbb{R} .
$$

Let

$$
\mathfrak{n}=\left\{\left(\begin{array}{ccc}
0 & y & 0 \\
-y^{T} & 0_{n-1} & y^{T} \\
0 & y & 0
\end{array}\right), y \in \mathbb{R}^{n-1}\right\} \simeq \mathbb{R}^{n-1}
$$

so that $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the Iwasawa decomposition of $\mathfrak{g}$. Here $y^{T}$ stands for the transpose of a vector $y \in \mathbb{R}^{n-1}$.

Let $N=\exp (\mathfrak{n})$ be the connected Lie subgroup of $G$ having $\mathfrak{n}$ as Lie algebra. According to the Iwasawa decomposition $G=K A N$, every element $g \in G$ can be uniquely written as

$$
g=\kappa(g) \mathrm{e}^{H(g)} n,
$$

where $\kappa(g) \in K, H(g) \in \mathfrak{a}$ and $n \in N$.
Let $\rho$ be the half sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$. Then $\rho\left(H_{0}\right)=\frac{n-1}{2}$ and we will write $\rho=\rho\left(H_{0}\right)$.

Let $P=M A N$ be the standard minimal parabolic subgroup of $G$, where $M$ is the centralizer of $A$ in $K$ given by

$$
\left\{\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & m & 0 \\
0 & 0 & 1
\end{array}\right): m \in \mathrm{SO}(n-1)\right\} \simeq \mathrm{SO}(n-1) .
$$

Then $G / P=K / M$ may be identified with the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^{n}$.
2.2. Differential forms on $\mathbb{H}^{n}$ and $\mathbb{S}^{n-1}$. Let $\langle\cdot, \cdot\rangle$ be the standard Euclidean scalar product in $\mathbb{R}^{n}$. Let $\left(e_{1}, e_{2}, \ldots, e_{n}\right)$ be the standard orthonormal basis of $\mathbb{R}^{n}$ and denote $\left(e_{1}^{*}, e_{2}^{*}, \ldots, e_{n}^{*}\right)$ its dual basis.

For an integer $p$ with $0 \leq p \leq n$, let $\Lambda^{p}\left(\mathbb{C}^{n}\right)^{*}=\Lambda^{p}\left(\mathbb{R}^{n}\right)^{*} \otimes \mathbb{C}$ be the space of complex-valued alternating multilinear $p$-forms on $\mathbb{R}^{n}$. A basis of $\Lambda^{p}\left(\mathbb{C}^{n}\right)^{*}$ is given by set of

$$
e_{I}^{*}:=e_{i_{1}}^{*} \wedge \cdots \wedge e_{i_{p}}^{*} \text { where }\left\{\begin{array}{l}
I=\left\{i_{1}, \cdots, i_{p}\right\} \\
1 \leq i_{1}<\cdots<i_{p} \leq n
\end{array}\right.
$$

The interior product $\iota_{v} \omega$ of a $p$-form $\omega$ with a vector $v \in \mathbb{R}^{n}$ is the ( $p-1$ )-form defined on the given basis by

$$
\iota_{e_{j}}\left(e_{i_{1}}^{*} \wedge \cdots \wedge e_{i_{p}}^{*}\right)= \begin{cases}0 & \text { if } j \neq \text { any } i_{r} \\ (-1)^{r-1} e_{i_{1}}^{*} \wedge \cdots \wedge \widehat{e_{i_{r}}^{*}} \wedge \cdots \wedge e_{i_{p}}^{*} & \text { if } j=i_{r}\end{cases}
$$

where ${ }^{\wedge}$ over $e_{i_{r}}^{*}$ means that it is deleted from the exterior product.
For a given $v \in \mathbb{R}^{n}$, the exterior product $\varepsilon_{v} \omega$ of a $p$-form $\omega$ with the linear form $v^{*}$ is the $(p+1)$-form defined by

$$
\varepsilon_{v} \omega=v^{*} \wedge \omega .
$$

For the reader's convenience and to keep the notations simple, we will identify $\left(\mathbb{C}^{n}\right)^{*}$ with $\mathbb{C}^{n}$ and $\Lambda^{p}\left(\mathbb{C}^{n}\right)^{*}$ with $\Lambda^{p} \mathbb{C}^{n}$.

We define an inner product $\langle\cdot, \cdot\rangle_{\Lambda^{p} \mathbb{C}^{n}}$ on $\Lambda^{p} \mathbb{C}^{n}$ as an extension of the one on $\mathbb{C}^{n}$ by setting

$$
\begin{equation*}
\left\langle v_{1} \wedge \cdots v_{p}, w_{1} \wedge \cdots w_{p}\right\rangle_{\Lambda^{p} \mathbb{C}^{n}}=\operatorname{det}\left(\left\langle v_{i}, w_{j}\right\rangle\right)_{i, j} \tag{2.1}
\end{equation*}
$$

It is easy to show that the basis of $\Lambda^{p} \mathbb{C}^{n}$ consisting of the $p$-vectors $e_{I}:=e_{i_{1}} \wedge$ $\cdots \wedge e_{i_{p}}$ (where $I=\left\{i_{1}, \cdots, i_{p}\right\}$, with $1 \leq i_{1}<\cdots<i_{p} \leq n$ ) is an orthonormal basis of $\Lambda^{p} \mathbb{C}^{n}$ with respect to (2.1). We have further the useful identity

$$
\begin{equation*}
\left\langle\iota_{v} \omega, \xi\right\rangle_{\Lambda^{p-1} \mathbb{C}^{n}}=\left\langle\omega, \varepsilon_{v} \xi\right\rangle_{\Lambda^{p} \mathbb{C}^{n}}, v \in \mathbb{R}^{n}, \omega \in \Lambda^{p} \mathbb{C}^{n}, \xi \in \Lambda^{p-1} \mathbb{C}^{n} . \tag{2.2}
\end{equation*}
$$

For $0 \leq p \leq n$, we let $\tau_{p}$ to be the $p$-exterior product $\Lambda^{p} \mathrm{Ad}^{*}$ of the coadjoint representation of $K=\operatorname{SO}(n)$ on $\mathfrak{p}_{\mathbb{C}}^{*}$. Its representation space being $V_{\tau_{p}}:=$ $\Lambda^{p}\left(\mathfrak{g}_{\mathbb{C}} / \mathfrak{k}_{\mathbb{C}}\right)^{*} \simeq \Lambda^{p} \mathbb{C}^{n}$. Notice that $\tau_{p}$ is unitary with respect to the inner product (2.1), and is equivalent to the standard representation of $K$ on $\Lambda^{p} \mathbb{C}^{n}$. By [7] or [13], the representation $\tau_{p}$ is irreducible for $p \neq \frac{n}{2}$ ( $n$ even), while $\tau_{\frac{n}{2}}=$ $\tau_{\frac{n}{2}}^{+} \oplus \tau_{\frac{n}{2}}^{-}$. The two factors $\tau_{\frac{n}{2}}^{ \pm}$being irreducible, inequivalent and act on the following eigenspaces of the Hodge star operator $\star$,

$$
\Lambda_{\frac{n}{2}}^{ \pm} \mathbb{C}^{n}=\left\{w \in \Lambda^{\frac{n}{2}} \mathbb{C}^{n}: \star w=\mu_{ \pm} w\right\}
$$

where $\mu_{ \pm}= \pm 1$ if $\frac{n}{2}$ is even and $\mu_{ \pm}= \pm i$ if $\frac{n}{2}$ is odd. Since the Hodge operator $\star$ induces the equivalence $\tau_{p} \simeq \tau_{n-p}$, we will restrict our attention to the case $0 \leq p<\frac{n}{2}$, without loss of generality.

For $0 \leq q \leq n-1$, let $\sigma_{q}$ be the standard representation of $M \simeq \operatorname{SO}(n-1)$ on $V_{\sigma_{q}}=\Lambda^{q} \mathbb{C}^{n-1}$. It is an irreducible representation for $q \neq \frac{n-1}{2}$, and as before $\sigma_{\frac{n-1}{2}}=\sigma_{\frac{n-1}{2}}^{+} \oplus \sigma_{\frac{n-1}{2}}^{-}$.

Lemma 2.1 (See, e.g. $[1,13])$. Let $\tau_{\left.\right|_{M}}$ be the restriction of $\tau_{p}$ to $M \simeq \operatorname{SO}(n-1)$. Then $\tau_{p_{M}}$ decomposes into inequivalent factors as follow:

1) For $p=0, \tau_{\left.\right|_{M}}=\sigma_{p}$.
2) For $0<p<\frac{n-1}{2}$,

$$
\tau_{\left.p\right|_{M}}=\sigma_{p-1} \oplus \sigma_{p} \text { with }
$$

$$
\begin{equation*}
\Lambda^{p} \mathbb{C}^{n}=e_{1} \wedge \Lambda^{p-1} \mathbb{C}^{n-1} \oplus \Lambda^{p} \mathbb{C}^{n-1} \simeq \Lambda^{p-1} \mathbb{C}^{n-1} \oplus \Lambda^{p} \mathbb{C}^{n-1} \tag{2.3}
\end{equation*}
$$

3) For $p=\frac{n-1}{2}, \tau_{\left.\right|_{\left.\right|_{M}}}=\sigma_{p-1} \oplus \sigma_{p}^{+} \oplus \sigma_{p}^{-}$.
4) For $p=\frac{n}{2}, \tau_{\left.\right|_{M}}=2 \sigma_{p-1} \sim 2 \sigma_{p}$.

Henceforth, we will assume along this paper that $0 \leq p<\frac{n-1}{2}$ (we say $p$ generic).

Remark 2.2. (1) In the decomposition (2.3) we have identified $\mathbb{C}^{n-1}$ with $\operatorname{span}\left\{e_{2}, \cdots, e_{n}\right\}$. The isomorphism (2.3) follows from the $\mathrm{SO}(n-1)$-equivariance of the decomposition

$$
\omega=e_{1} \wedge \omega^{\prime}+\omega^{\prime \prime} \quad \text { with } \omega^{\prime} \in \Lambda^{p-1} \mathbb{C}^{n-1} \text { and } \omega^{\prime \prime} \in \Lambda^{p} \mathbb{C}^{n-1}
$$

for any $\omega \in \Lambda^{p} \mathbb{C}^{n}$.
(2) The scalar products on $\Lambda^{q} \mathbb{C}^{n-1}, q \in\{p-1, p\}$, are induced from the one on $\Lambda^{p} \mathbb{C}^{n}$ defined in (2.1).
(3) For $q \in\{p-1, p\}$, we will consider the following natural isometric embedding

$$
\begin{equation*}
\iota_{q}^{p}: V_{\sigma_{q}}=\Lambda^{q} \mathbb{C}^{n-1} \rightarrow V_{\tau_{p}}=\Lambda^{p} \mathbb{C}^{n} . \tag{2.4}
\end{equation*}
$$

Notice that $\iota_{q}^{p} \in \operatorname{Hom}_{M}\left(V_{\sigma_{q}}, V_{\tau_{p}}\right)$ and it is given by

$$
\begin{aligned}
\iota_{p-1}^{p}: \Lambda^{p-1} \mathbb{C}^{n-1} & \rightarrow e_{1} \wedge \Lambda^{p-1} \mathbb{C}^{n-1} \oplus \Lambda^{p} \mathbb{C}^{n-1} \\
\xi & \mapsto e_{1} \wedge \xi+0
\end{aligned}
$$

and

$$
\begin{aligned}
\iota_{p}^{p}: \Lambda^{p} \mathbb{C}^{n-1} & \rightarrow e_{1} \wedge \Lambda^{p-1} \mathbb{C}^{n-1} \oplus \Lambda^{p} \mathbb{C}^{n-1} \\
\xi & \mapsto 0+\xi
\end{aligned}
$$

In particular, for any $\omega, \omega^{\prime} \in \Lambda^{p-1} \mathbb{C}^{n-1}$,

$$
\left\langle\omega, \omega^{\prime}\right\rangle_{\Lambda^{p-1} \mathbb{C}^{n-1}}=\left\langle e_{1} \wedge \omega, e_{1} \wedge \omega^{\prime}\right\rangle_{\Lambda^{p} \mathbb{C}^{n}}
$$

(4) For $q \in\{p-1, p\}$, let $\pi_{p}^{q}$ denotes the natural projection

$$
\pi_{p}^{q}: V_{\tau_{p}} \rightarrow V_{\sigma_{q}}
$$

Then one can see from (2.2) that $\left(\pi_{p}^{q}\right)^{*}=\iota_{q}^{p}$.

## 3. Poisson transform on differential forms

In this section we shall define the Poisson transform for differential forms on $\partial \mathbb{H}^{n}$. We will follow the definition of Okamoto [21], see also Minemura [20], Yang [32], Juhl [15], Van der ven [28], Olbrich [22] and Pedon [24, 25]. There is also another approach to define the differential forms-valued Poisson transforms initiated by Gaillard [8] and generalized by Harrach [11].

