On well-posedness for some Korteweg-De Vries type equations with variable coefficients

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ON WELL-POSEDNESS FOR SOME KORTEWEG-DE VRIES TYPE EQUATIONS WITH VARIABLE COEFFICIENTS.

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Abstract. In this paper, KdV-type equations with time- and space-dependent coefficients are considered. Assuming that the dispersion coefficient in front of u_{xxx} is positive and uniformly bounded away from the origin and that a primitive function of the ratio between the anti-dissipation and the dispersion coefficients is bounded from below, we prove the existence and uniqueness of a solution u such that hu belongs to a classical Sobolev space, where h is a function related to this ratio. The LWP in $H^s(\mathbb{R})$, $s > 1/2$, in the classical (Hadamard) sense is also proven under an assumption on the integrability of this ratio. Our approach combines a change of unknown with dispersive estimates. Note that previous results were restricted to $H^s(\mathbb{R})$, $s > 3/2$, and only used the dispersion to compensate the anti-dissipation and not to lower the Sobolev index required for well-posedness.

1. Introduction and Main Results

1.1. Presentation of the problem. In this paper, we study the Cauchy problem for the KdV-type equation with variable coefficients

$$\begin{cases} u_t + \alpha(t,x)u_{3x} + \beta(t,x)u_{2x} + \gamma(t,x)u_x + \delta(t,x)u \\ \qquad = \epsilon(t,x)uu_x \quad \text{for} \quad (t,x) \in (0,T)\times\mathbb{R} \\ u_{|t=0} = u_0, \end{cases} \tag{1.1}$$ KdV1

where $u = u(t,x)$, from $[0,T]\times\mathbb{R}$ into $\mathbb{R}$, is the unknown function of the problem, $u_0 = u_0(x)$, from $\mathbb{R}$ into $\mathbb{R}$, is the given initial condition, $\alpha = \alpha(t,x) \geq \alpha_0 > 0\ \forall\,(t,x) \in [0,T]\times\mathbb{R}$, and β, γ, δ, ϵ are real-valued smooth and bounded given functions with exact regularities that will be precised later. Of course, we will also require a strong condition on the relation between α and the positive part of β. This equation covers several important unidirectional models for the water waves problems at different regimes which take into account the variations of the bottom. We have in view in particular the example of the KdV equation with variable coefficients (see for instance [10], [13]) for which $\beta \equiv 0$. Looking for solutions of (1.1) plays an important and significant role in the study of unidirectional limits for water wave problems with variable depth and topographies.

The study of equations of this type with variable coefficients goes back to the seminal paper of Craig-Kappeler-Strauss [7] where the local well-posedness (LWP) in high regularity Sobolev spaces is established under the condition that $-\beta \geq 0$. Actually their results even concern quasilinear version of (1.1). In [2], Akhunov proved that the associated linear equation is LWP under an assumption on the boundedness uniformly in time and space of the primitive function $(t,x) \mapsto \int_0^x r(t,z)dz$ where $r(\cdot,\cdot)$ is the ratio function $r(t,z) = \beta(t,z)/\alpha(t,z)$. He also showed some evidences on the sharpness of this assumption. Adaptation of the LWP in high

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regularity Sobolev spaces under this hypothesis for quasilinear and fully nonlinear generalizations of (1.1) can be found in respectively [1] and [3]. In [8], Israwi and the second author proved the LWP of (1.1) in $H^s(\mathbb{R})$, $s > 3/2$, under the same type of integrability assumption on the ratio function $r(t, x)$. Their method of proof uses weighted energy estimates.

Up to our knowledge, our approach is the first one that enables to treat low regularity solutions. Note that, in sharp contrast to [8], we use in a crucial way the dispersive nature of the equation driven by the third order term not only to compensate the anti-diffusion term but also to lower the regularity of the resolution space. We proceed in two steps. In a first step we make a change of unknown in order to rely the solutions of (1.1) to the solutions of the following KdV-type equation with a constant coefficient in front of u_{3x} :

$$u_t + u_{3x} - b(t,x)u_{2x} + c(t,x)u_x + d(t,x)u = e(t,x)uu_x + f(t,x)u^2 \quad \text{for} \quad (t,x) \in (0,T)\times\mathbb{R} \tag{1.2}$$

KdV2

where b, c, d, e, f are real-valued smooth given functions with this time $b \geq 0$. Note that this change of unknown is related to the gauge method that is used in similar contexts as in [2], [5], [8]. Actually, at this stage, to ensure that the coefficients e and f of the nonlinear terms are bounded we will require the boundedness from above uniformly in $[0, T]\times\mathbb{R}$ of $-\int_0^x r_1(t, z)\, dz$ where $r_1 = \beta_1/\alpha$ is, roughly speaking, the ratio function between the positive part β_1 of β and α (see Hypothesis 3 in Section 3).

We then prove that the Cauchy problem associated with (1.2) is locally well-posed [(1)]in $H^s(\mathbb{R})$, $s > 1/2$, by using the method recently introduced by the first author and S. Vento in [11] that combines energy's and Bourgain's type estimates. It is worth noticing that terms as $c(t, x)u_x$ and $-b(t, x)u_{2x}$ may not be treated by a classical fixed point argument in Bourgain's spaces associated with the KdV linear flow. We would like also to emphasize that we will not require a coercive condition on b in $[0, T]\times\mathbb{R}$ ($b \geq \beta > 0$ on $[0, T]\times\mathbb{R}$) but only the non negativity of b. Actually we even obtain the unconditional uniqueness in $H^s(\mathbb{R})$ in the case $b = 0$.

Coming back to (1.1) this proves the existence of a solution u such that $hu \in C([0, T]; H^s)$ with $T = T(\|hu_0\|_{H^s})$, where $h > 0$ defined in (3.8) is a function related to the ratio function $r(\cdot, \cdot)$ (see Theorem 3.1). This solution is the unique solution of (1.1) such that hu belongs in $L^\infty(0, T; H^s)$. It is worth pointing out that we do not need any assumption (except to be bounded and "smooth") on the coefficient β outside a neighborhood of $-\infty$. Actually, as noticed in Remark 3.1, any smooth and bounded β that is non positive uniformly in time at $-\infty$ would satisfy our assumption.

Finally to get the LWP of (1.1) in classical Sobolev spaces $H^s(\mathbb{R})$, $s > 1/2$, we need not only h but also $1/h$ to be bounded, that corresponds to require h to be a classical gauge. This leads to an integrability conditon on $\mathbb{R}$ uniformly in time of the ratio function $r_1(\cdot, \cdot)$. Note that this type of condition, that already appears in other works on the subject as [2] and [8], is proven to be sharp for the LWP in $H^s(\mathbb{R})$ of the linear equation in [2]. In particular, it turns out that anti-diffusion on a compact set will not avoid the local well-posedness of the equation.

To end this introduction, let us recall the linear explanation of this last result that can be found for instance in [5]. To simplify we concentrate on the linear

[(1)]In a forthcoming paper we will show how to enhance the LWP result to $H^s(\mathbb{R})$, $s \geq 0$, that will enable to prove a global well-posedness result for a KdV equation with a variable bottom that is non increasing.

equation

$$u_t + \alpha u_{3x} + \beta u_{2x} = 0 \ .$$

and we assume that α and β are constant on $[0,T]\times[-R,R]$ with $\alpha>0$ and $\beta\geq 0$. Since a wave packet of amplitude close to A and frequencies close to ξ_0 moves to the left with a speed close to $\frac{d\omega}{d\xi}(\xi_0)=3\alpha\xi_0^2$, this wave packet will stays in $[-R,R]$ during about an interval of time $\Delta t=\frac{2R}{\alpha\xi_0^2}$ and thus the effect of the anti-diffusion will make its amplitude growths to $A\exp(2R\frac{\beta}{\alpha})$ that does not depend on ξ_0. This shows that the speed of propagation of wave packets induced by the dispersion term of order three ∂_x^3 is just sufficient to compensate the growth of the amplitude of this wave packet induced by the anti-diffusion on a compact set.

1.2. **Main results.** In the sequel $[s]$ denotes the integer part of the real number s and for any $N\in\mathbb{N}$, $C_b^N(\mathbb{R})$ denotes the space of functions $f\in C^N(\mathbb{R})$ with f, f', .., $f^{(N)}$ bounded.

We first introduce our notion of solutions to (1.1) and (1.2).

Definition 1.1. *Assume that $\alpha\in L^\infty_T C^3_b$, $\beta\in L^\infty_T C^2_b$, $\gamma,\epsilon\in L^\infty_T C^1_b$ and $\delta\in L^\infty(]0,T[\times\mathbb{R})$.*

We say that $u\in L^\infty_T L^2_x$ is a weak solution to (1.1) if for any $\phi\in C^\infty_c(]-T,T[\times\mathbb{R})$ it holds

$$\int_0^T\int_{\mathbb{R}} u\Big[-\phi_t-\partial_x^3(\alpha\phi)+\partial_x^2(\beta\phi)-\partial_x(\gamma\phi)+\delta\phi\Big]\,dx\,dt+\frac{1}{2}\int_0^T\int_{\mathbb{R}} u^2\partial_x(\epsilon\phi)\,dx\,dt$$
$$+\int_{\mathbb{R}} u_0(x)\phi(0,x)\,dx=0 \qquad (1.3)$$ weak1

rem1 **Remark 1.1.** *Note that if $u\in L^\infty_T L^2_x$ is a weak solution to (1.1) then (1.1) is satisfied in the distributional sense on $]0,T[\times\mathbb{R}$ and thus $u_t\in L^\infty_T H^{-3}_x$. This forces u to belong to $C_w([0,T];L^2(\mathbb{R}))$ and (1.3) ensures that $u(0)=u_0$.*

We define in the same way the weak solutions to (1.2).

Definition 1.2. *Assume that $b\in L^\infty_T C^2_b$, $c,e\in L^\infty_T C^1_b$ and $d,f\in L^\infty(]0,T[\times\mathbb{R})$.*

We say that $u\in L^\infty_T L^2_x$ is a weak solution to (1.2) if for any $\phi\in C^\infty_c(]-T,T[\times\mathbb{R})$ it holds

$$\int_0^T\int_{\mathbb{R}} u\Big[-\phi_t-\phi_{3x}-\partial_x^2(b\phi)-\partial_x(c\phi)+d\phi\Big]\,dx\,dt+\int_0^T\int_{\mathbb{R}} u^2\Big[\frac{1}{2}\partial_x(e\phi)+f\Big]\,dx\,dt$$
$$+\int_{\mathbb{R}} u_0(x)\phi(0,x)\,dx=0 \qquad (1.4)$$ weak2

Let us now state our first result.

th2 **Theorem 1.1.** *Let $s>\frac{1}{2}$ and $T\in]0,+\infty]$. Assume that b,c,e in $L^\infty(]0,T[;C_b^{[s]+2}(\mathbb{R}))$ with e_t in $L^\infty(]0,T[\times\mathbb{R})$ and $d,f\in L^\infty(]0,T[;C_b^{[s]+1}(\mathbb{R}))$. Assume moreover that*

$$b\geq 0 \quad \text{on } [0,T]\times\mathbb{R}\ . \qquad (1.5)$$

Then for all $u_0\in H^s(\mathbb{R})$, there exist a time $0<T_0=T_0(\|u_0\|_{H^{\frac{1}{2}+}})\leq T$ and a solution u to (1.2) in $C([0,T_0];H^s)\cap L^2_{[b]}(0,T_0;H^{s+1})$. This solution is the unique weak solution of (1.2) that belongs respectively to $L^\infty(0,T_0;H^s)\cap L^2_{[b]}(0,T_0;H^{s+1})$ and $L^\infty(0,T_0;H^s)$ in respectively the cases $b\not\equiv 0$ and $b\equiv 0$. Moreover, for any $R>0$ the solution-map $u_0\mapsto u$ is continuous from the ball of $H^s(\mathbb{R})$ centered at the origin with radius R into $C([0,T_0(R)];H^s)$.

Remark 1.2. *$L^2_{[b]}(0,T_0;H^{s+1})$ is defined in Subsection 2.2.*

Remark 1.3. *The hypotheses on the coefficients b, c, d, e and f given in the above statement are not optimal. More accurate hypotheses on the coefficients b, c, d, e and f involving norms in Zygmund spaces can be found in Remark 4.1.*

By a suitable change of unknown we will be able to link the solutions of (1.1) to the ones of (1.2). As a consequence of the above theorem we then get the following result for (1.1).

th1 **Theorem 1.2.** *Let $s > \frac{1}{2}$, $T \in]0, +\infty]$ and assume that $\alpha \in L^\infty(]0, T[; C_b^{[s]+4}(\mathbb{R}))$ with $\alpha_t \in L^\infty(]0, T[; C_b^{[s]+1}(\mathbb{R}))$ β, γ, ϵ in $L^\infty(]0, T[; C_b^{[s]+2}(\mathbb{R}))$ with ϵ_t in $L^\infty(]0, T[\times\mathbb{R})$ and $\delta \in L^\infty(]0, T[; C_b^{[s]+1}(\mathbb{R}))$. Assume moreover that*

1. *There exists $\alpha_0 > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$,*
$$\alpha_0 \le \alpha(t, x) \le \alpha_0^{-1} \; .$$
2. $$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \Big| \int_0^x (\alpha^{-4/3}\alpha_t)(t, y) dy \Big| < \infty \; .$$
3. *β can be decomposed as $\beta = \beta_1 + \beta_2$ with $\beta_2 \le 0$, $\beta_1, \beta_2 \in L^\infty(]0, T[; C_b^{[s]+2})$ such that*
$$(t, x) \mapsto \int_0^x (\alpha^{-1}\beta_1)(t, y)\, dy \in W^{1,\infty}([0, T]; L^\infty(\mathbb{R})) \; .$$

We set $g(t, x) = -\beta_2(t, x)\alpha^{1/3}(t, A(x))$ Then for all $u_0 \in H^s(\mathbb{R})$, there exist a time $0 < T_0 = T_0(\|u_0\|_{H^{\frac{1}{2}+}}) \le T$ and a solution u to (1.1) in $C([0, T_0]; H^s) \cap L^2_{[g]}(0, T_0; H^{s+1})$. This solution is the unique weak solution of (1.1) that belongs to $L^\infty(0, T_0; H^s) \cap L^2_{[g]}(0, T_0; H^{s+1})$. For any $R > 0$ the solution-map $u_0 \mapsto u$ is continuous from the ball of $H^s(\mathbb{R})$ centered at the origin with radius R into $C([0, T_0(R)]; H^s)$.