Let $G \times{ }_{K} V_{\tau_{p}}$ be the homogeneous vector bundle over $G / K$ associated with $\tau_{p}$. The space of its smooth sections is identified with

$$
C^{\infty}\left(G / K ; \tau_{p}\right)=\left\{f: G \rightarrow V_{\tau_{p}} \text { smooth } \mid f(g k)=\tau_{p}\left(k^{-1}\right) f(g) \forall g \in G, \forall k \in K\right\} .
$$

As a homogeneous vector bundle, we have $\Lambda^{p} \mathbb{H}^{n}:=\Lambda^{p} T_{\mathbb{C}}^{*} \mathbb{H}^{n}=G \times_{K} V_{\tau_{p}}$ and therfore we identify the space $C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ of its smooth sections (i.e., smooth differential $p$-forms on $\mathbb{H}^{n}$ ) with the space $C^{\infty}\left(G / K ; \tau_{p}\right)$.

Consider the exterior differentiation operator $d: C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right) \rightarrow C^{\infty}\left(\Lambda^{p+1} \mathbb{H} \mathbb{H}^{n}\right)$ and the co-differentiation $d^{*}=(-1)^{n(p+1)+1} \star d \star: C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right) \rightarrow C^{\infty}\left(\Lambda^{p-1} \mathbb{H}^{n}\right)$. Let $\Delta=d d^{*}+d^{*} d$ be the Hodge-de Rham Laplacian on $C^{\infty}\left(\Lambda \mathbb{H}^{n}\right)$. Let $\mathbf{D}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ be the algebra of $G$-invariant differential operators acting on $C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)$. Its known by [10] that for generic $p, \mathbf{D}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ is a commutative algebra generated by $d d^{*}$ and $d^{*} d$.

Next, we shall describe the eigenforms for differential operators in $\mathbf{D}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ by means of Poisson transforms.

For $q \in\{p-1, p\}$ and $\lambda \in \mathbb{C}$, we consider the following irreducible representation of $P=M A N$,

$$
\sigma_{q, \lambda}: m a_{t} n \mapsto \sigma_{q}(m) \mathrm{e}^{(\rho-i \lambda) t}
$$

Let $E_{q, \lambda}$ be the homogeneous vector bundle over $\partial \mathbb{H}^{n}$ corresponding to $\sigma_{q, \lambda}$. We denote by $C^{-\omega}\left(\partial \mathbb{H}^{n} ; E_{q, \lambda}\right)$ the space of its hyperfunction sections and we identify it with the space $C^{-\omega}\left(G / P ; \sigma_{q, \lambda}\right)$ of $V_{q}$-valued hyperfunctions $\phi$ on $G$ such that

$$
f\left(g m a_{t} n\right)=\mathrm{e}^{(i \lambda-\rho) t} \sigma_{q}\left(m^{-1}\right) f(g)
$$

for all $g \in G, m \in M, n \in N, a_{t} \in A$. Then, define the Poisson transform

$$
\mathcal{P}_{q, \lambda}^{p}: C^{-\omega}\left(G / P ; \sigma_{q, \lambda}\right) \rightarrow C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)
$$

by

$$
\mathcal{P}_{q, \lambda}^{p} f(g)=c_{p, q} \int_{K} \tau_{p}(k) \iota_{q}^{p}(f(g k)) \mathrm{d} k, \quad g \in G,
$$

where $\iota_{q}^{p}$ is the embedding given by (2.4), $\mathrm{d} k$ denotes the normalized Haar measure on $K$, and where the constant factor $c_{p, q}$ is given by

$$
c_{p, q}=\sqrt{\frac{\operatorname{dim} \tau_{p}}{\operatorname{dim} \sigma_{q}}}= \begin{cases}\sqrt{\frac{n}{n-p}} & \text { if } q=p,  \tag{3.1}\\ \sqrt{\frac{n}{p}} & \text { if } q=p-1\end{cases}
$$

Let us mention that for $q=p, E_{p, \lambda}$ can be seen as the vector bundle $G \times_{P}$ $V_{\sigma_{p}} \otimes \mathcal{E}[\rho-i \lambda]$, where $\sigma_{p}$ is extended to a representation of $P$ and $\mathcal{E}[\rho-i \lambda]$ is the density line bundle associated to the character $m a_{t} n \mapsto e^{(\rho-i \lambda) t}$ of $P$. Then $C^{-\omega}\left(\partial \mathbb{H}^{n} ; E_{p, \lambda}\right)$ can be viewed as the space of $p$-hyperforms on $\partial \mathbb{H}^{n}$ with value in $\mathcal{E}[\rho-i \lambda]$. In view of this observation, $\mathcal{P}_{p, \lambda}^{p}=c_{p, p} \Phi_{p}^{\rho-i \lambda}$, where $\Phi_{p}^{\rho-i \lambda}$ is the Poisson transform considered in [8]. When $i \lambda=\rho-p$ (which corresponds to the harmonic case, see below) the space $C^{-\omega}\left(\partial \mathbb{H}^{n} ; E_{p,-i(\rho-p)}\right)$ consists of $p$-hyeprforms with value in $\mathcal{E}[p]$.

By the Iwasawa decomposition, the restriction map of $f \mapsto f_{\mid K}$ gives an isomorphism from $C^{-\omega}\left(G / P ; \sigma_{q, \lambda}\right)$ onto the space $C^{-\omega}\left(K / M ; \sigma_{q}\right)$ of $V_{q}$-valued hyperfunctions $f$ on $K$ satisfying $f(k m)=\sigma_{q}\left(m^{-1}\right) f(k)$, for all $k \in K, m \in M$.

In this compact model, the Poisson transform

$$
\mathcal{P}_{q, \lambda}^{p}: C^{-\omega}\left(K / M ; \sigma_{q}\right) \rightarrow C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right)
$$

takes the form

$$
\mathcal{P}_{q, \lambda}^{p} f(g)=c_{p, q} \int_{K} \mathrm{e}^{-(i \lambda+\rho) H\left(g^{-1} k\right)} \tau_{p}\left(\kappa\left(g^{-1} k\right)\right) \iota_{q}^{p}(f(g k)) \mathrm{d} k, \quad g \in G .
$$

Below, we shall give the explicit action of the algebra $\mathbf{D}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ on the Poisson transform of elements in $C^{-\omega}\left(K / M ; \sigma_{q}\right)$. The following result is due to Gaillard [8,9], see also Pedon [25].

Proposition 3.1. For $f \in C^{-\omega}\left(K / M ; \sigma_{q}\right)$ with $q \in\{p-1, p\}$, we have

$$
\begin{array}{ll}
d^{*} \mathcal{P}_{p, \lambda}^{p}(f)=0, & d \mathcal{P}_{p-1, \lambda}^{p}(f)=0, \\
d^{*} d \mathcal{P}_{p, \lambda}^{p}(f)=\left(\lambda^{2}+(\rho-p)^{2}\right) \mathcal{P}_{p, \lambda}^{p}(f), & d d^{*} \mathcal{P}_{p-1, \lambda}^{p}(f)=\left(\lambda^{2}+(\rho-p+1)^{2}\right) \mathcal{P}_{p-1, \lambda}^{p}(f) .
\end{array}
$$

For a character $\chi: \mathbf{D}\left(\Lambda^{p} \mathbb{H}^{n}\right) \rightarrow \mathbb{C}$, let $\mathcal{E}_{\chi}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ be the corresponding eigenspace,

$$
\mathcal{E}_{\chi}\left(\Lambda^{p} \mathbb{H}^{n}\right):=\left\{f \in C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right) \mid D f=\chi(D) f, \forall D \in \mathbf{D}\left(\Lambda^{p} \mathbb{H}^{n}\right)\right\}
$$

Put $\chi(\Delta)=\gamma$ and suppose $\gamma \neq 0$. Similarly, denote $\chi\left(d d^{*}\right)=\gamma_{1}$ and $\chi\left(d^{*} d\right)=\gamma_{2}$. Consider the eigenspace

$$
\mathcal{E}_{\gamma}\left(\Lambda^{p} \mathbb{H}^{n}\right):=\left\{f \in C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right) \mid \Delta f=\gamma f\right\}
$$

Since $\left(d^{*} d\right)\left(d d^{*}\right)=0$, we have $\gamma_{1} \gamma_{2}=0$. As $\gamma \neq 0$ and $\gamma=\gamma_{1}+\gamma_{2}$, therefore, we have either ( $\gamma_{1}=0$ and $\gamma_{2}=\gamma$ ) or ( $\gamma_{2}=0$ and $\gamma_{1}=\gamma$ ). We denote $\chi$ by $\chi_{1}$ in the first case and by $\chi_{2}$ the second case. Thus,

$$
\mathcal{E}_{\gamma}\left(\Lambda^{p} \mathbb{H}^{n}\right)=\mathcal{E}_{\chi_{1}}\left(\Lambda^{p} \mathbb{H}^{n}\right) \oplus \mathcal{E}_{\chi_{2}}\left(\Lambda^{p} \mathbb{H}^{n}\right)
$$

In view of Proposition 3.1, we deduce that $\gamma_{1}=\lambda^{2}+(\rho-p+1)^{2}, \gamma_{2}=\lambda^{2}+(\rho-p)^{2}$ and

$$
\begin{gathered}
\mathcal{E}_{\chi_{1}}\left(\Lambda^{p} \mathbb{H}^{n}\right)=\left\{f \in C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right) \left\lvert\,\left\{\begin{array}{ll}
\Delta f & =\left(\lambda^{2}+(\rho-p)^{2}\right) f \\
d^{*} f & =0
\end{array}\right\}\right.,\right. \\
\mathcal{E}_{\chi_{2}}\left(\Lambda^{p} \mathbb{H}^{n}\right)=\left\{f \in C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right) \left\lvert\,\left\{\begin{array}{ll}
\Delta f & =\left(\lambda^{2}+(\rho-p+1)^{2}\right) f \\
d f & =0
\end{array}\right\} .\right.\right.
\end{gathered}
$$

Under the identification $C^{\infty}\left(\Lambda^{p} \mathbb{H}^{n}\right) \simeq C^{\infty}\left(G / K ; \tau_{p}\right)$, we let $D, D^{*}$ and $-\mathcal{C}$ to be the counterpart of $d, d^{*}$ and $\Delta$ acting on $C^{\infty}\left(G / K ; \tau_{p}\right)$, given by

$$
\begin{equation*}
D=\sum_{j} X_{j} \varepsilon_{X_{j}}, \quad D^{*}=-\sum_{j} X_{j} \iota_{X_{j}}, \quad \mathcal{C}=\sum_{j} X_{j}^{2}-\sum_{j} Y_{j}^{2}, \tag{3.2}
\end{equation*}
$$

where $\left(X_{i}\right)$ and $\left(Y_{i}\right)$ are orthonormal ${ }^{2}$ bases of $\mathfrak{p}$ and $\mathfrak{k}$ respectively. Thus, the spaces $\mathcal{E}_{\chi_{1}}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ and $\mathcal{E}_{\chi_{2}}\left(\Lambda^{p} \mathbb{H}^{n}\right)$ are identified respectively with

[^2]\[

$$
\begin{gather*}
\mathcal{E}_{p, \lambda}\left(G / K ; \tau_{p}\right)=\left\{f \in C^{\infty}\left(G / K ; \tau_{p}\right) \left\lvert\,\left\{\begin{array}{ll}
\mathcal{C} f & =-\left(\lambda^{2}+(\rho-p)^{2}\right) f \\
D^{*} f=0
\end{array}\right\}\right.,\right.  \tag{3.3}\\
\mathcal{E}_{p-1, \lambda}\left(G / K ; \tau_{p}\right)=\left\{f \in C^{\infty}\left(G / K ; \tau_{p}\right) \left\lvert\,\left\{\begin{array}{ll}
\mathcal{C} f=-\left(\lambda^{2}+(\rho-p+1)^{2}\right) f \\
D f=0
\end{array}\right\} .\right.\right. \tag{3.4}
\end{gather*}
$$
\]

Notice that $\mathcal{C}$ is the Casimir operator of $\mathfrak{g}$ acting on $C^{\infty}\left(G / K ; \tau_{p}\right)$.
Proposition 3.2 (see [9]). Let $0 \leq p<(n-1) / 2, q \in\{p-1, p\}$ and let $\lambda \in \mathbb{C}$ such that

$$
\begin{cases}i \lambda \notin\{-\rho+p\} \cup\left(\mathbb{Z}_{\leq 0}-\rho\right) & \text { if } q=p \\ i \lambda \notin\{\rho-p+1\} \cup\left(\mathbb{Z}_{\leq 0}-\rho\right) & \text { if } q=p-1 .\end{cases}
$$

The Poisson transform $\mathcal{P}_{q, \lambda}^{p}$ is a topological isomorphism from the space $C^{-\omega}\left(K / M ; \sigma_{q}\right)$ onto the space $\mathcal{E}_{q, \lambda}\left(G / K ; \tau_{p}\right)$.