Remark 1.4. *It is worth noticing that point 3. of the above theorem is satisfies if there exists $R > 0$ such that*
$$\beta \le 0 \quad on \quad [0, T_0] \times (\mathbb{R} \setminus [-R, R]) \; .$$
Indeed, we can then decompose β as $\beta = \beta_1 + \beta_2$ with $\beta_1 \equiv 0$ on $\mathbb{R} \setminus [-R_0, R_0[$ with $R_0 > R$, that clearly satisfies point 3. This means that, when the anti-dissipation is confined in a fixed compact set for all $t \in [0, T]$, the Cauchy problem associated to (1.1) is locally well-posed in the Hadamard sense in H^s.

Remark 1.5. *If Hypothesis 3. in Theorem 1.2 holds with $\beta_1 = \beta$ (i.e. $\beta_2 = 0$) then the change of unknown does link the solution to (1.1) to a solution of (1.2) with $b \equiv 0$ on $\mathbb{R}$. Therefore, on account of Theorem 1.1, we obtain that in this case (1.1) is actually unconditionally locally well-posed in $H^s(\mathbb{R})$.*

The rest of this paper is organized as follows. In the next section we introduce some notations, define our resolution spaces and recall some technical lemmas that will be used in Section 4 to prove estimates on solutions to (1.1). Note that the proof of some of these lemmas are postponed to the appendix. In Section 3 we establish the links between the problems (1.1) and (1.2) that enables us to prove Theorem 1.2 assuming Theorem 1.1. Finally, Sections 4 and 5 are devoted to the proof of Theorem 1.1.

2. Notations, function spaces and technical lemmas

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2.1. Notations. For any $s \in \mathbb{R}$, we denote $[s]$ the integer part of s. For $\alpha \in \mathbb{R}$, α_+, respectively α_-, will denote a number slightly greater, respectively lesser, than α.

For $(a,b) \in (\mathbb{R}_+)^2$, We denote by respectively $a \vee b$ and $a \wedge b$ the maximum and the minimum of a and b.

We denote by $C(\lambda_1, \lambda_2, \dots)$ a nonnegative constant depending on the parameters λ_1, λ_2,... and whose dependence on the λ_j is always assumed to be nondecreasing.

Let p be any constant with $1 \le p < \infty$ and denote $L^p = L^p(\mathbb{R})$ the space of all Lebesgue-measurable functions f with the standard norm

$$\|f\|_{L^p} = \Big(\int_{\mathbb{R}} |f(x)|^p dx \Big)^{1/p} < \infty.$$

The real inner product of any two functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R})$ is denoted by

$$(f_1, f_2) = \int_{\mathbb{R}} f_1(x) f_2(x) dx.$$

The space $L^\infty = L^\infty(\mathbb{R})$ consists of all essentially bounded and Lebesgue-measurable functions f with the norm

$$\|f\|_{L^\infty} = \sup |f(x)| < \infty.$$

We denote by $W^{1,\infty}(\mathbb{R}) = \{f \in \mathcal{D}'(\mathbb{R}), \text{ s.t. } f, \partial_x f \in L^\infty(\mathbb{R})\}$ endowed with its canonical norm.

For any real constant $s \ge 0$, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space of all tempered distributions f with the norm $\|f\|_{H^s} = \|\Lambda^s f\|_{L^2} < \infty$, where Λ is the pseudo-differential operator $\Lambda = (1 - \partial_x^2)^{1/2}$.

For any two functions $u = u(t,x)$ and $v(t,x)$ defined on $[0,T) \times \mathbb{R}$ with $T > 0$, we denote the H^s inner product, the L^p-norm and especially the L^2-norm, as well as the Sobolev norm, with respect to the spatial variable x, by $(u,v) = (u(t,\cdot), v(t,\cdot))_{H^s}$, $\|u\|_{L^p} = \|u(t,\cdot)\|_{L^p}$, $\|u\|_{L^2} = \|u(t,\cdot)\|_{L^2}$, and $\|u\|_{H^s} = \|u(t,\cdot)\|_{H^s}$, respectively.

We denote $L^\infty([0,T); H^s(\mathbb{R}))$ the space of functions such that $u(t,\cdot)$ is controlled in H^s, uniformly for $t \in [0,T)$: $\|u\|_{L^\infty([0,T);H^s(\mathbb{R}))} = \sup_{t\in[0,T)} |u(t,\cdot)|_{H^s} < \infty$.

Finally, $C^k(\mathbb{R})$ denotes the space of k-times continuously differentiable functions.

Throughout the paper, we fix a smooth even bump function η such that

$$\eta \in C_0^\infty(\mathbb{R}), \quad 0 \le \eta \le 1, \quad \eta_{|[-1,1]} = 1 \quad \text{and} \quad \text{supp}(\eta) \subset [-2,2]. \tag{2.1}$$ defeta

We set $\phi(\xi) := \eta(\xi) - \eta(2\xi)$. For $l \in \mathbb{N} \setminus \{0\}$, we define

$$\phi_{2^l}(\xi) := \phi(2^{-l}\xi) \quad \text{and} \quad \psi_{2^l}(\xi,\tau) = \phi_{2^l}(\tau - \xi^3)\ .$$

By convention, we also denote

$$\phi_1(\xi) := \eta(\xi) \quad \text{and} \quad \psi_1(\xi,\tau) := \eta(\tau - \xi^3).$$

Any summations over capitalized variables such as N, L, K or M are presumed to be dyadic. Unless stated otherwise, we work with non-homogeneous decompositions for space, time and modulation variables, i.e. these variables range over numbers of the form $\{2^k : k \in \mathbb{N}\}$ respectively. Then, we have that

$$\sum_{N \ge 1} \phi_N(\xi) = 1 \quad \forall \xi \in \mathbb{R}, \quad \text{supp}\,(\phi_N) \subset \{\frac{N}{2} \le |\xi| \le 2N\},\ N \in \{2^k : k \in \mathbb{N} \setminus \{0\}\},$$

and

$$\sum_{L \ge 1} \psi_L(\xi,\tau) = 1 \quad \forall (\xi,\tau) \in \mathbb{R}^2, \quad L \in \{2^k : k \in \mathbb{N}\}.$$

Let us now define the following Littlewood-Paley multipliers :

$$P_N u = \mathcal{F}_x^{-1}\big(\phi_N \mathcal{F}_x u\big), \quad Q_L u = \mathcal{F}^{-1}\big(\psi_L \mathcal{F} u\big), \quad R_K u = \mathcal{F}_t^{-1}\big(\phi_K \mathcal{F}_t u\big) \ . \tag{2.2}$$ project

We then set

$$\tilde{P}_N := \sum_{N/4\le K\le 4N} P_K, \quad P_{\ge N} := \sum_{K\ge N} P_K, \quad P_{\le N} := \sum_{1\le K\le N} P_K, \quad P_{\ll N} := \sum_{1\le K\ll N} P_K,$$

$$P_{\gtrsim N} := \sum_{K\gtrsim N} P_K, \quad Q_{\ge L} := \sum_{K\ge L} Q_K, \quad Q_{\le L} := \sum_{1\le K\le L} Q_K \text{ and } Q_{\sim L} := \sum_{K\sim N} Q_K \ .$$

For brevity we also write $u_N = P_N u$, $u_{\le N} = P_{\le N} u$, $u_{\ge N} = P_{\ge N} u$, $u_{\ll N} = P_{\ll N} u$ and $u_{\gtrsim N} = P_{\gtrsim N} u$.