We point out that the above statement was stated in [15] for $q=p$ and $n$ even.
For $1<r<\infty$, we denote by $L^{r}\left(K / M ; \sigma_{q}\right)$ the space of $\Lambda^{q} \mathbb{C}^{n-1}$-valued functions on $K$ which are covariant of type $\sigma_{q}$, i.e.,

$$
f(k m)=\sigma_{q}\left(m^{-1}\right) f(k), \quad \forall k \in K, \forall m \in M,
$$

and such that

$$
\|f\|_{L^{r}\left(K / M ; \sigma_{q}\right)}:=\left(\int_{K}\|f(k)\|_{\Lambda^{q} \mathbb{C}^{n-1}}^{r} \mathrm{~d} k\right)^{\frac{1}{r}}<\infty
$$

Note that, for any $F: K \rightarrow \Lambda^{\kappa} \mathbb{C}^{N}$ we have

$$
\begin{equation*}
\left\|\int_{K} F(k) \mathrm{d} k\right\|_{\Lambda^{\kappa} \mathbb{C}^{N}} \leq \int_{K}\|F(k)\|_{\Lambda^{\kappa} \mathbb{C}^{N}} \mathrm{~d} k . \tag{3.5}
\end{equation*}
$$

From above, it follows that the Poisson transform $\mathcal{P}_{q, \lambda}^{p}$ maps $L^{r}\left(K / M ; \sigma_{q}\right)$ into $\mathcal{E}_{q, \lambda}\left(G / K ; \tau_{p}\right)$. Our aim is to characterize the exact image of the space $L^{r}\left(K / M ; \sigma_{q}\right)$ by the Poisson transform $\mathcal{P}_{q, \lambda}^{p}$ for generic $p$ and $q \in\{p-1, p\}$.

## 4. Fatou-type theorem and the Harish-Chandra $c$-function

For $\lambda \in \mathbb{C}$, generic $p$, and $q \in\{p-1, p\}$, we define for $1<r<\infty$, the space $\mathcal{E}_{q, \lambda}^{r}\left(G / K ; \tau_{p}\right)$ to be the subspace of all $F$ in $\mathcal{E}_{q, \lambda}\left(G / K ; \tau_{p}\right)$ for which

$$
\|F\|_{\mathcal{E}_{q, \lambda}^{r}}:=\sup _{t>0} \mathrm{e}^{(\rho-\Re(i \lambda)) t}\left(\int_{K}\left\|F\left(k a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{r} \mathrm{~d} k\right)^{\frac{1}{r}}
$$

is finite.
Proposition 4.1. For every $\lambda \in \mathbb{C}$ with $\Re(i \lambda)>0$, there exists a positive constant $\gamma_{\lambda}$ such that, for any $f \in L^{r}\left(K / M ; \sigma_{q}\right)$ we have

$$
\begin{equation*}
\left(\int_{K}\left\|\mathcal{P}_{q, \lambda}^{p} f\left(k a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{r} \mathrm{~d} k\right)^{1 / r} \leq \gamma_{\lambda} c_{p, q} \mathrm{e}^{(\Re(i \lambda)-\rho) t}\|f\|_{L^{r}\left(K / M ; \sigma_{q}\right)} \tag{4.1}
\end{equation*}
$$

Proof. By (3.5) we have

$$
\begin{aligned}
& \left\|\mathcal{P}_{q, \lambda}^{p} f\left(k a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}} \\
& \leq c_{p, q} \int_{K} \mathrm{e}^{-(\Re(i \lambda)+\rho) H\left(a_{t}^{-1} k^{-1} h\right)} \| \tau_{p}\left(\kappa\left(a_{t}^{-1} k^{-1} h\right) \iota_{q}^{p}(f(h)) \|_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} h\right. \\
& \leq c_{p, q} \int_{K} \mathrm{e}^{-(\Re(i \lambda)+\rho) H\left(a_{t}^{-1} k^{-1} h\right)}\left\|\iota_{q}^{p}(f(h))\right\|_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} h,
\end{aligned}
$$

where the last inequality follows from the unitarity of $\tau_{p}$. Since $\iota_{q}^{p}$ is an isometric embedding, we can deduce that

$$
\begin{aligned}
\left\|\mathcal{P}_{q, \lambda}^{p} f\left(k a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}} & \leq c_{p, q} \int_{K} \mathrm{e}^{-(\Re(i \lambda)+\rho) H\left(a_{t}^{-1} k^{-1} h\right)}\|f(h)\|_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} h \\
& =c_{p, q} e_{\lambda, t}(\cdot) *\|f(\cdot)\|_{\Lambda^{q} \mathbb{C}^{n-1}}(k)
\end{aligned}
$$

where $e_{\lambda, t}(k)=\mathrm{e}^{-(\Re(i \lambda)+\rho) H\left(a_{t}^{-1} k^{-1}\right)}$, and $*$ is the convolution over $K$. Therefore, by Young's inequality, we obtain

$$
\left(\int_{K}\left\|\mathcal{P}_{q, \lambda}^{p} f\left(k a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{r} \mathrm{~d} k\right)^{1 / r} \leq c_{p, q}\left\|e_{\lambda, t}\right\|_{L^{1}\left(K / M ; \sigma_{q}\right)}\|f\|_{L^{r}\left(K / M ; \sigma_{q}\right)}
$$

Further,

$$
\left\|e_{\lambda, t}\right\|_{L^{1}\left(K / M ; \sigma_{q}\right)}=\int_{K} \mathrm{e}^{-(\Re(i \lambda)+\rho) H\left(a_{t}^{-1} k^{-1}\right)} \mathrm{d} k=\phi_{-i \Re(i \lambda)}^{\left(\rho-\frac{1}{2},-\frac{1}{2}\right)}(t),
$$

where $\phi_{\nu}^{(\alpha, \beta)}$ is the Jacobi function, see (4.7). Since $\Re(i \lambda)>0$, by (4.8) we have

$$
\phi_{-i \Re(i \lambda)}^{\left(\rho-\frac{1}{2},-\frac{1}{2}\right)}(t)=e^{(\Re(i \lambda)-\rho) t}\left(c_{\rho-\frac{1}{2},-\frac{1}{2}}(-i \Re(i \lambda))+o(1)\right) \text { as } t \rightarrow \infty,
$$

where $c_{\rho-\frac{1}{2},-\frac{1}{2}}(-i \Re(i \lambda))$ is given by (4.9). This proves the estimate (4.1) and consequently that the Poisson transform is continuous from $L^{r}\left(K / M ; \sigma_{q}\right)$ into $\mathcal{E}_{q, \lambda}^{r}\left(G / K ; \tau_{p}\right)$.

Let $\bar{N}=\theta(N)$, where $\theta$ is the Cartan involution of $G$. For $\lambda \in \mathbb{C}$ and $0 \leq p<$ $\frac{n-1}{2}$, define the generalized Harish-Chandra $c$-function by

$$
\begin{equation*}
\mathbf{c}(\lambda, p)=\int_{\bar{N}} \mathrm{e}^{-(i \lambda+\rho) H(\bar{n})} \tau_{p}(\kappa(\bar{n})) \mathrm{d} \bar{n} \in \operatorname{End}\left(\Lambda^{p} \mathbb{C}^{n}\right) \tag{4.2}
\end{equation*}
$$

Here $\mathrm{d} \bar{n}$ is the Haar measure on $\bar{N}$ with the normalization

$$
\int_{\bar{N}} \mathrm{e}^{-2 \rho(H(\bar{n}))} \mathrm{d} \bar{n}=1 .
$$

The integral (4.2) converges for $\lambda$ such that $\Re(i \lambda)>0$ and has a meromorphic continuation to $\mathbb{C}$ (see, e.g. [31]). Since the restriction $\mathbf{c}(\lambda, p)_{\mid V_{\sigma_{q}}}$ commutes with $\sigma_{q}$, then by Schur's lemma, there exists a complex scalar $c_{q}(\lambda, p)$ such that $\mathbf{c}(\lambda, p)_{\mid V_{\sigma_{q}}}=c_{q}(\lambda, p) \operatorname{Id}_{\Lambda^{q} \mathbb{C}^{n-1}}$. Therefore,

$$
\begin{equation*}
\mathbf{c}(\lambda, p)=c_{p-1}(\lambda, p) \operatorname{Id}_{\Lambda^{p-1}} \mathbb{C}^{n-1}+c_{p}(\lambda, p) \operatorname{Id}_{\Lambda^{p} \mathbb{C}^{n-1}} . \tag{4.3}
\end{equation*}
$$

In [29], an explicit expression of $c_{p-1}(\lambda, p)$ and $c_{p}(\lambda, p)$ are given by a direct computation of the integral (4.2). However, below in Proposition 4.6, we will recover their expressions by using a different approach.

The following lemma is needed for later use.
Lemma 4.2. (1) For every $v \in V_{\sigma_{q}}$,

$$
\begin{equation*}
\left\|\mathbf{c}(\lambda, p) \iota_{q}^{p}(v)\right\|_{\Lambda^{p} \mathbb{C}^{n}}=\left|c_{q}(\lambda, p)\right|\|v\|_{\Lambda^{q} \mathbb{C}^{n-1}} \tag{4.4}
\end{equation*}
$$

(2) For every linear operator $L$ form a vector space $V$ to $V_{\sigma_{q}}$,

$$
\begin{equation*}
\left\|\mathbf{c}(\lambda, p) \iota_{q}^{p} L\right\|_{\mathrm{HS}}=\left|c_{q}(\lambda, p)\right|\|L\|_{\mathrm{HS}} \tag{4.5}
\end{equation*}
$$

Proof. Using Remark 2.2, the first statement follows directly from

$$
\mathbf{c}(\lambda, p) \iota_{q}^{p}(v)= \begin{cases}c_{p-1}(\lambda, p) e_{1} \wedge v, & q=p-1 \\ c_{p}(\lambda, p) v, & q=p .\end{cases}
$$

On the other hand,

$$
\begin{aligned}
\left\|\mathbf{c}(\lambda, p) \iota_{q}^{p} L\right\|_{\mathrm{HS}}^{2} & =\operatorname{tr}\left(\left(\mathbf{c}(\lambda, p) \iota_{q}^{p} L\right)^{*}\left(\mathbf{c}(\lambda, p) \iota_{q}^{p} L\right)\right) \\
& =\operatorname{tr}\left(L^{*}\left(\pi_{p}^{q} \mathbf{c}(\lambda, p)^{*} \mathbf{c}(\lambda, p) \iota_{q}^{p}\right) L\right) .
\end{aligned}
$$

Notice that $\pi_{p}^{q} \mathbf{c}(\lambda, p)^{*} \mathbf{c}(\lambda, p) \iota_{q}^{p} \in \operatorname{End}_{\mathrm{M}}\left(\mathrm{V}_{\sigma_{\mathrm{q}}}\right)$, (hence is scalar). By (4.3), we deduce that

$$
\mathbf{c}(\lambda, p)^{*} \mathbf{c}(\lambda, p)=\left(\begin{array}{cc}
\left|c_{p-1}(\lambda, p)\right|^{2} \operatorname{Id}_{\Lambda^{p-1}} \mathbb{C}^{n-1} & 0 \\
0 & \left|c_{p}(\lambda, p)\right|^{2} \operatorname{Id}_{\Lambda^{p} \mathbb{C}^{n-1}}
\end{array}\right) .
$$

Thus $\pi_{p}^{q} \mathbf{c}(\lambda, p)^{*} \mathbf{c}(\lambda, p) \iota_{q}^{p}=\left|c_{q}(\lambda, p)\right|^{2} \operatorname{Id}_{\Lambda^{q} \mathbb{C}^{n-1}}$, and this proves the second statement.

Theorem 4.3. Let $\lambda \in \mathbb{C}$ such that $\Re(i \lambda)>0$. Then

$$
\lim _{t \rightarrow \infty} \mathrm{e}^{(\rho-i \lambda) t} \mathcal{P}_{q, \lambda}^{p} f\left(k a_{t}\right)=c_{p, q} \mathbf{c}(\lambda, p) \iota_{q}^{p}(f(k))
$$

(i) uniformly for $f \in C^{\infty}\left(K / M ; \sigma_{q}\right)$,
(ii) in the $L^{r}\left(K ; \Lambda^{p} \mathbb{C}^{n}\right)$-sens, for every $f \in L^{r}\left(K / M ; \sigma_{q}\right)$.

Proof. The statement $(i)$ has been proved earlier, see for instance [28] and [32].
(ii) Let $f \in L^{r}\left(K / M ; \sigma_{q}\right)$ and $\varepsilon>0$. By density argument, there exists a $K-$ finite vector $\varphi$ in $C^{\infty}\left(K / M ; \sigma_{q}\right)$ such that $\|f-\varphi\|_{L^{r}\left(K / M ; \sigma_{q}\right)}<\varepsilon$. Put $p_{\lambda}^{t}(f)(k)=$ $\mathcal{P}_{q, \lambda}^{p} f\left(k a_{t}\right)$, then

$$
\begin{aligned}
\left\|\mathrm{e}^{-(i \lambda-\rho) t} p_{\lambda}^{t}(f)(k)-c_{p, q} \mathbf{c}(\lambda, p) \iota_{q}^{p} f(k)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{r} & \leq\left\|\mathrm{e}^{-(i \lambda-\rho) t} p_{\lambda}^{t}(f-\varphi)(k)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{r} \\
& +\left\|\mathrm{e}^{-(i \lambda-\rho) t} p_{\lambda}^{t}(\varphi)(k)-c_{p, q} \mathbf{c}(\lambda, p) \iota_{q}^{p} \varphi(k)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{r} \\
& +c_{p q}^{r}\left\|\mathbf{c}(\lambda, p) \iota_{q}^{p} \varphi(k)-\mathbf{c}(\lambda, p) \iota_{q}^{p} f(k)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{r} .
\end{aligned}
$$

From Proposition 4.1 we obtain

$$
\int_{K}\left\|\mathrm{e}^{-(i \lambda-\rho) t} p_{\lambda}^{t}(f-\varphi)(k)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{r} \mathrm{~d} k \leq \gamma_{\lambda}^{r} c_{p, q}^{r}\|f-\varphi\|_{L^{r}\left(K / M ; \sigma_{q}\right)}^{r},
$$

and form part $(i)$ above it follows that

$$
\lim _{t \rightarrow \infty} \int_{K}\left\|\mathrm{e}^{-(i \lambda-\rho) t} p_{\lambda}^{t}(\varphi)(k)-c_{p, q} \mathbf{c}(\lambda, p) \iota_{q}^{p} \varphi(k)\right\|_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} k=0 .
$$

Further, according to (4.4) we obtain

$$
\int_{K}\left\|\mathbf{c}(\lambda, p) \iota_{q}^{p} \varphi(k)-\mathbf{c}(\lambda, p) \iota_{q}^{p} f(k)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{r} \mathrm{~d} k \leq\left|c_{q}(\lambda, p)\right|^{r}\|f-\varphi\|_{L^{r}\left(K / M ; \sigma_{q}\right)}^{r} .
$$

In conclusion we have
$\lim _{t \rightarrow \infty} \int_{K}\left\|\mathrm{e}^{-(i \lambda-\rho) t} p_{\lambda}^{t}(f)(k)-c_{p, q} \mathbf{c}(\lambda, p) \iota_{q}^{p} f(k)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{r} \mathrm{~d} k \leq \varepsilon^{r} c_{p, q}^{r}\left(\gamma_{\lambda}^{r}+\left|c_{q}(\lambda, p)\right|^{r}\right)$, and this proves the desired statement.

The following inequalities are crucial.
Proposition 4.4. For every $\lambda \in \mathbb{C}$ such that $\Re(i \lambda)>0$, there exists a positive constant $\gamma_{\lambda}$ such that for all $f \in L^{r}\left(K / M ; \sigma_{q}\right), 1<r<\infty$, we have

$$
\begin{equation*}
c_{p, q}\left|c_{q}(\lambda, p)\right|\|f\|_{L^{r}\left(K / M ; \sigma_{q}\right)} \leq\left\|\mathcal{P}_{q, \lambda}^{p} f\right\|_{\mathcal{E}_{q, \lambda}^{r}} \leq c_{p, q} \gamma_{\lambda}\|f\|_{L^{r}\left(K / M ; \sigma_{q}\right)} . \tag{4.6}
\end{equation*}
$$

Proof. The right-hand side inequality is noting but the estimate (4.1). For the left-hand side inequality, by Theorem $4.3[(i i)]$, there exists a sequence $\left(t_{j}\right)_{j}$ with $t_{j} \rightarrow \infty$ such that

$$
\lim _{j \rightarrow \infty}\left\|\mathrm{e}^{(\rho-i \lambda) t_{j}} \mathcal{P}_{q, \lambda}^{p} f\left(k a_{t_{j}}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}=\left\|c_{p, q} \mathbf{c}(\lambda, p) \iota_{q}^{p}(f(k))\right\|_{\Lambda^{p} \mathbb{C}^{n}}
$$

almost every where in $K$. Consequently, by the classical Fatou theorem and (4.4) we get

$$
c_{p, q}^{r}\left|c_{q}(\lambda, p)\right|^{r} \int_{K}\|f(k)\|_{\Lambda^{q} \mathbb{C}^{n-1}}^{r} \mathrm{~d} k \leq \sup _{j} \mathrm{e}^{r \Re(\rho-i \lambda) t_{j}} \int_{K}\left\|p_{\lambda}^{t_{j}}(f)(k)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{r} \mathrm{~d} k
$$

which implies

$$
c_{p, q}\left|c_{q}(\lambda, p)\right|\|f\|_{L^{r}\left(K / M ; \sigma_{q}\right)} \leq\left\|\mathcal{P}_{q, \lambda}^{p} f\right\|_{\mathcal{E}_{q, \lambda}^{r}}^{r}
$$

In the rest of this section we will see how the asymptotic behavior formula given in Theorem 4.3 will allows us to give explicitly the Harish-Chandra $c$-function

$$
\mathbf{c}(\lambda, p)=\int_{\bar{N}} \mathrm{e}^{-(i \lambda+\rho) H(\bar{n})} \tau_{p}(\kappa(\bar{n})) \mathrm{d} \bar{n} .
$$

To this aim, recall the Jacobi functions, see, e.g. [17],

$$
\begin{equation*}
\phi_{\lambda}^{(\alpha, \beta)}(t)={ }_{2} F_{1}\left(\frac{i \lambda+\alpha+\beta+1}{2}, \frac{-i \lambda+\alpha+\beta+1}{2} ; \alpha+1 ;-\sinh ^{2} t\right), \tag{4.7}
\end{equation*}
$$

with $\Re(\alpha+1)>0$ and ${ }_{2} F_{1}$ is the classical hypergeometric function. We shall need the following asymptotic behavior of Jacobi functions,

$$
\begin{equation*}
\phi_{\lambda}^{(\alpha, \beta)}(t)=\mathrm{e}^{(i \lambda-\alpha-\beta-1) t}\left(c_{\alpha, \beta}(\lambda)+o(1)\right) \text { as } t \rightarrow \infty \tag{4.8}
\end{equation*}
$$

for $\Re(i \lambda)>0$, where

$$
\begin{equation*}
c_{\alpha, \beta}(\lambda)=\frac{2^{\alpha+\beta+1-i \lambda} \Gamma(\alpha+1) \Gamma(i \lambda)}{\Gamma\left(\frac{i \lambda+\alpha+\beta+1}{2}\right) \Gamma\left(\frac{i \lambda+\alpha-\beta+1}{2}\right)} \tag{4.9}
\end{equation*}
$$

A continuous function $F: G \rightarrow \operatorname{End}\left(V_{\tau_{p}}\right)$ is called elementary $\tau_{p}$-spherical if $F$ satisfies
(i) $\left(\tau_{p}\right.$-radial function) $F\left(k_{1} g k_{2}\right)=\tau_{p}\left(k_{2}\right)^{-1} F(g) \tau_{p}\left(k_{1}^{-1}\right), \quad \forall g \in G, \forall k_{1}, k_{2} \in$ $K$,
(ii) $F$ is a joint-eigenfunction of all $D \in \mathbf{D}\left(G / K ; \tau_{p}\right)$ with $F(e)=I d$.

A $\tau_{p}$-radial function $F: G \rightarrow \operatorname{End}\left(V_{\tau_{p}}\right)$ (i.e. satisfying $\left.(i)\right)$ is determined by its restriction $F_{\left.\right|_{A}}$ to the subgroup $A$ of $G$. Since $A$ and $M$ commute, $F_{\left.\right|_{A}}$ becomes an $M$-morphism of $V_{\tau_{p}}=\Lambda^{p} \mathbb{C}^{n}$. Now, in the generic case, $\tau_{p_{\left.\right|_{M}}}$ is multiplicity free, therefore by Schur's lemma, $F_{\left.\right|_{A}}$ is scalar on each $M$-irreducible component $V_{\sigma_{p}}=\Lambda^{p} \mathbb{C}^{n-1}$ and $V_{\sigma_{p-1}}=\Lambda^{p-1} \mathbb{C}^{n-1}$. Thus

$$
F_{\left.\right|_{A}}\left(a_{t}\right)=f_{p-1}(t) \operatorname{Id}_{\Lambda^{p-1}} \mathbb{C}^{n-1}+f_{p}(t) \operatorname{Id}_{\Lambda^{p} \mathbb{C}^{n-1}}
$$

the coefficients $f_{p-1}$ and $f_{p}$ are called the scalar components of $F$.
For $\lambda \in \mathbb{C}$, we define the Eisenstein integral $\Phi_{q}^{p}(\lambda, g) \in \operatorname{End}\left(V_{\tau_{p}}\right)$ by

$$
\begin{equation*}
\Phi_{q}^{p}(\lambda, g)=c_{p, q}^{2} \int_{K} \mathrm{e}^{-(i \lambda+\rho) H\left(g^{-1} k\right)} \tau_{p}\left(\kappa\left(g^{-1} k\right)\right) \iota_{q}^{p}\left(\pi_{p}^{q}\left(\tau_{p}(k)^{-1}\right)\right) \mathrm{d} k \tag{4.10}
\end{equation*}
$$

Proposition 4.5 (see [25, Theorem 5.4]). Assume that $0 \leq p<\frac{n-1}{2}$.
(1) The set $\left\{\Phi_{q}^{p}(\lambda, \cdot), q=p-1, p ; \lambda \in \mathbb{C} \backslash\{ \pm 1\}\right\}$ exhausts the class of $\tau_{p}$ elementary spherical functions.
(2) The scalar components $\varphi_{q, p-1}(\lambda, t), \varphi_{q, p}(\lambda, t)$ of $\Phi_{q}^{p}\left(\lambda, a_{t}\right)$ are given by

$$
\Phi_{p}^{p}\left(\lambda, a_{t}\right):\left\{\begin{array}{l}
\varphi_{p, p-1}(\lambda, t)=\phi_{\lambda}^{\left(\frac{n}{2},-\frac{1}{2}\right)}(t),  \tag{4.11}\\
\varphi_{p, p}(\lambda, t)=\frac{n}{n-p} \phi_{\lambda}^{\left(\frac{n}{2}-1,-\frac{1}{2}\right)}(t)-\frac{p}{n-p}(\cosh t) \phi_{\lambda}^{\left(\frac{n}{2},-\frac{1}{2}\right)}(t),
\end{array}\right.
$$

and

$$
\Phi_{p-1}^{p}\left(\lambda, a_{t}\right):\left\{\begin{array}{l}
\varphi_{p-1, p-1}(\lambda, t)=\frac{n}{p} \phi_{\lambda}^{\left(\frac{n}{2}-1,-\frac{1}{2}\right)}(t)-\frac{n-p}{p}(\cosh t) \phi_{\lambda}^{\left(\frac{n}{2},-\frac{1}{2}\right)}(t)  \tag{4.12}\\
\varphi_{p-1, p}(\lambda, t)=\phi_{\lambda}^{\left(\frac{n}{2},-\frac{1}{2}\right)}(t)
\end{array}\right.
$$

For $q \in\{p-1, p\}$, let us introduce the notation $\rho_{q}=\rho-q=\frac{n-1}{2}-q$.
Proposition 4.6. Let $\lambda \in \mathbb{C}$ such that $\Re(i \lambda)>0$. The generalized HarishChandra $c$-function is given by

$$
\mathbf{c}(\lambda, p)=c_{p-1}(\lambda, p) \operatorname{Id}_{\Lambda^{p-1} \mathbb{C}^{n-1}}+c_{p}(\lambda, p) \operatorname{Id}_{\Lambda^{p} \mathbb{C}^{n-1}}
$$

where the scalar coefficients are explicitly given by

$$
c_{p-1}(\lambda, p)=\frac{i \lambda-\rho_{p-1}}{i \lambda+\rho} c(\lambda)
$$

and

$$
c_{p}(\lambda, p)=\frac{i \lambda+\rho_{p}}{i \lambda+\rho} c(\lambda)
$$

with

$$
c(\lambda)=2^{\rho-i \lambda} \frac{\Gamma(i \lambda) \Gamma\left(\rho+\frac{1}{2}\right)}{\Gamma\left(\frac{i \lambda+\rho}{2}\right) \Gamma\left(\frac{i \lambda+\rho+1}{2}\right)} .
$$

Proof. Let $\lambda \in \mathbb{C}$ such that $\Re(i \lambda)>0$. Since

$$
\Phi_{q}^{p}\left(\lambda, k a_{t}\right)=\mathcal{P}_{q, \lambda}^{p}\left(c_{p, q} \pi_{p}^{q}\left(\tau\left(k^{-1}\right)\right)\right)\left(a_{t}\right),
$$

Theorem 4.3 implies

$$
\begin{equation*}
\Phi_{q}^{p}\left(\lambda, a_{t}\right)=c_{p, q}^{2} \mathbf{c}(\lambda, p) \mathrm{e}^{(i \lambda-\rho) t}\left(\pi_{p}^{q}+o(1)\right) \quad \text { as } t \rightarrow \infty, \tag{4.13}
\end{equation*}
$$

with

$$
c_{p, q}^{2}= \begin{cases}\frac{n}{n-p} & \text { if } q=p \\ \frac{n}{p} & \text { if } q=p-1\end{cases}
$$

Let us first consider the case $q=p$. Using the asymptotic behavior of Jacobi functions (4.8) together with the relation

$$
c_{\frac{n}{2},-\frac{1}{2}}(\lambda)=\frac{2 n}{i \lambda+\rho} c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda),
$$

we obtain

$$
\begin{aligned}
& \varphi_{p, p}(\lambda, t) \underset{t \rightarrow \infty}{=} \\
& \frac{1}{n-p} \mathrm{e}^{(i \lambda-\rho) t}\left(n c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda)-\frac{p}{2} c_{\frac{n}{2},-\frac{1}{2}}(\lambda)+o(1)\right), \\
&= \mathrm{e}^{(i \lambda-\rho) t} \frac{n}{n-p} c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda)\left(\frac{i \lambda+\rho-p}{i \lambda+\rho}+o(1)\right) .
\end{aligned}
$$