Following [10], to handle coefficient that are not asymptotically flat we will use the classical Zygmund spaces : for $s \in \mathbb{R}$, $C^s_*(\mathbb{R})$ is the set of all $v \in \mathcal{S}'(\mathbb{R})$ such that

$$\|v\|_{C^s_*} := \sup_{N\ge 1} N^s \|P_N v\|_{L^\infty} < \infty \ . \tag{2.3}$$ defZyg

Note that, for all $k \in \mathbb{N}$,

$$C^{k+}_*(\mathbb{R}) \hookrightarrow W^{k,\infty}(\mathbb{R}) \hookrightarrow C^k_*(\mathbb{R}) \ .$$

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2.2. **Function Spaces.** Let $T > 0$, $b \in L^\infty(]0,T[\times\mathbb{R})$ with $b \ge 0$ and $\theta > -1/2$. We define the sub vector space $L^2_{[b]}(]0,T[;H^{\theta+1})$ of $L^\infty(0,T;L^2(\mathbb{R}))$ as

$$L^2_{[b]}(]0,T[;H^{\theta+1}) = \Big\{ u \in L^\infty(0,T;L^2(\mathbb{R})), \quad \|u\|_{L^2_{[b]}(]0,T[;H^{\theta+1})} < +\infty \| \Big\}$$

with

$$\|u\|^2_{L^2_{[b]}(]0,T[;H^{\theta+1})} = \sum_{N>0} \langle N\rangle^{2\theta} \Big\| \sqrt{b}\, P_N u_x \Big\|^2_{L^2_T L^2_x} \tag{2.4}$$ defL2b

For s, $\theta \in \mathbb{R}$, we introduce the Bourgain spaces $X^{s,\theta}$ related to the linear KdV equation as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|v\|_{X^{s,\theta}} := \left(\int_{\mathbb{R}^2} \langle \tau - \xi^3 \rangle^{2\theta} \langle \xi \rangle^{2s} |\widehat{v}(\xi,\tau)|^2 d\xi d\tau \right)^{\frac{1}{2}}, \tag{2.5}$$ Bourgain

where $\langle x \rangle := 1 + |x|$. Recall that

$$\|v\|_{X^{s,\theta}} = \|U(-t)v\|_{H^{s,\theta}_{x,t}}$$

where $U(t) = \exp(-t\partial_x^3)$ is the generator of the free evolution associated with the linear KdV equation and where $\|\cdot\|_{H^{s,\theta}_{x,t}}$ is the usual space-time Sobolev norm given by

$$\|u\|_{H^{s,\theta}_{x,t}} := \left(\int_{\mathbb{R}^2} \langle \tau \rangle^{2\theta} \langle \xi \rangle^{2s} |\widehat{u}(\xi,\tau)|^2 d\xi d\tau \right)^{\frac{1}{2}} \ .$$

We define the function space Y^s by $Y^s = L^\infty_t H^s_x \cap X^{s-1,1}$ equipped with its natural norm

$$\|u\|_{Y^s} = \|u\|_{L^\infty_t H^s_x} + \|u\|_{X^{s-1,1}} \ . \tag{2.6}$$ defZs

Finally, we will use restriction in time versions of these spaces. Let $T > 0$ be a positive time and Y be a normed space of space-time functions. The restriction space Y_T will be the space of functions $v : \mathbb{R}\times]0,T[\to \mathbb{R}$ satisfying

$$\|v\|_{Y_T} := \inf\{ \|\tilde{v}\|_Y \mid \tilde{v} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}, \ \tilde{v}|_{\mathbb{R}\times]0,T[} = v \} < \infty \ .$$

2.3. **Technical Lemmas.** We first recall the following technical lemmas that were proven in [11].

continuiteQ **Lemma 2.1.** *Let $L \geq 1$, $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. The operator $Q_{\leq L}$ is bounded in $L^p_t H^s$ uniformly in $L \geq 1$.*

For any $T > 0$, we consider 1_T the characteristic function of $[0, T]$ and use the decomposition

$$1_T = 1^{low}_{T,R} + 1^{high}_{T,R}, \quad \widehat{1^{low}_{T,R}}(\tau) = \eta(\tau/R)\widehat{1_T}(\tau) \tag{2.7}$$ ind-dec

for some $R > 0$.

ihigh-lem **Lemma 2.2.** *For any $R > 0$ and $T > 0$ it holds*

$$\|1^{high}_{T,R}\|_{L^1} \lesssim T \wedge R^{-1}. \tag{2.8}$$ high

and, for any $p \in [1, +\infty]$,

$$\|1^{low}_{T,R}\|_{L^p} + \|1^{high}_{T,R}\|_{L^p} \lesssim T^{1/p} \tag{2.9}$$ low

ilow-lem **Lemma 2.3.** *Let $u \in L^2(\mathbb{R}^2)$. Then for any $T > 0$, $R > 0$ and $L \gg R$ it holds*

$$\|Q_L(1^{low}_{T,R} u)\|_{L^2} \lesssim \|Q_{\sim L} u\|_{L^2}$$

We will need product estimates in Sobolev spaces for functions in Sobolev and in Zygmund spaces (see [4] for (2.10) and [10] for (2.12). The proof of (2.11) follows exactly the same lines as the one of (2.10)).

product **Lemma 2.4.** *1. Let $(t, s, r) \in \mathbb{R}^3$ with $s + r > t + 1/2$, $s + r > 0$ and $s, r \geq t$. Then for any $f \in H^s(\mathbb{R})$ and $g \in H^r(\mathbb{R})$, it holds $fg \in H^t(\mathbb{R})$ with*

$$\|fg\|_{H^t} \lesssim \|f\|_{H^s}\|g\|_{H^r} \tag{2.10}$$ estsobo

2. Let $(t, s, r) \in \mathbb{R}^3$ with $s + r > t$, $s + r > 0$ and $s, r \geq t$. Then for any $f \in C^s_*(\mathbb{R})$ and $g \in H^r(\mathbb{R})$, it holds $fg \in H^t(\mathbb{R})$ with

$$\|fg\|_{H^t} \lesssim \|f\|_{C^s_*}\|g\|_{H^r} \tag{2.11}$$ estC

In particular, let $s \in \mathbb{R}$, then for any $f \in C^{|s|+}_*(\mathbb{R})$ and $g \in H^s(\mathbb{R})$, it holds $fg \in H^s(\mathbb{R})$ with

$$\|fg\|_{H^s} \lesssim \|f\|_{C^{|s|+}_*}\|g\|_{H^s} \tag{2.12}$$ estCs

We will also need the following lemma on commutator and double commutator estimates (see ([[10], p. 288] the remark in the footnote for (2.13)) that we prove in the Appendix.

commutator **Lemma 2.5.** *Let $f \in L^\infty(\mathbb{R})$ and $g \in L^2(\mathbb{R})$. For any $N > 0$ it holds*

$$\|[P_N, P_{\ll N} f]g\|_{L^2_x} \lesssim N^{-1}\|P_{\ll N} f_x\|_{L^\infty_x}\|\tilde{P}_N g\|_{L^2_x} \tag{2.13}$$ commu

and

$$\Big\|\Big[P_N, [P_N, P_{\ll N} f]\Big]g\Big\|_{L^2_x} \lesssim N^{-2}\|P_{\ll N} f_{xx}\|_{L^\infty_x}\|\tilde{P}_N g\|_{L^2_x} \tag{2.14}$$ commu2

Moreover, it holds

$$\int_{\mathbb{R}} [P_N, P_{\ll N} f]g \, P_N g = \frac{1}{2}\int_{\mathbb{R}} \Big[P_N, [P_N, P_{\ll N} f]\Big]\tilde{P}_N g \, \tilde{P}_N g \tag{2.15}$$ comcom