Similarly, we get

$$
\varphi_{p, p-1}(\lambda, t) \underset{t \rightarrow \infty}{=} \mathrm{e}^{(i \lambda-\rho-1) t}\left(c_{\frac{n}{2},-\frac{1}{2}}(\lambda)+o(1)\right) .
$$

Thus

$$
\begin{aligned}
\Phi_{p}^{p}\left(\lambda, a_{t}\right) & =\mathrm{e}^{(i \lambda-\rho-1) t}\left(c_{\frac{n}{2},-\frac{1}{2}}(\lambda)+o(1)\right) \operatorname{Id}_{\Lambda^{p-1}} \mathbb{C}^{n-1} \\
& +\mathrm{e}^{(i \lambda-\rho) t} \frac{n}{n-p} c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda)\left(\frac{i \lambda+\rho-p}{i \lambda+\rho}+o(1)\right) \operatorname{Id}_{\Lambda^{p} \mathbb{C}^{n-1}}
\end{aligned}
$$

from which we deduce that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \mathrm{e}^{(\rho-i \lambda) t} \Phi_{p}^{p}\left(\lambda, a_{t}\right)=\frac{n}{n-p}\left(\frac{i \lambda+\rho-p}{i \lambda+\rho}\right) c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda) \operatorname{Id}_{\Lambda^{p} \mathbb{C}^{n-1}} . \tag{4.14}
\end{equation*}
$$

Finally, by identification of (4.13) and (4.14) it follows that

$$
c_{p}(\lambda, p)=\frac{i \lambda+\rho-p}{i \lambda+\rho} c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda)=\frac{i \lambda+\rho-p}{i \lambda+\rho} c(\lambda) .
$$

Similarly, for $q=p-1$ we can prove that

$$
\lim _{t \rightarrow \infty} \mathrm{e}^{(i \lambda-\rho) t} \Phi_{p-1}^{p}\left(\lambda, a_{t}\right)=\frac{n}{p}\left(\frac{i \lambda-\rho+p-1}{i \lambda+\rho}\right) c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda) \operatorname{Id}_{\Lambda^{p-1}} \mathbb{C}^{n-1},
$$

from which we deduce that

$$
c_{p-1}(\lambda, p)=\frac{i \lambda-\rho+p-1}{i \lambda+\rho} c_{\frac{n}{2}-1,-\frac{1}{2}}(\lambda)=\frac{i \lambda-\rho+p-1}{i \lambda+\rho} c(\lambda) .
$$

## 5. The $L^{2}$-range of the Poisson transform

Recall that our main goal is to characterize the image of the space $L^{r}\left(K / M ; \sigma_{q}\right)$ under the Poisson transform $\mathcal{P}_{q, \lambda}^{p}$, for $1<r<\infty$. To do so, we will start with the case $r=2$.
Fix $\sigma_{q} \in \widehat{M}$ acting on the space $V_{\sigma_{q}}=\Lambda^{q} \mathbb{C}^{n-1}$ of dimension $d_{\sigma_{q}}$. To simplify notations, we will write sometimes ( $\sigma, V_{\sigma}$ ) instead of ( $\sigma_{q}, V_{\sigma_{q}}$ ).

Let $\left(\delta, V_{\delta}\right)$ be an element in $\widehat{K}(\sigma)$, where $\widehat{K}(\sigma) \subset \widehat{K}$ denotes the subset of those classes containing $\sigma$ upon restriction to $K$. It follows from Frobenius reciprocity theorem together with [13] that $\sigma$ occurs in $\delta_{\mid M}$ with multiplicity one and therefore $\operatorname{dim} \operatorname{Hom}_{M}\left(V_{\delta}, V_{\sigma}\right)=1$. We choose the orthogonal projection $P_{\delta}: V_{\delta} \rightarrow V_{\sigma}$ as a generator of $\operatorname{Hom}_{M}\left(V_{\delta}, V_{\sigma}\right)$.
let $\left(v_{j}\right)_{j=1}^{d_{\delta}}$ be an orthonormal basis for $V_{\delta}$, where $d_{\delta}=\operatorname{dim} V_{\delta}$. Then the functions

$$
k \mapsto \phi_{j}^{\delta}(k)=P_{\delta}\left(\delta\left(k^{-1}\right) v_{j}\right), \quad 1 \leq j \leq d_{\delta}, \quad \delta \in \widehat{K}(\sigma)
$$

define an orthogonal basis of the space $L^{2}\left(K / M ; \sigma_{q}\right)$, see, e.g. [30]. Thus, the Fourier expansion of every $f \in L^{2}\left(K / M ; \sigma_{q}\right)$ is given by

$$
f(k)=\sum_{\delta \in \widehat{K}(\sigma)} \sum_{j=1}^{d_{\delta}} a_{j}^{\delta} \phi_{j}^{\delta}(k),
$$

with

$$
\begin{equation*}
\|f\|_{L^{2}(K / M ; \sigma)}^{2}=\sum_{\delta \in \widehat{K}(\sigma)} \frac{d_{\delta}}{d_{\sigma}} \sum_{j=1}^{d_{\delta}}\left|a_{j}^{\delta}\right|^{2} . \tag{5.1}
\end{equation*}
$$

Next, we will prove a general result giving the Poisson integral representation of a joint eigensections of the algebra $\mathbf{D}\left(G / K ; \tau_{p}\right)$ of $G$-invariant differential operators acting on $C^{\infty}\left(G / K ; \tau_{p}\right)$.

By a functional on $E_{q, \lambda}=G \times{ }_{P} V_{\sigma_{q}}$ we shall mean a linear form $T$ on $C^{\infty}\left(G / P ; \sigma_{q, \bar{\lambda}}\right)$. For a such functional $T$, we define $\widetilde{\mathcal{P}_{q, \lambda}^{p}}(T)$ by

$$
\begin{equation*}
\left\langle v, \widetilde{\mathcal{P}_{q, \lambda}^{p}} T(g)\right\rangle_{\Lambda^{p} \mathbb{C}^{n}}=c_{p, q}\left(T, \pi_{p}^{q} L_{g} \Phi_{\lambda} v\right), \quad \forall v \in \Lambda^{p} \mathbb{C}^{n} \tag{5.2}
\end{equation*}
$$

where $L_{g}$ is the left regular action, and $\Phi_{\lambda}: G \rightarrow \operatorname{End}\left(V_{\tau_{p}}\right)$ is given by

$$
\begin{equation*}
\Phi_{\lambda}(g)=\mathrm{e}^{(i \bar{\lambda}-\rho) H(g)} \tau_{p}^{-1}(\kappa(g)) \tag{5.3}
\end{equation*}
$$

Notice that $\Phi_{\lambda}\left(g^{-1} k\right)^{*}=P_{q, \lambda}^{p}(g, k)$, where $P_{q, \lambda}^{p}: G \times K \rightarrow \operatorname{End}\left(V_{\tau_{p}}\right)$ is the Poisson kernel given by

$$
\begin{equation*}
P_{q, \lambda}^{p}(g, k)=\mathrm{e}^{-(i \lambda+\rho) H\left(g^{-1} k\right)} \tau_{p}\left(\kappa\left(g^{-1} k\right)\right) \tag{5.4}
\end{equation*}
$$

If $T=T_{f}$ is a functional given by $f \in C^{\infty}\left(G / P ; \sigma_{q, \lambda}\right)$, then

$$
\begin{equation*}
\widetilde{\mathcal{P}_{q, \lambda}^{p}}\left(T_{f}\right)=\mathcal{P}_{q, \lambda}^{p}(f) \tag{5.5}
\end{equation*}
$$

Indeed,

$$
\begin{aligned}
\left\langle v, \widetilde{\mathcal{P}_{q, \lambda}^{p}} T_{f}(g)\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} & =c_{p, q}\left(T, \pi_{p}^{q} L_{g} \Phi_{\lambda} v\right) \\
& =c_{p, q} \int_{K}\left\langle f(k), \pi_{p}^{q} L_{g} \Phi_{\lambda}(k) v\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k \\
& =c_{p, q} \int_{K}\left\langle f(k), \pi_{p}^{q} \Phi_{\lambda}\left(g^{-1} k\right) v\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k \\
& =c_{p, q} \int_{K}\left\langle\Phi_{\lambda}^{*}\left(g^{-1} k\right) \iota_{q}^{p} f(k), v\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} k \\
& =c_{p, q} \int_{K}\left\langle P_{q, \lambda}^{p}(g, k) \iota_{q}^{p} f(k), v\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} k \\
& =\left\langle v, \mathcal{P}_{q, \lambda}^{p} f(g)\right\rangle_{\Lambda^{p} \mathbb{C}^{n}}
\end{aligned}
$$

Proposition 5.1. For every eigensection $F$ of $\mathbf{D}\left(G / K ; \tau_{p}\right)$, there exists a functional $T$ on $C^{\infty}\left(G / P ; \sigma_{q, \bar{\lambda}}\right)$ such that $F=\widetilde{\mathcal{P}_{q, \lambda}^{p}} T$.

Proof. Let $F$ be an arbitrary joint eigensection of all $D \in \mathbf{D}\left(G / K ; \tau_{p}\right)$. Then $F$ has an expansion

$$
F(g)=\sum_{\delta \in \widehat{K}(\sigma)} F_{\delta}(g)
$$

in $C^{\infty}\left(G / K ; \tau_{p}\right)$. Since $F_{\delta}$ is $K$-finite of type $\delta$, then, by [32, Corollary 10.8], there exists a $K$-finite vector $f_{\delta}$ in $C^{\infty}\left(G / P ; \sigma_{q, \lambda}\right)$ such that $F_{\delta}=\mathcal{P}_{q, \lambda}^{p} f_{\delta}$. We have

$$
f_{\delta}(k)=\sum_{j=1}^{d_{\delta}} a_{j}^{\delta} P_{\delta}\left(\delta\left(k^{-1}\right) v_{j}\right)
$$

Define a functional $T$ by

$$
\begin{equation*}
(T, \varphi)=\sum_{\delta \in \widehat{K}(\sigma)} \sum_{j=1}^{d_{\delta}} \overline{a_{j}^{\delta}} \int_{K}\left\langle\varphi(k), P_{\delta}\left(\delta\left(k^{-1}\right) v_{j}\right)\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k \tag{5.6}
\end{equation*}
$$

for all $\varphi \in C^{\infty}\left(G / P ; \sigma_{q, \bar{\lambda}}\right)$ for which the above sum converges. Choose $\varphi$ in (5.6) to be $\varphi: k \mapsto c_{p, q} \pi_{p}^{q}\left(\Phi_{\lambda}\left(g^{-1} k\right) w\right)$ with $w \in V_{\tau_{p}}$, then we get

$$
\begin{aligned}
(T, \varphi) & =c_{p, q} \sum_{\delta \in \widehat{K}(\sigma)} \sum_{j=1}^{d_{\delta}} \overline{a_{j}^{\delta}} \int_{K}\left\langle\pi_{p}^{q} \Phi_{\lambda}\left(g^{-1} k\right) w, P_{\delta}\left(\delta\left(k^{-1}\right) v_{j}\right)\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k \\
& =c_{p, q} \sum_{\delta} \sum_{j} \overline{a_{j}^{\delta}} \int_{K}\left\langle w, \Phi_{\lambda}\left(g^{-1} k\right)^{*}\left(\pi_{p}^{q}\right)^{*} P_{\delta}\left(\delta\left(k^{-1}\right) v_{j}\right)\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} k, \\
& =c_{p, q} \sum_{\delta} \sum_{j} \overline{a_{j}^{\delta}} \int_{K}\left\langle w, \mathrm{e}^{-(i \lambda+\rho) H\left(g^{-1} k\right)} \tau_{p}\left(\kappa\left(g^{-1} k\right)\right) \iota_{q}^{p} P_{\delta}\left(\delta\left(k^{-1}\right) v_{j}\right)\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} k, \\
& =\left\langle w, \sum_{\delta} c_{p, q} \int_{K} \mathrm{e}^{-(i \lambda+\rho) H\left(g^{-1} k\right)} \tau_{p}\left(\kappa\left(g^{-1} k\right)\right) \iota_{q}^{p} \sum_{j} a_{j}^{\delta} P_{\delta}\left(\delta\left(k^{-1}\right) v_{j}\right) \mathrm{d} k\right\rangle_{\Lambda^{p} \mathbb{C}^{n}}, \\
& =\left\langle w, \sum_{\delta \in \widehat{K}(\sigma)} \mathcal{P}_{\lambda, p}^{q} f_{\delta}(g)\right\rangle_{\Lambda^{p} \mathbb{C}^{n}}, \\
& =\langle w, F(g)\rangle_{\Lambda^{p} \mathbb{C}^{n}} .
\end{aligned}
$$

On the other hand, by the definition (5.2) of the Poisson transform on functionals, we have

$$
\left(T, c_{p, q} \pi_{p}^{q}\left(L_{g} \Phi_{\lambda} w\right)\right)=\left\langle w, \widetilde{\mathcal{P}_{q, \lambda}^{p}} T(g)\right\rangle_{\Lambda^{p} \mathbb{C}^{n}}
$$

from which we deduce that $F(g)=\widetilde{\mathcal{P}_{q, \lambda}^{p}} T(g)$, since the vector $w$ is arbitrary.
Theorem 5.2. Assume that $\lambda \in \mathbb{C}$ such that

$$
\begin{cases}\Re(i \lambda)>0 & \text { if } q=p,  \tag{5.7}\\ \Re(i \lambda)>0 \text { and } i \lambda \neq \rho-p+1 & \text { if } q=p-1 .\end{cases}
$$

The Poisson transform $\mathcal{P}_{q, \lambda}^{p}$ is a topological isomorphism from the space $L^{2}\left(K / M ; \sigma_{q}\right)$ onto the space $\mathcal{E}_{q, \lambda}^{2}\left(G / K ; \tau_{p}\right)$. Moreover, there exists a positive constant $\gamma_{\lambda}$ such that

$$
c_{p, q}\left|c_{q}(\lambda, p)\right|\|f\|_{L^{2}\left(K / M ; \sigma_{q}\right)} \leq\left\|\mathcal{P}_{q, \lambda}^{p} f\right\|_{\mathcal{E}_{q, \lambda}^{2}} \leq c_{p, q} \gamma_{\lambda}\|f\|_{L^{2}\left(K / M ; \sigma_{q}\right)}
$$

for every $f \in L^{2}\left(K / M ; \sigma_{q}\right)$.
Proof. On one hand, by Proposition 3.2 and Proposition 4.4 it follows that $\mathcal{P}_{q, \lambda}^{p}$ is a continuous map from $L^{2}\left(K / M ; \sigma_{q}\right)$ into $\mathcal{E}_{q, \lambda}^{2}\left(G / K ; \tau_{p}\right)$.

On the other hand, for $F \in \mathcal{E}_{q, \lambda}^{2}\left(G / K ; \tau_{p}\right)$, by Proposition 5.1, there exists a functional $T$ on $C^{\infty}\left(G / P ; \sigma_{q, \bar{\lambda}}\right)$ defined by (5.6) such that $F=\widetilde{\mathcal{P}_{q, \lambda}^{p}} T$. From the proof of Proposition 5.1, it follows that

$$
F(g)=c_{p, q} \sum_{\delta \in \widehat{K}(\sigma)} \sum_{j=1}^{d_{\delta}} a_{j}^{\delta} \int_{K} \mathrm{e}^{-(i \lambda+\rho) H\left(g^{-1} k\right)} \tau_{p}\left(\kappa\left(g^{-1} k\right)\right) \iota_{q}^{p} P_{\delta}\left(\delta\left(k^{-1}\right) v_{j}\right) \mathrm{d} k
$$

Define $\Phi_{\lambda, \delta}$ by

$$
\begin{equation*}
\Phi_{\lambda, \delta}(g)(v)=c_{p, q} \int_{K} \mathrm{e}^{-(i \lambda+\rho) H\left(g^{-1} k\right)} \tau_{p}\left(\kappa\left(g^{-1} k\right)\right) \iota_{q}^{p} P_{\delta}\left(\delta\left(k^{-1}\right) v\right) \mathrm{d} k, \tag{5.8}
\end{equation*}
$$

for $g \in G$ and $v \in V_{\delta}$. Clearly $\Phi_{\lambda, \delta}\left(k_{1} g k_{2}\right)=\tau_{p}\left(k_{2}^{-1}\right) \Phi_{\lambda, \delta}(g) \delta\left(k_{1}^{-1}\right)$ for every $g \in G$ and $k_{1}, k_{2} \in K$. Further

$$
\int_{K}\left\langle F\left(k a_{t}\right), F\left(k a_{t}\right)\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} k=\sum_{\delta, \delta^{\prime}} \sum_{j, \ell} a_{j}^{\delta} \overline{a_{\ell}^{\delta^{\prime}}} \int_{K}\left\langle\Phi_{\lambda, \delta}\left(k a_{t}\right) v_{j}, \Phi_{\lambda, \delta^{\prime}}\left(k a_{t}\right) v_{\ell}\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} k
$$

By the covariance property and Schur's lemma, we obtain

$$
\begin{aligned}
\int_{k}\left\langle\Phi_{\lambda, \delta}\left(k a_{t}\right) v_{j}, \Phi_{\lambda, \delta^{\prime}}(k a) v_{\ell}\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} k & =\int_{K}\left\langle\Phi_{\lambda, \delta^{\prime}}\left(a_{t}\right)^{*} \Phi_{\lambda, \delta}\left(a_{t}\right) \delta\left(k^{-1}\right) v_{j}, \delta^{\prime}\left(k^{-1}\right) v_{\ell}\right\rangle_{V_{\delta}} \mathrm{d} k \\
& = \begin{cases}0 & \text { if } \delta^{\prime} \nsim \delta \\
\frac{1}{d_{\delta}} \operatorname{tr}\left(\Phi_{\lambda, \delta}\left(a_{t}\right)^{*} \Phi_{\lambda, \delta}\left(a_{t}\right)\right)\left\langle v_{j}, v_{\ell}\right\rangle_{V_{\delta}} & \text { otherwise }\end{cases}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_{K}\left\|F\left(k a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{2} \mathrm{~d} k & =\sum_{\delta \in \widehat{K}(\sigma)} \frac{1}{d_{\delta}} \sum_{j=1}^{d_{\delta}}\left|a_{j}^{\delta}\right|^{2} \operatorname{tr}\left(\Phi_{\lambda, \delta}\left(a_{t}\right)^{*} \Phi_{\lambda, \delta}\left(a_{t}\right)\right) \\
& =\sum_{\delta} \frac{1}{d_{\delta}}\left\|\Phi_{\lambda, \delta}\left(a_{t}\right)\right\|_{\mathrm{HS}}^{2} \sum_{j}\left|a_{j}^{\delta}\right|^{2}
\end{aligned}
$$

where $\|\cdot\|_{\text {HS }}$ is the Hilbert-Schmidt norm. Hence, for a finite subset $\Lambda \subset \widehat{K}(\sigma)$ we get

$$
\begin{aligned}
\sum_{\delta \in \Lambda} \frac{1}{d_{\delta}} \sum_{j}\left\|a_{j}^{\delta} \mathrm{e}^{(\rho-i \lambda) t} \Phi_{\lambda, \delta}\left(a_{t}\right)\right\|_{\mathrm{HS}}^{2} & \leq \sup _{t>0} \mathrm{e}^{2(\rho-\Re(i \lambda)) t} \int_{K}\left\|F\left(k a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{2} \mathrm{~d} k, \\
& =\|F\|_{\mathcal{E}_{2, \lambda}^{2}}^{2}
\end{aligned}
$$

Under the assumption (5.7) we may use Theorem 4.3, i.e.,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \mathrm{e}^{(\rho-i \lambda) t} \Phi_{\lambda, \delta}\left(a_{t}\right)=c_{p, q} \mathbf{c}(\lambda, p) \iota_{q}^{p} P_{\delta} \tag{5.9}
\end{equation*}
$$

and (4.5) to obtain

$$
c_{p, q}^{2}\left|c_{q}(\lambda, p)\right|^{2} \sum_{\delta \in \Lambda} \frac{1}{d_{\delta}} \sum_{j}\left\|a_{j}^{\delta} P_{\delta}\right\|_{\mathrm{HS}}^{2} \leq\|F\|_{\mathcal{E}_{2, \lambda}^{2}}^{2} .
$$

That is

$$
c_{p, q}^{2}\left|c_{q}(\lambda, p)\right|^{2} \sum_{\delta \in \Lambda} \frac{1}{d_{\delta}} \sum_{j} d_{\sigma}\left|a_{j}^{\delta}\right|^{2} \leq\|F\|_{\mathcal{E}_{2, \lambda}^{2}}^{2} .
$$

Since the subset $\Lambda \subset \widehat{K}(\sigma)$ is arbitrary, it follows that

$$
c_{p, q}^{2}\left|c_{q}(\lambda, p)\right|^{2} \sum_{\delta \in \widehat{K}(\sigma)} \frac{d_{\sigma}}{d_{\delta}} \sum_{j}\left|a_{j}^{\delta}\right|^{2} \leq\|F\|_{\mathcal{E}_{2, \lambda}^{2}}^{2}<\infty
$$

This shows that the functional $T(k) \sim \sum_{\delta \in \widehat{K}(\sigma)} \sum_{j=1}^{d_{\delta}} a_{j}^{\delta} P_{\delta} \delta\left(k^{-1}\right) v_{j}$ defines a function $f \in L^{2}(K / M ; \sigma)$ and by (5.5), we deduce that $F=\mathcal{P}_{q, \lambda}^{p} f$ with

$$
c_{p, q}\left|c_{q}(\lambda, p)\right|\|f\|_{L^{2}(K / M ; \sigma)} \leq\left\|\mathcal{P}_{q, \lambda}^{p} f\right\|_{\mathcal{E}_{2, \lambda}^{2}} .
$$

Lemma 5.3. We have

$$
\sup _{t>0} \mathrm{e}^{(\rho-\Re(i \lambda)) t}\left\|\Phi_{\lambda, \delta}\left(a_{t}\right)\right\|_{\mathrm{HS}} \leq \gamma_{\lambda} c_{p, q}\left\|P_{\delta}\right\|_{\mathrm{HS}}=\gamma_{\lambda} c_{p, q} \sqrt{d_{\sigma}} .
$$

Proof. By Proposition 4.1 we have
$\sup _{t>0} \mathrm{e}^{(\rho-\Re(i \lambda)) t}\left(\int_{K}\left\|\mathcal{P}_{q, \lambda}^{p}\left(P_{\delta}\left(\delta^{-1}(\cdot) v\right)\right)\left(k a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{2} \mathrm{~d} k\right)^{1 / 2} \leq \gamma_{\lambda} c_{p, q}\left\|P_{\delta}\left(\delta^{-1}(\cdot) v\right)\right\|_{L^{2}\left(K / M ; \sigma_{q}\right)}$.
Since $\mathcal{P}_{q, \lambda}^{p}\left(P_{\delta}\left(\delta^{-1}(\cdot) v\right)\right)\left(k a_{t}\right)=\Phi_{\lambda, \delta}\left(k a_{t}\right)(v)$, we get

$$
\begin{aligned}
\int_{K}\left\|\mathcal{P}_{q, \lambda}^{p}\left(P_{\delta}\left(\delta^{-1}(\cdot) v\right)\left(k a_{t}\right)\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{2} \mathrm{~d} k & =\int_{K}\left\langle\Phi_{\lambda, \delta}\left(a_{t}\right) \delta\left(k^{-1}\right) v, \Phi_{\lambda, \delta}\left(a_{t}\right) \delta\left(k^{-1}\right) v\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} k, \\
& =\frac{1}{d_{\delta}} \operatorname{tr}\left(\Phi_{\lambda, \delta}\left(a_{t}\right)^{*} \Phi_{\lambda, \delta}\left(a_{t}\right)\right)\|v\|_{V_{\delta}}^{2}, \\
& =\frac{1}{d_{\delta}}\left\|\Phi_{\lambda, \delta}\left(a_{t}\right)\right\|_{\mathrm{HS}}^{2}\|v\|_{V_{\delta}}^{2} .
\end{aligned}
$$

Now the desired inequality follows from

$$
\left\|P_{\delta}\left(\delta^{-1}(\cdot) v\right)\right\|_{L^{2}\left(K / M ; \sigma_{q}\right)}^{2}=\frac{d_{\sigma}}{d_{\delta}}\|v\|_{V_{\delta}}^{2} .
$$

Lemma 5.4. We have

$$
\lim _{t \rightarrow \infty} \mathrm{e}^{2(\rho-\Re(i \lambda)) t}\left\|\Phi_{\lambda, \delta}\left(a_{t}\right)\right\|_{\mathrm{HS}}^{2}=c_{p, q}^{2}\left|c_{q}(\lambda, p)\right|^{2} d_{\sigma_{q}},
$$

where $c_{q}(\lambda, p)$ is the scalar component of $\mathbf{c}(\lambda, p)$ on $V_{\sigma_{q}}=\Lambda^{q} \mathbb{C}^{n-1}$.
Proof. Recall that $\Phi_{\lambda, \delta}\left(a_{t}\right)=\mathcal{P}_{q, \lambda}^{p}\left(P_{\delta}\left(\delta^{-1}(\cdot)\right)\right)\left(a_{t}\right)$. Then

$$
\begin{aligned}
e^{2(\rho-\Re(i \lambda)) t}\left\|\Phi_{\lambda, \delta}\left(a_{t}\right)\right\|_{\mathrm{HS}}^{2} & =\sum_{j=1}^{d_{\delta}}\left\|e^{(\rho-\Re(i \lambda)) t} \Phi_{\lambda, \delta}\left(a_{t}\right) v_{j}\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{2}, \\
& =\sum_{j=1}^{d_{\delta}}\left\|e^{(\rho-\Re(i \lambda)) t} \mathcal{P}_{q, \lambda}^{p}\left(P_{\delta}\left(\delta^{-1}(\cdot) v_{j}\right)\right)\left(a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{2} .
\end{aligned}
$$