Finally we construct a bounded linear operator from $X^{s-1,1}_T \cap L^\infty_T H^s_x$ into Y^s with a bound that does not depend on s and T . For this we follow [12] and introduce the extension operator ρ_T defined by

$$\rho_T(u)(t) := U(t)\eta(t)U(-\mu_T(t))u(\mu_T(t)) , \tag{2.16}$$ defrho

where η is the smooth cut-off function defined in Section 2.1 and μ_T is the continuous piecewise affine function defined by

$$\mu_T(t) = \begin{cases} 0 & \text{for} \quad t \notin]0, 2T[\\ t & \text{for} \quad t \in [0, T] \\ 2T - t & \text{for} \quad t \in [T, 2T] \end{cases} \tag{2.17}$$ defext

extension **Lemma 2.6.** *Let $0 < T \le 2$ and $s \in \mathbb{R}$. Then,*

$$\begin{aligned} \rho_T : X_T^{s-1,1} \cap L_T^\infty H_x^s &\longrightarrow Y^s \\ u &\mapsto \rho_T(u) \end{aligned}$$

is a bounded linear operator, i.e.

$$\|\rho_T(u)\|_{L_t^\infty H_x^s} + \|\rho_T(u)\|_{X^{s-1,1}} \lesssim \|u\|_{L_T^\infty H_x^s} + \|u\|_{X_T^{s-1,1}} , \tag{2.18}$$ extension.1

for all $u \in X_T^{s-1} \cap L_T^\infty H_x^s$.

Moreover, the implicit constant in (2.18) can be chosen independent of $0 < T \le 2$ and $s \in \mathbb{R}$.

sect3 3. Transformation of the problem and proof of Theorem 1.2.

3.1. **Link between solutions of (1.1) and (1.2).** The main assumption on the coefficient of the third order term is that it is bounded from above and from below by positive constants. Of course, we can also treat the case of a negative coefficient by making the trivial change of unknwon $\tilde{u}(t,x) = u(t,-x)$ but this will also change the sens of the real axis. This would play no role in Theorem 1.2 but would change the assumption $\sup_{(t,x)\in[0,T]\times\mathbb{R}} -\int_0^x \frac{\beta_1}{\alpha}(t,y)dy < \infty$ by $\sup_{(t,x)\in[0,T]\times\mathbb{R}} \int_0^x \frac{\beta_1}{\alpha}(t,y)dy < \infty$ in Theorem 3.1 below.

hyp1 **Hypothesis 1.** *There exists $\alpha_0 > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}$,*

$$\alpha_0 \le \alpha(t,x) \le \alpha_0^{-1} .$$

prop31 **Proposition 3.1.** *Assume that Hypothesis 1 is satisfied and that $\alpha \in L^\infty(]0,T[; C_b^3(\mathbb{R}))$ with $\alpha_t \in L^\infty(]0,T[; C_b(\mathbb{R}))$ and $\beta \in L^\infty(]0,T[; C_b^2(\mathbb{R}))$. Let $A \in L^\infty(]0,T[; C_b^4(\mathbb{R}))$ with $A_t \in L^\infty(]0,T[; C_b^1(\mathbb{R}))$ be defined for $(t,x) \in [0,T] \times \mathbb{R}$ by*

$$A(t,x) = \int_0^x \alpha^{-1/3}(t,y)\, dy \tag{3.1}$$ defA

and let $h > 0$ with $h \in L^\infty(]0,T[; C_b^3(\mathbb{R}))$ with $h_t \in L^\infty(]0,T[; C_b(\mathbb{R}))$. For each $t \in [0,T]$ we denote by $A^{-1}(t,\cdot)$ the increasing reciproqual bijection of $A(t,\cdot)$.

Then $u \in L_T^\infty L_x^2$ is a weak solution to (1.1) if and only if

$$(t,x) \mapsto v(t,x) = h(t, A^{-1}(t,x))\, u(t, A^{-1}(t,x))$$

is a weak solution to (1.2) with

$$\begin{cases} b(t,x) & = \alpha^{1/3}\Big(-\beta\alpha^{-1} + \alpha_x\alpha^{-1} + 3h^{-1}h_x\Big) \\ c(t,x) & = A_t + \alpha^{-1/3}\Big(6h_x^2h^{-2}\alpha + \frac{4}{9}\alpha_x^2\alpha^{-1} + \alpha_x h_x h^{-1} - 3h_{2x}h^{-1}\alpha - \frac{1}{3}\alpha_{2x} \\ & \quad -2h_xh^{-1}\beta - \frac{1}{3}\alpha^{-1}\alpha_x\beta + \gamma\Big) \\ d(t,x) & = \alpha\Big(-6h_x^3h^{-3} + 6h_{2x}h^{-2}h_x - h_{3x}h^{-1}\Big) + \beta\Big(2h_x^2h^{-2} - h_{2x}h^{-1}\Big) \\ & \quad -\gamma h_xh^{-1} - h_th^{-1} + \delta \\ e(t,x) & = \epsilon\alpha^{-1/3}h^{-1} \qquad \text{and } f(t,x) = -\epsilon h_x h^{-2} . \end{cases} \tag{3.2}$$ defb

where all the functions in the right-hand side are evaluated at $(t, A^{-1}(t,x))$.

Proof. Since $\alpha \geq \alpha_0 > 0$ on $[0,T]\times\mathbb{R}$, for each $t\in[0,T]$, $A(t,\cdot)$ is an increasing bijection of $\mathbb{R}$ with no critical point and thus its reciprocal bijection $A^{-1}(t,\cdot)$ is well-defined and belong to the same C^n-space. Therefore, since $\alpha \in L^\infty(]0,T[;C^3_b(\mathbb{R}))$ with $\alpha_t \in L^\infty(]0,T[;C_b(\mathbb{R}))$, it is clear that A and A^{-1} belong to $L^\infty(]0,T[;C^4_b(\mathbb{R}))\cap W^{1,\infty}([0,T];C^1_b(\mathbb{R}))$

We first assume that $u\in C([0,T];H^\infty)$ with $u_t \in L^\infty(]0,T[;H^\infty)$ and we set

$$V(t,x) = h(t,A^{-1}(t,x))\, u(t,A^{-1}(t,x)) \tag{3.3}$$ defV

so that

$$u(t,x) = \frac{V(t,A(t,x))}{h(t,x)}$$

In the calculus below the functions u, h ,α, β, γ, δ ϵ will be evaluated at (t,x) whereas V is evaluated at $(t,A(t,x))$. Then it holds

$$\begin{aligned}
u_t(t,x) =& - h_t h^{-2} V + h^{-1} V_t + A_t h^{-1} V_x \\
u_x(t,x) =& - \frac{h_x}{h^2} V + \frac{\alpha^{-1/3}}{h} V_x \\
u_{2x}(t,x) =& \alpha^{-2/3} h^{-1} V_{2x} - \Big(\frac{h^{-1}}{3}\alpha^{-4/3}\alpha_x + 2h_x h^{-2}\alpha^{-1/3}\Big) V_x \\
&+ \Big(2h_x^2 h^{-3} - h_{2x} h^{-2}\Big) V \\
u_{3x}(t,x) =& \alpha^{-1} h^{-1} V_{3x} + V_{2x}\Big(-h^{-1}\alpha^{-5/3}\alpha_x - 3h_x h^{-2}\alpha^{-2/3}\Big) \\
&+ V_x\Big(h_x h^{-2}\alpha^{-4/3}\alpha_x + \frac{4}{9} h^{-1}\alpha^{-7/3}\alpha_x^2 - \frac{1}{3} h^{-1}\alpha^{-4/3}\alpha_{2x} \\
&\qquad - 3h_{2x} h^{-2}\alpha^{-1/3} + 6h_x^2 h^{-3}\alpha^{-1/3}\Big) \\
&+ V\Big(6h_{2x} h_x h^{-3} - 6h_x^3 h^{-4} - h_{3x} h^{-2}\Big) \\
(uu_x)(t,x) =& - h^{-3} h_x V^2 + \alpha^{-1/3} h^{-2} V V_x\ .
\end{aligned}$$

Gathering the above identity we thus obtain

$$\begin{aligned}
&h(t,x)\Big(u_t + \alpha u_{3x} + \beta u_{2x} + \gamma u_x + \delta u - \epsilon u u_x\Big)(t,x) \\
&\quad = [V_t + V_{3x} - bV_{2x} + cV_x + dV - eVV_x - fV^2](t,A(t,x))
\end{aligned} \tag{3.4}$$

with b,c,d,e given by (3.2).