Using Theorem 4.3 and (4.5), we obtain

$$
\begin{aligned}
\lim _{t \rightarrow \infty} e^{2(\rho-\Re(i \lambda)) t}\left\|\Phi_{\lambda, \delta}\left(a_{t}\right)\right\|_{\mathrm{HS}}^{2} & =c_{p, q}^{2} \sum_{j=1}^{d_{\delta}}\left\langle\mathbf{c}(\lambda, p) i_{q}^{p} P_{\delta} v_{j}, \mathbf{c}(\lambda, p) \iota_{q}^{p} P_{\delta} v_{j}\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} . \\
& =c_{p, q}^{2}\left\|\mathbf{c}(\lambda, p) \iota_{q}^{p} P_{\delta}\right\|_{\mathrm{HS}} \\
& =c_{p, q}^{2}\left|c_{q}(\lambda, p)\right|^{2} d_{\sigma_{q}} .
\end{aligned}
$$

Theorem 5.5 (Inversion formula). Assume $\lambda \in \mathbb{C}$ such that

$$
\begin{cases}\Re(i \lambda)>0 & \text { if } q=p \\ \Re(i \lambda)>0 \text { and } i \lambda \neq \rho-p+1 & \text { if } q=p-1 .\end{cases}
$$

Let $F \in \mathcal{E}_{q, \lambda}^{2}\left(G / K ; \tau_{p}\right)$ and let $f \in L^{2}\left(K / M ; \sigma_{q}\right)$ be its boundary value. Then the following inversion formula holds in $L^{2}\left(K / M ; \sigma_{q}\right)$

$$
f(k)=c_{p, q}^{-1}\left|c_{q}(\lambda, p)\right|^{-2} \lim _{t \rightarrow \infty} e^{2(\rho-\Re(i \lambda)) t} \pi_{p}^{q}\left(\int_{K} P_{q, \lambda}^{p}\left(h a_{t}, k\right)^{*} F\left(h a_{t}\right) \mathrm{d} h\right)
$$

where $P_{q, \lambda}^{p}(\cdot, \cdot)$ is the Poisson kernel given in (5.4).
Proof. Let $F \in \mathcal{E}_{q, \lambda}^{2}\left(G / K ; \tau_{p}\right)$. By Theorem 5.2, there exists a unique $f \in$ $L^{2}\left(K / M ; \sigma_{q}\right)$ such that $F=\mathcal{P}_{q, \lambda}^{p} f$. Write

$$
f(k)=\sum_{\delta \in \widehat{K}\left(\sigma_{q}\right)} \sum_{j=1}^{d_{\delta}} a_{j}^{\delta} P_{\delta}\left(\delta\left(k^{-1}\right)\right) v_{j}
$$

Then

$$
F\left(k a_{t}\right)=\sum_{\delta} \sum_{j} a_{j}^{\delta} \Phi_{\lambda, \delta}\left(a_{t}\right) \delta\left(k^{-1}\right) v_{j}
$$

and therefore

$$
\int_{K}\left\|F\left(k a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{2} \mathrm{~d} k=\sum_{\delta} \sum_{j} \frac{\left|a_{j}^{\delta}\right|^{2}}{d_{\delta}}\left\|\Phi_{\lambda, \delta}\left(a_{t}\right)\right\|_{\mathrm{HS}}^{2}
$$

From Lemma 5.4 we deduce

$$
\lim _{t \rightarrow \infty} \mathrm{e}^{2(\rho-\mathbb{R}(i \lambda)) t} \int_{K}\left\|\mathcal{P}_{q, \lambda}^{p} f\left(k a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{2} \mathrm{~d} k=c_{p, q}^{2}\left|c_{q}(\lambda, p)\right|^{2}\|f\|_{L^{2}\left(K / M ; \sigma_{q}\right)}^{2}
$$

which implies

$$
\lim _{t \rightarrow \infty}\left(g_{t}, \varphi\right)_{L^{2}\left(K / M ; \sigma_{q}\right)}=(f, \varphi)_{L^{2}\left(K / M ; \sigma_{q}\right)}, \quad \forall \varphi \in L^{2}\left(K / M ; \sigma_{q}\right)
$$

where $g_{t}$ is the $V_{\sigma_{q}}$-valued function defined by

$$
g_{t}(k)=c_{p, q}^{-1}\left|c_{q}(\lambda, p)\right|^{-2} \mathrm{e}^{2(\rho-\Re(i \lambda)) t} \pi_{p}^{q} \int_{K} P_{q, \lambda}^{p}\left(h a_{t}, k\right)^{*} F\left(h a_{t}\right) \mathrm{d} h
$$

To obtain the inversion formula, it is only required to show that

$$
\lim _{t \rightarrow \infty}\left\|g_{t}\right\|_{L^{2}\left(K / M ; \sigma_{q}\right)}=\|f\|_{L^{2}\left(K / M ; \sigma_{q}\right)}
$$

To do so, let us first compute the Fourier coefficients $c_{j}^{\delta}\left(g_{t}\right)$ of $g_{t}$ :

$$
\begin{aligned}
& c_{j}^{\delta}\left(g_{t}\right)=\frac{d_{\delta}}{d_{\sigma}} \int_{K}\left\langle g_{t}(k), P_{\delta} \delta\left(k^{-1}\right) v_{j}\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k \\
= & c_{p, q}^{-1}\left|c_{q}(\lambda, p)\right|^{-2} \mathrm{e}^{2(\rho-\Re(i \lambda)) t} \\
\times & \frac{d_{\delta}}{d_{\sigma}} \sum_{\delta^{\prime}, \ell} a_{\ell}^{\delta^{\prime}} \int_{K}\left\langle\pi_{p}^{q} \int_{K} P_{q, \lambda}^{p}\left(h a_{t}, k\right)^{*} \Phi_{\lambda, \delta^{\prime}}\left(a_{t}\right) \delta^{\prime}\left(h^{-1}\right) v_{\ell} \mathrm{d} h, P_{\delta} \delta\left(k^{-1}\right) v_{j}\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k .
\end{aligned}
$$

Since $\left(\pi_{p}^{q}\right)^{*}=\iota_{q}^{p}$, we get

$$
\begin{aligned}
c_{j}^{\delta}\left(g_{t}\right) & =c_{p, q}^{-1}\left|c_{q}(\lambda, p)\right|^{-2} \mathrm{e}^{2(\rho-\Re(i \lambda)) t} \\
& \times \frac{d_{\delta}}{d_{\sigma}} \sum_{\delta^{\prime}, \ell} a_{\ell}^{\delta^{\prime}} \int_{K} \int_{K}\left\langle\Phi_{\lambda, \delta^{\prime}}\left(a_{t}\right) \delta^{\prime}\left(h^{-1}\right) v_{\ell}, P_{q, \lambda}^{p}\left(h a_{t}, k\right) \iota_{q}^{p} P_{\delta} \delta\left(k^{-1}\right) v_{j}\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} h \mathrm{~d} k
\end{aligned}
$$

As $\int_{K} P_{q, \lambda}^{p}\left(h a_{t}, k\right) \iota_{q}^{p} P_{\delta} \delta\left(k^{-1}\right) \mathrm{d} k=c_{p, q}^{-1} \Phi_{\lambda, \delta}\left(h a_{t}\right)$, we obtain

$$
\begin{aligned}
c_{j}^{\delta}\left(g_{t}\right) & =c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2} \mathrm{e}^{2(\rho-\Re(i \lambda)) t} \\
& \times \frac{d_{\delta}}{d_{\sigma}} \sum_{\delta^{\prime}, \ell} a_{\ell}^{\delta^{\prime}} \int_{K}\left\langle\Phi_{\lambda, \delta^{\prime}}\left(a_{t}\right) \delta^{\prime}\left(h^{-1}\right) v_{\ell}, \Phi_{\lambda, \delta}\left(a_{t}\right) \delta\left(h^{-1}\right) v_{j}\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} h \\
& =c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2} \mathrm{e}^{2(\rho-\Re(i \lambda)) t} \\
& \times \frac{d_{\delta}}{d_{\sigma}} \sum_{\delta^{\prime}, \ell} a_{\ell}^{\delta^{\prime}} \int_{K}\left\langle\delta(h) \Phi_{\lambda, \delta}\left(a_{t}\right)^{*} \Phi_{\lambda, \delta^{\prime}}\left(a_{t}\right) \delta^{\prime}\left(h^{-1}\right) v_{\ell}, v_{j}\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} h .
\end{aligned}
$$

By the Schur lemma, we get

$$
\begin{aligned}
c_{j}^{\delta}\left(g_{t}\right) & =c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2} \mathrm{e}^{2(\rho-\Re(i \lambda)) t} \\
& \times \frac{d_{\delta}}{d_{\sigma}} \sum_{\ell} a_{\ell}^{\delta} \int_{K} \frac{1}{d_{\delta}} \operatorname{tr}\left(\Phi_{\lambda, \delta}\left(a_{t}\right)^{*} \Phi_{\lambda, \delta}\left(a_{t}\right)\right)\left\langle v_{\ell}, v_{j}\right\rangle_{V_{\delta}} \mathrm{d} h, \\
& =c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2} \mathrm{e}^{2(\rho-\Re(i \lambda)) t} \frac{1}{d_{\sigma}} a_{j}^{\delta}\left\|\Phi_{\lambda, \delta}\left(a_{t}\right)\right\|_{\mathrm{HS}}^{2} .
\end{aligned}
$$

From all the above computations, we conclude that,

$$
\left\|g_{t}\right\|_{L^{2}(K / M, \sigma)}^{2}=\left(\mathrm{e}^{2(\rho-\Re(i \lambda)) t}\left|c_{p, q} c_{q}(\lambda, p)\right|^{-2}\right)^{2} \sum_{\delta} \frac{d_{\sigma}}{d_{\delta}} \sum_{j} \frac{1}{d_{\sigma}^{2}}\left|a_{j}^{\delta}\right|^{2}\left\|\Phi_{\lambda, \delta}\left(a_{t}\right)\right\|_{\mathrm{HS}}^{4}
$$

and by Lemma 5.4 we get

$$
\lim _{t \rightarrow \infty}\left\|g_{t}\right\|_{L^{2}(K / M ; \sigma)}^{2}=\sum_{\delta} \frac{d_{\sigma}}{d_{\delta}} \sum_{j}\left|a_{j}^{\delta}\right|^{2}=\|f\|_{L^{2}(K / M ; \sigma)}^{2}
$$

To finish the proof, we have to justify that we can reverse $\lim _{t \rightarrow \infty}$ and $\sum_{\delta}$ by proving that the serie

$$
\sum_{\delta \in \widehat{K}(\sigma)} \frac{1}{d_{\delta}} \sum_{j=1}^{d_{\delta}}\left|a_{j}^{\delta}\right|^{2}\left\|\Phi_{\lambda, \delta}\left(a_{t}\right)\right\|_{\mathrm{HS}}^{4},
$$

is uniformly convergent. This follows easily from Lemma 5.3.

## 6. The $L^{r}$-range of the Poisson transform

In this section we shall generalize Theorem 5.2 to $L^{r}\left(K / M ; \sigma_{q}\right)$ with $1<r<\infty$.
Theorem 6.1. Let $0 \leq p<(n-1) / 2$, and $\lambda \in \mathbb{C}$ such that

$$
\begin{cases}\Re(i \lambda)>0 & \text { if } q=p \\ \Re(i \lambda)>0 \text { and } i \lambda \neq \rho-p+1 & \text { if } q=p-1 .\end{cases}
$$

For $1<r<\infty$, the Poisson transform $\mathcal{P}_{q, \lambda}^{p}$ is a topological isomorphism from the space $L^{r}\left(K / M ; \sigma_{q}\right)$ onto the space $\mathcal{E}_{q, \lambda}^{r}\left(G / K ; \tau_{p}\right)$. Moreover, there exists a positive constant $\gamma_{\lambda}$ such that

$$
c_{p, q} \mid c_{q}(\lambda, p)\|f\|_{L^{r}\left(K / M ; \sigma_{q}\right)} \leq\left\|\mathcal{P}_{q, \lambda}^{p} f\right\|_{\mathcal{E}_{q, \lambda}^{r}} \leq c_{p, q} \gamma_{\lambda}\|f\|_{L^{r}\left(K / M ; \sigma_{q}\right)}
$$

for every $f \in L^{r}\left(K / M ; \sigma_{q}\right)$.
Proof. The necessary condition follows from Proposition 3.2 and Proposition 4.4. For the sufficiency condition, let $F \in \mathcal{E}_{q, \lambda}^{r}\left(G / K ; \tau_{p}\right)$ and write $F(g)=\sum_{i} F_{i}(g) u_{i}$ where $\left(u_{i}\right)_{i}$ is an orthonormal basis of $\Lambda^{p} \mathbb{C}^{n}$. Fix $\left(\chi_{m}\right)_{m}$ to be an approximation of the identity in $C^{\infty}(K)$ and let $F_{i, m}(g)=\int_{K} \chi_{m}(k) F_{i}\left(k^{-1} g\right) \mathrm{d} k$. Then $\left(F_{i, m}\right)_{m}$ converges point-wise to $F_{i}$. Define $F_{m}: G \rightarrow \Lambda^{p} \mathbb{C}^{n}$ by $F_{m}(g)=\sum_{i} F_{i, m}(g) u_{i}$. Then

$$
\begin{aligned}
F_{m}(g) & =\sum_{i}\left(\int_{K} \chi_{m}(k) F_{i}\left(k^{-1} g\right) \mathrm{d} k\right) u_{i} \\
& =\int_{K} \chi_{m}(k) \sum_{i} F_{i}\left(k^{-1} g\right) u_{i} \mathrm{~d} k \\
& =\int_{K} \chi_{m}(k) F\left(k^{-1} g\right) \mathrm{d} k .
\end{aligned}
$$