Therefore for $\phi \in L^\infty(]0,T[;C^3_b(\mathbb{R}))$ with $\phi_t \in L^\infty(]0,T[;C_b(\mathbb{R}))$ and compact support in $[0,T[\times\mathbb{R}$, making use at any fixed $t\in[0,T]$ of the change of variable $y=A^{-1}(t,x)$ and noticing that $A^{-1}_x(t,x) = \alpha^{1/3}(t,A^{-1}(t,x))$ we observe that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \Big(u_t + \alpha u_{3x} + \beta u_{2x} + \gamma u_x + \delta u - \epsilon u u_x\Big)(t,y)\phi(t,y)\, dy \\
&\quad = \int_0^T \int_{\mathbb{R}} h\Big(u_t + \alpha u_{3x} + \beta u_{2x} + \gamma u_x + \delta u - \epsilon u u_x\Big)(t,y)\frac{\phi}{h}(t,y)\, dy \\
&= \int_0^T \int_{\mathbb{R}} \Big[h\Big(u_t + \alpha u_{3x} + \beta u_{2x} + \gamma u_x + \delta u - \epsilon u u_x\Big)\frac{\phi}{h}\Big](t,A^{-1}(t,x))\ \alpha^{1/3}(t,A^{-1}(t,x))\, dx\, dt \\
&\quad = \int_0^T \int_{\mathbb{R}} \Big(V_t + V_{3x} - bV_{2x} + cV_x + dV - eVV_x - fV^2\Big)(t,x)\psi(t,x)\, dx\, dt \\
&\quad = \int_0^T \int_{\mathbb{R}} V\Big[-\psi_t - \psi_{3x} - \partial_x^2(b\psi) - \partial_x(c\psi) + d\psi\Big] + V^2\Big[\frac{1}{2}\partial_x(e\psi) + f\Big]\, dx\, dt \\
&\qquad + \int_{\mathbb{R}} V(0,x)\psi(0,x)\, dx
\end{aligned} \tag{3.5}$$ weak11

with $\psi(t,x) = \dfrac{\alpha^{1/3}\,\phi}{h}(t, A^{-1}(t,x))$.

Now let $u \in L^\infty_T L^2_x$ be a weak solution to (1.1). Recall that by Remark 1.1, $u_t \in L^\infty_T H^{-3}_x$. Then by using mollifiers we can approximate u in $L^\infty_T L^2_x$ by $u_n \in C([0,T] : H^\infty)$ with $u_t \in L^\infty(]0,T[; H^\infty)$ such that $u_n(0) \to u_0$ in $L^2(\mathbb{R})$ and $u_n \to u \in L^\infty_T L^2_x$. Note that by defining V_n in the same way as V in (3.3) we also have $V_n(0) \to V_0$ in $L^2(\mathbb{R})$ and $V_n \to V \in L^\infty_T L^2_x$. Making use of (3.5) and that u is a weak solution to (1.1) we thus get

$$
\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}} \Big(u\Big[-\phi_t - \partial_x^3(\alpha\phi) + \partial_x^2(\beta\phi) - \partial_x(\gamma\phi) + \delta\phi\Big] + \frac{1}{2}u^2 \partial_x(\epsilon\phi)\Big)(t,x)\,dx\,dt \\
&\qquad\qquad + \int_{\mathbb{R}} u_0(x)\phi(0,x)\,dx \\
&= \lim_{n\to+\infty} \int_0^T \int_{\mathbb{R}} \Big(u_n\Big[-\phi_t - \partial_x^3(\alpha\phi) + \partial_x^2(\beta\phi) - \partial_x(\gamma\phi) + \delta\phi\Big] + \frac{1}{2}u_n^2 \partial_x(\epsilon\phi)\Big)(t,x)\,dx\,dt \\
&\qquad\qquad + \int_{\mathbb{R}} u_n(0,x)\phi(0,x)\,dx \\
&= \lim_{n\to+\infty} \int_0^T \int_{\mathbb{R}} \Big(u_{n,t} + \alpha u_{n,3x} + \beta u_{n,2x} + \gamma u_{n,x} + \delta u_n - \epsilon u_n u_{n,x}\Big)(t,x)\phi(t,x)\,dx\,dt \\
&= \lim_{n\to+\infty} \int_0^T \int_{\mathbb{R}} V_n\Big[-\psi_t - \psi_{3x} - \partial_x^2(b\psi) - \partial_x(c\psi) + d\psi\Big] + V_n^2\Big[\frac{1}{2}\partial_x(e\psi) + f\Big]\,dx\,dt \\
&\qquad\qquad + \int_{\mathbb{R}} V_n(0,x)\psi(0,x)\,dx \\
&= \int_0^T \int_{\mathbb{R}} V\Big[-\psi_t - \psi_{3x} - \partial_x^2(b\psi) - \partial_x(c\psi) + d\psi\Big] + V^2\Big[\frac{1}{2}\partial_x(e\psi) + f\Big]\,dx\,dt \\
&\qquad\qquad + \int_{\mathbb{R}} V(0,x)\psi(0,x)\,dx \qquad (3.6)
\end{aligned}
$$

that proves that u is a weak solution to (1.1) if and only if :
$(t,x) \mapsto V(t,x) = h(t, A^{-1}(t,x))u(t, A^{-1}(t,x))$ is a weak solution to (1.2). Indeed since $\alpha, h \in L^\infty(]0,T[; C^3_b(\mathbb{R}))$, $\alpha_t, h_t \in L^\infty(]0,T[; C_b(\mathbb{R}))$ with $h > 0$ and $\alpha \ge \alpha_0 > 0$, the map

$$\Theta \quad : \quad \phi \mapsto \Big(\frac{\phi\,\alpha^{1/3}}{h}\Big)(t, A^{-1}(t,x))$$

is a bijection from the space of functions in $L^\infty(]0,T[; C^3_b(\mathbb{R}))$ with time derivative in $L^\infty(]0,T[; C_b(\mathbb{R}))$ and compact support in $[0,T[\times\mathbb{R}$ into itself. The reciprocal bijection is given by

$$\Theta^{-1} \quad : \quad \psi \mapsto \Big(\frac{\psi h}{\alpha^{1/3}}\Big)(t, A(t,x))\ .$$

(1.3) is thus satisfied by all $\psi \in L^\infty(]0,T[; C^3_b(\mathbb{R}))$ with $\psi_t \in L^\infty(]0,T[; C_b(\mathbb{R}))$ and compact support in $[0,T[\times\mathbb{R}$ that leads to the desired result. □