We have $\left\|F_{m}(g)-F(g)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{2} \underset{m \rightarrow \infty}{\rightarrow 0}$ and since the operators $\mathcal{C}, D$ and $D^{*}$ in (3.2) are $K$-invariant, then $F_{m} \in \mathcal{E}_{q, \lambda}\left(G / K ; \tau_{p}\right)$ for every $m$. Further,

$$
\begin{aligned}
F_{m}\left(k a_{t}\right) & =\int_{K} \chi_{m}(h) F\left(h^{-1} k a_{t}\right) \mathrm{d} h, \\
& =\left(\chi_{m} * F^{t}\right)(k),
\end{aligned}
$$

where $F^{t}: K \rightarrow \Lambda^{p} \mathbb{C}^{n}$ is defined for any $t>0$ by $F^{t}(k)=F\left(k a_{t}\right)$ for every $F$. By (3.5) we have

$$
\left\|\left(\chi_{m} * F^{t}\right)(k)\right\|_{\Lambda^{p} \mathbb{C}^{n}} \leq \int_{K}\left|\chi_{m}(h)\right|\left\|F^{t}\left(h^{-1} k\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} h,
$$

that is

$$
\left\|F_{m}^{t}(k)\right\|_{\Lambda^{p} \mathbb{C}^{n}} \leq\left(\left|\chi_{m}(\cdot)\right| *\left\|F^{t}(\cdot)\right\|_{\Lambda^{p} \mathbb{C}^{n}}\right)(k)
$$

Therefore

$$
\left\|F_{m}^{t}\right\|_{L^{r}\left(K ; \Lambda^{p} \mathbb{C}^{n}\right)} \leq\left\|\left|\chi_{m}(\cdot)\right| *\right\| F^{t}(\cdot)\left\|_{\Lambda^{p} \mathbb{C}^{n}}\right\|_{L^{r}(K)}
$$

By Young's inequalities we obtain

$$
\begin{align*}
\left\|F_{m}^{t}\right\|_{L^{r}\left(K ; \Lambda^{p} \mathbb{C}^{n}\right)} & \leq\left\|\chi_{m}\right\|_{L^{1}(K)}\| \| F^{t}(\cdot)\left\|_{\Lambda^{p} \mathbb{C}^{n}}\right\|_{L^{r}(K)}, \\
& =\left\|F^{t}\right\|_{L^{r}\left(K ; \Lambda^{p} \mathbb{C}^{n}\right)}, \tag{6.1}
\end{align*}
$$

and

$$
\begin{align*}
\left\|F_{m}^{t}\right\|_{L^{2}\left(K ; \Lambda^{p} \mathbb{C}^{n}\right)} & \leq\left\|\chi_{m}\right\|_{L^{2}(K)}\| \| F^{t}(\cdot)\left\|_{\Lambda^{p} \mathbb{C}^{n}}\right\|_{L^{1}(K)} \\
& =\left\|\chi_{m}\right\|_{L^{2}(K)}\left\|F^{t}\right\|_{L^{r}\left(K ; \Lambda^{p} \mathbb{C}^{n}\right)} \tag{6.2}
\end{align*}
$$

The inequality (6.2) implies

$$
\sup _{t>0} \mathrm{e}^{(\rho-\Re(i \lambda)) t}\left(\int_{K}\left\|F_{m}\left(k a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}^{2}\right)^{1 / 2} \leq\left\|\chi_{m}\right\|_{L^{2}(K)}\|F\|_{\mathcal{E}_{q, \lambda}^{r}}<\infty .
$$

Hence, for each $m, F_{m} \in \mathcal{E}_{q, \lambda}^{2}\left(G / K ; \tau_{p}\right)$ and from Theorem 5.2 it follows that there exists $f_{m} \in L^{2}\left(K / M ; \sigma_{q}\right)$ such that $F_{m}=\mathcal{P}_{q, \lambda}^{p} f_{m}$. To prove that $f_{m} \in$ $L^{r}\left(K / M ; \sigma_{q}\right)$ we will follow the same method as in [5]. According to Theorem 5.5 we have, for any $\varphi \in C^{\infty}\left(K / M ; \sigma_{q}\right)$,

$$
\int_{K}\left\langle f_{m}(k), \varphi(k)\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k=\lim _{t \rightarrow \infty} \int_{K}\left\langle g_{m}^{t}(k), \varphi(k)\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k,
$$

where

$$
g_{m}^{t}(k):=c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2} e^{2(\rho-\Re(i \lambda)) t} \pi_{p}^{q} \int_{K} P_{\lambda}\left(h a_{t}, k\right)^{*} F_{m}\left(h a_{t}\right) \mathrm{d} h .
$$

Further,

$$
\begin{aligned}
& \int_{K}\left\langle g_{m}^{t}(k), \varphi(k)\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k \\
& =c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2} e^{2(\rho-\Re(i \lambda)) t} \int_{K}\left\langle\pi_{p}^{q} \int_{K} P_{\lambda}\left(h a_{t}, k\right)^{*} F_{m}\left(h a_{t}\right) \mathrm{d} h, \varphi(k)\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k, \\
& =c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2} e^{2(\rho-\Re(i \lambda)) t} \int_{K} \int_{K}\left\langle F_{m}\left(h a_{t}\right), P_{\lambda}\left(h a_{t}, k\right) i_{q}^{p} \varphi(k) \mathrm{d} k\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} h, \\
& =c_{p, q}^{-3}\left|c_{q}(\lambda, p)\right|^{-2} e^{2(\rho-\Re(i \lambda)) t} \int_{K}\left\langle F_{m}\left(h a_{t}\right),\left(\mathcal{P}_{q, \lambda}^{p} \varphi\right)\left(h a_{t}\right)\right\rangle_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} h .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \left|\int_{K}\left\langle g_{m}^{t}(k), \varphi(k)\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k\right| \\
& \leq c_{p, q}^{-3}\left|c_{q}(\lambda, p)\right|^{-2} e^{2(\rho-\Re(i \lambda)) t} \int_{K}\left\|F_{m}\left(h a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}}\left\|\mathcal{P}_{q, \lambda}^{p} \varphi\left(h a_{t}\right)\right\|_{\Lambda^{p} \mathbb{C}^{n}} \mathrm{~d} h
\end{aligned}
$$

By Hölder's inequality (with $\frac{1}{r}+\frac{1}{s}=1$ ), we deduce

$$
\begin{aligned}
& \left|\int_{K}\left\langle g_{m}^{t}(k), \varphi(k)\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k\right| \\
& \leq c_{p, q}^{-3}\left|c_{q}(\lambda, p)\right|^{-2} e^{2(\rho-\Re(i \lambda)) t}\left\|F_{m}^{t}\right\|_{L^{r}\left(K ; \Lambda^{p} \mathbb{C}^{n}\right)}\left\|\left(\mathcal{P}_{q, \lambda}^{p} \varphi\right)^{t}\right\|_{L^{s}\left(K ; \Lambda^{p} \mathbb{C}^{n}\right)}
\end{aligned}
$$

where $\left(\mathcal{P}_{q, \lambda}^{p} \varphi\right)^{t}(k)=\left(\mathcal{P}_{q, \lambda}^{p} \varphi\right)\left(k a_{t}\right)$. Using (6.1) and Proposition 4.1 we get

$$
\begin{aligned}
\left|\int_{K}\left\langle f_{m}(k), \varphi(k)\right\rangle_{\Lambda q \mathbb{C}^{n-1}} \mathrm{~d} k\right| & \leq \gamma_{\lambda} c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2}\left\|F_{m}\right\| \mathcal{E}_{q, \lambda}^{r}\|\varphi\|_{L^{s}\left(K / M ; \sigma_{q}\right)} \\
& \leq \gamma_{\lambda} c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2}\|F\|_{\mathcal{E}_{q, \lambda}^{r}}\|\varphi\|_{L^{s}\left(K / M ; \sigma_{q}\right)}
\end{aligned}
$$

By taking the supremum over all $\varphi \in C^{\infty}\left(K / M ; \sigma_{q}\right)$ with $\|\varphi\|_{L^{s}\left(K / M ; \sigma_{q}\right)}=1$ we obtain

$$
\left\|f_{m}\right\|_{L^{r}\left(K / M ; \sigma_{q}\right)} \leq \gamma_{\lambda} c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2}\|F\|_{\mathcal{E}_{q, \lambda}^{r}}^{r},
$$

which implies $f_{m}$, initially belongs to $L^{2}\left(K / M ; \sigma_{q}\right)$, is in fact in $L^{r}\left(K / M ; \sigma_{q}\right)$.

For every $m$, define the linear form $T_{m}$ on $L^{s}\left(K / M ; \sigma_{q}\right)$ by

$$
T_{m}(\varphi)=\int_{K}\left\langle f_{m}(k), \varphi(k)\right\rangle_{\Lambda^{q} \mathbb{C}^{n-1}} \mathrm{~d} k
$$

Clearly, $T_{m}$ is continuous and

$$
\left|T_{m}(\varphi)\right| \leq \gamma_{\lambda} c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2}\|F\|_{\mathcal{E}_{q, \lambda}^{r}}^{r}\|\varphi\|_{L^{s}\left(K / M ; \sigma_{q}\right)} .
$$

This shows that $\left(T_{m}\right)_{m}$ is uniformly bounded in $L^{s}\left(K / M ; \sigma_{q}\right)$, with

$$
\sup _{m}\left\|T_{m}\right\|_{\mathrm{op}} \leq \gamma_{\lambda} c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2}\|F\|_{\mathcal{E}_{q, \lambda}^{r}}^{\varepsilon^{r}} .
$$

The Banach-Alaouglu-Bourbaki theorem will then ensures the existence of a subsequence of bounded operators $\left(T_{m_{j}}\right)$ which converges to a bounded operator $T$ under the weak-* topology, with

$$
\|T\|_{\mathrm{op}} \leq \gamma_{\lambda} c_{p, q}^{-2}\left|c_{q}(\lambda, p)\right|^{-2}\|F\|_{\mathcal{E}_{q, \lambda}^{r}} .
$$

Thus, Riesz's representation theorem guarantees the existence of a unique $f \in$ $L^{r}\left(K / M ; \sigma_{q}\right)$ such that

$$
T(\varphi)=\int_{K}\langle\varphi(k), f(k)\rangle_{\Lambda q \mathbb{C}^{n-1}} \mathrm{~d} k .
$$

By means of the Poisson kernel (5.4), we consider the test function $\varphi_{g}(k)=$ $P_{q, \lambda}^{p}(g, k) v$ with $v \in \Lambda^{p} \mathbb{C}^{n}$, then

$$
T\left(\varphi_{g}\right)=\left\langle v, \mathcal{P}_{q, \lambda}^{p} f(g)\right\rangle_{\Lambda^{p} \mathbb{C}^{n}}
$$

On the other hand

$$
T_{m_{j}}\left(\varphi_{g}\right)=\left\langle v, \mathcal{P}_{q, \lambda}^{p} f_{m_{j}}(g)\right\rangle_{\Lambda^{p} \mathbb{C}^{n}}=\left\langle v, F_{m_{j}}(g)\right\rangle_{\Lambda^{p} \mathbb{C}^{n}}
$$

Taking the limit of the above identity when $j \rightarrow \infty$ we conclude that $F(g)=$ $\mathcal{P}_{q, \lambda}^{p} f(g)$ for every $g \in G$.

As an immediate consequence of Theorem 6.1 we obtain the following characterization of co-closed harmonic $p$-forms on $\mathbb{H}^{n}$ :

Corollary 6.2. Let $p$ be an integer with $0 \leq p<(n-1) / 2$. For $1<r<$ $\infty$, the Poisson transform $\mathcal{P}_{p, i(p-\rho)}^{p}$ is a topological isomorphism from the space $L^{r}\left(K / M ; \sigma_{p}\right)$ onto the space $\mathcal{E}_{p, i(p-\rho)}^{r}\left(G / K ; \tau_{p}\right)$. Moreover, for every $f \in L^{r}\left(K / M ; \sigma_{p}\right)$ the following estimates hold,

$$
\frac{2(\rho-p)}{2 \rho-p} c_{p}(\rho)\|f\|_{L^{r}\left(K / M ; \sigma_{p}\right)} \leq\left\|\mathcal{P}_{p, i(p-\rho)}^{p} f\right\|_{\mathcal{E}_{p, i(p-\rho)}^{r}} \leq c_{p}(\rho)\|f\|_{L^{r}\left(K / M ; \sigma_{p}\right)}
$$

where

$$
c_{p}(\rho)=c_{p, p} \frac{2^{p} \Gamma\left(\rho+\frac{1}{2}\right) \Gamma(\rho-p)}{\Gamma\left(\rho-\frac{p}{2}\right) \Gamma\left(\rho-\frac{p}{2}+\frac{1}{2}\right)}
$$

In the case where $p=0$, we recover the classical fact that the Poisson transform is an isometric isomorphism from $L^{r}\left(\partial \mathbb{H}^{n}\right)$ onto the Hardy-harmonic space on $\mathbb{H}^{n}$ (see [27]).

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[^1]:    ${ }^{1}$ The parameter $\lambda$ in Yang [32] corresponds in our notation to $i \lambda$.

[^2]:    