3.2. **Proof of Theorem 1.2 assuming Theorem 1.1.** We want to choose h such that $b \ge 0$. For this we decompose $\beta(\cdot,\cdot)$ as $\beta_1 + \beta_2$ with β_1 and β_2 bounded and $\beta_2 \le 0$ (Note that we can always take $\beta_1 = \beta$ and $\beta_2 = 0$). According to (3.2) it suffices to take h that satisfies

$$\frac{h_x}{h} = \frac{1}{3}(\beta_1\alpha^{-1} - \alpha_x\alpha^{-1}) \qquad (3.7)$$

abiir

so that

$$b = -\beta\alpha^{-\frac{2}{3}} + \alpha_x\alpha^{-\frac{2}{3}} + 3\frac{h_x}{h}\alpha^{1/3} = -\beta_2\alpha^{-\frac{2}{3}} \ge 0.$$

Equation (3.7) is satisfied for

$$h(t,x) = \Big[\frac{\alpha(t,0)}{\alpha(t,x)}\Big]^{1/3} \exp\Big(\frac{1}{3}\int_0^x (\beta_1\alpha^{-1})(t,y)\,dy\Big)\ . \qquad (3.8)$$

defh

For this choice of h we need the coefficients b, c, d, e, f to be bounded to solve the equation with the help of Theorem 1.1. First we notice that the coefficient c contains A_t. The requirement that A_t is bounded leads to the following hypothesis.

hyp2 **Hypothesis 2.**

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}}\Big|\int_0^x(\alpha^{-4/3}\alpha_t)(t,y)dy\Big|<\infty\ .$$

Now, since $\alpha\geq\alpha_0$ one can check that all the terms $\frac{h_x}{h}$, $\frac{h_{2x}}{h}$ that appear in c and d are bounded. On the other hand the boundedness of $h_t h^{-1}$ that appears in the coefficient d requires a new hypothesis. Moreover, in the coefficient e and f of the nonlinear part, h^{-1} appears alone. To force $h_t h^{-1}$, e and f to be bounded we thus add the following hypothesis that ensures in particular that there exists $h_0>0$ such that for $(t,x)\in[0,T_0]\times\mathbb{R}$, $h(t,x)\geq h_0$.

hyp3 **Hypothesis 3.** *β can be decomposed as $\beta=\beta_1+\beta_2$ with $\beta_2\leq 0$, $\beta_1,\beta_2\in L^\infty([0,T];C_b^2)$, $\partial_t\beta_1\in L^\infty(]0,T[;L^\infty)$ such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}}\Big|\int_0^x\partial_t(\alpha^{-1}\beta_1)(t,y)dy\Big|<\infty\ .$$

and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}}-\int_0^x\frac{\beta_1}{\alpha}(t,y)dy<\infty\ .$$

Now, according to Theorem 1.1, for $s>1/2$, (1.2) is locally well-posed in $H^s(\mathbb{R})$, whenever $b\geq 0$ on $[0,T]\times\mathbb{R}$ with b,c,e in $L^\infty(0,T;C_b^{[s]+2}(\mathbb{R}))$, e_t in $L^\infty(]0,T[\times\mathbb{R})$ and $d,f\in L^\infty(]0,T[;C_b^{[s]+1}(\mathbb{R}))$.

In view of (3.2), (3.8) and Hypotheses 1-3, one can easily check that the function spaces to which $\alpha,\beta,\gamma,\delta,\epsilon$ and β_1,β_2 belong in the statement of Theorem 1.2 ensure that b,c,e,d and f belong to the above function spaces. Moreover, this ensures that $u\in C([0,T_0];H^s)$ if and only if $V(t,x)=h(t,A^{-1}(t,x))\,u(t,A^{-1}(t,x))$ belongs also to this space. Therefore, gathering Theorem 1.1 and Proposition 3.1 leads to the existence of a solution to (1.1) with uniqueness in the space of functions u such that $hu\in L^\infty(0,T_0;H^s)$. More precisely, we can state the following slightly less restrictive version of Theorem 1.2.

th3 **Theorem 3.1.** *Let $s>1/2$ and $T\in]0,+\infty]$ and assume that $\alpha\in L^\infty(]0,T[;C_b^{[s]+4}(\mathbb{R}))$ with $\alpha_t\in L^\infty(]0,T[;C_b^{[s]+1}(\mathbb{R}))$ β,γ,ϵ in $L^\infty(]0,T[;C_b^{[s]+2}(\mathbb{R}))$ with ϵ_t in $L^\infty(]0,T[\times\mathbb{R})$ and $\delta\in L^\infty(]0,T[;C_b^{[s]+1}(\mathbb{R}))$. Assume moreover that*

- *There exists $\alpha_0>0$ such that for all $(t,x)\in[0,T]\times\mathbb{R}$,*

$$\alpha_0\leq\alpha(t,x)\leq\alpha_0^{-1}\ .$$

-

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}}\Big|\int_0^x\partial_t(\alpha^{-1/3})(t,y)dy\Big|<\infty\ .$$

- *β can be decomposed as $\beta=\beta_1+\beta_2$ with $\beta_2\leq 0$, $\beta_1,\beta_2\in L^\infty(]0,T[;C_b^{[s]+2})$ such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}}\Big|\int_0^x\partial_t(\alpha^{-1}\beta_1)(t,y)dy\Big|<\infty\ .$$

and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}}-\int_0^x\frac{\beta_1}{\alpha}(t,y)dy<\infty\ .$$

We set

$$h(t,x)=\Big[\frac{\alpha(t,0)}{\alpha(t,x)}\Big]^{1/3}\exp\Big(\frac{1}{3}\int_0^x \beta_1\alpha^{-1}\Big) \text{ and } \quad g(t,x)=-\beta_2(t,x)\alpha^{1/3}(t,A(x)) \; .$$

Then for all $u_0\in H^s(\mathbb{R})$, there exist a time $0<T_0=T_0(\|u_0\|_{H^{\frac{1}{2}+}})\le T$ and a solution u to (1.3) in $C([0,T_0];H^s)\cap L^2_{[g]}(0,T_0;H^{s+1})$. This solution is the unique weak solution of (1.1) *such that hu belongs to $L^\infty(0,T_0;H^s)\cap L^2_{[g]}(0,T_0;H^{s+1})$.*

remark31 **Remark 3.1.** *It is worth noticing that we can always choose (β_1,β_2) such that the hypothesis of integrability on $\beta_1\alpha^{-1}$ in the above theorem is satisfied in $+\infty$. Indeed, β being bounded by hypothesis, taking β_2 such that $\beta_2=-\sup_{\mathbb{R}}|\beta|$ on $\mathbb{R}_+$ it follows that $\beta_1=\beta-\beta_2\ge 0$ on $\mathbb{R}_+$ and thus $\int_0^x\frac{\beta_1}{\alpha}(t,y)dy\ge 0$ for any $x\in\mathbb{R}_+$. That means that this existence and uniqueness result works with a uniform anti-diffusion in the neighborhood of $+\infty$. For instance a coefficient β such that $\beta\ge 1$ on $[0,T]\times\mathbb{R}_+$. This lost of symmetry between $+\infty$ and $-\infty$ is linked to the fact that we imposed that $\alpha>0$ so that linear waves solutions of $u_t+\alpha u_{3x}=0$ are travelling only to the left.*

Finally, if we want to get the well-posedness in the Hadamard sense of (1.1) we need to require a little more on h so that $\|u(t)\|_{H^s}\sim\|(hu)(t)\|_{H^s}$ uniformly on $[0,T_0]$. This forces h to be situated between two positive values, i.e. there exists $h_0,h_1>0$ such that for any $(t,x)\in[0,T]\times\mathbb{R}$, $h_0\le h(t,x)\le h_1$.

For this it suffices to replace Hypothesis 3 by the following one :

hyp4 **Hypothesis 4.** *β can be decomposed as $\beta=\beta_1+\beta_2$ with $\beta_2\le 0$, $\beta_1,\beta_2\in L^\infty([0,T];C^2_b)$, $\partial_t\beta_1\in L^\infty(]0,T[;L^\infty)$ such that*

$$(t,x)\mapsto\int_0^x(\alpha^{-1}\beta_1)(t,y)\,dy\in W^{1,\infty}([0,T];L^\infty(\mathbb{R}))\; .$$

which leads to Theorem 1.1.

4. Estimates on the solutions to (1.2)

sect4

In this section, we prove the needed estimates on solutions to (1.2) to get the local well-posedness of (1.2) in $H^s(\mathbb{R})$ for $s>1/2$. For this purpose we use the approach introduced in [11] that mix energy's and Bourgain's type estimates.

4.1. An estimate using Bourgain's type spaces.

We start by proving the only estimate where we need Bourgain's type spaces. This estimate will be used to bound the contribution of the nonlinear KdV term euu_x in the energy estimate. First we check that under suitable space projections on the functions, we have a good lower bound on the resonance relation that appears in this contribution.

resolem **Lemma 4.1.** *Let $L_i\ge 1$ and $N_i\ge 1$ be dyadic numbers and $u_i\in L^2(\mathbb{R}^2)$ for $i\in\{1,2,3,4\}$. If $N_1\ll\min(N_2,N_3,N_4)$ then it holds*

$$\int_{\mathbb{R}^2}P_{N_4}\Big(Q_{L_1}P_{\le N_1}u_1\,Q_{L_2}P_{N_2}u_2\,Q_{L_3}P_{N_3}u_3\Big)Q_{L_4}P_{N_4}u_4=0$$

whenever the following relation is not satisfied :

$$L_{max}\sim N_2N_3N_4 \text{ or } (L_{max}\gg N_2N_3N_4 \text{ and } L_{max}\sim L_{med}) \qquad (4.1)$$ resonance3

where $L_{max}=\max_{i=1,..,4}L_i$ and $L_{med}=\max(\{L_1,L_2,L_3,L_4\}-\{L_{\max}\})$.

Proof. Applying Plancherel identity, this is a direct consequence of the condition $N_1 \ll \min(N_2, N_3, N_4)$ together with the cubic resonance relation associated with the KdV propagator :

$$\Omega_3(\xi_1,\xi_2,\xi_3) = \sigma\Big(-\sum_{i=1}^{3}\tau_i, -\sum_{i=1}^{3}\xi_i\Big) + \sum_{i=1}^{3}\sigma(\tau_i,\xi_i) = -3(\xi_2+\xi_3)(\xi_1+\xi_3)(\xi_1+\xi_2)$$

where $\sigma(\tau,\xi) := \tau - \xi^3$. Note that the conditions on the N_i's ensure that the above integrals vanish for $L_{max} \lesssim 1$. □

Now we can give our main estimate that uses Bourgain's type spaces.

lemtriest **Lemma 4.2.** *Assume $0 < T < 1$, $e \in L^\infty_{Tx}$ with $e_t \in L^\infty_{Tx}$ and $u_i \in L^\infty_T H^{-1/2} \cap X_T^{-\frac{3}{2},1}$, $i = 2,3,4$. Let $N_j \in 2^{\mathbb{N}}$, $j = 1,2,3,4$ with $N_1 \ll \min(N_2,N_3,N_4)$. Setting, for all $0 < t < T$,*

$$I_t^3 = I_t(e,u_2,u_3,u_4) = \int_0^t \int_{\mathbb{R}} P_{N_4}(P_{\le N_1} e\, P_{N_2}u_2 \partial_x P_{N_3}u_3) P_{N_4}u_4 \ , \tag{4.2}$$ II

it holds

$$|I_t^3| \lesssim (\|e\|_{L^\infty_{Tx}} + \|e_t\|_{L^\infty_{Tx}})\Big[\|P_{N_r}u_r\|_{L^\infty_T L^2_x}\Big(\sum_{i=p,q}\|P_{N_i}u_i\|_{L^2_{Tx}}\Big)\Big(\sum_{i=p,q}\|P_{N_i}u_i\|_{X_T^{-1,1}}\Big)$$
$$+ T^{\frac{1}{16}} N_p^{-\frac14}\sum_{i=2}^{4}\Big(\|P_{N_i}u_i\|_{X_T^{-1,1}} + \|P_{N_i}u_i\|_{L^\infty_T L^2_x}\Big)\prod_{\substack{j=2\\ j\neq i}}^{4}\|P_{N_j}u_j\|_{L^\infty_T L^2_x}\Big] \tag{4.3}$$ f1

whenever $N_p \sim N_q \gtrsim N_r$ where (p,q,r) is a permutation of $(2,3,4)$.

Proof. We start by noticing that we may also assume that e and e_t belong to $L^2_T L^2_x$. Indeed, approximating e by $e_R = e\,\eta_R$ with $\eta_R = \eta(\cdot/R)$ where η is the smooth non negative compactly supported function defined in (2.1), we notice that for any $t \in [0,T]$, Lebesgue dominated convergence theorem leads for any $N \in 2^{\mathbb{N}}$ to

$$\mathcal{F}_x^{-1}(\phi_{\le N}) * e_R \to \mathcal{F}_x^{-1}(\phi_{\le N}) * e = P_{\le N} e \quad \text{ on } \mathbb{R},$$

since $\mathcal{F}_x^{-1}(\phi_{\le N}) \in L^1(\mathbb{R})$ and $|e(t)\,\eta_R| \le |e(t)| \in L^\infty(\mathbb{R})$. Applying again the Lebesgue dominated convergence theorem we get

$$\int_0^t\int_{\mathbb{R}} P_{N_4}(P_{\le N_1}e_R\, P_{N_2}u_2\partial_x P_{N_3}u_3)P_{N_4}u_4 = \int_0^t\int_{\mathbb{R}} P_{\le N_1}e_R\, P_{N_2}u_2\partial_x P_{N_3}u_3 P^2_{N_4}u_4$$
$$\underset{R\to+\infty}{\longrightarrow} \int_0^t\int_{\mathbb{R}} P_{\le N_1}e\, P_{N_2}u_2\partial_x P_{N_3}u_3 P^2_{N_4}u_4$$
$$= I_t^3 \ ,$$

by using that, for any fixed $j \in \mathbb{N}$, $P_{2^j}u_i \in L^\infty_{Tx} \cap L^2_{Tx}$. This proves the desired result since

$$\|e_R\|_{L^\infty_{tx}} + \|\partial_t e_R\|_{L^\infty_{tx}} \le \|e\|_{L^\infty_{tx}} + \|\partial_t e\|_{L^\infty_{tx}} \ , \forall R \ge 1.$$

Now we extend the functions e, u_2, u_3, u_4 on the whole time axis. For u_2, u_3, u_4 we use the extension operator ρ_T defined in Lemma 2.6. On the other hand for e we use the extension operator $\tilde\rho_T$ defined by $\tilde\rho_T(e)(t) = \eta(t)e(\mu_T(t))$ with μ_T defined in (2.17) and η defined in (2.1). This extension operator is bounded from $W^{1,\infty}_T L^\infty_x$ into $W^{1,\infty}_t L^\infty_x$ with a bound that does not depend on $T > 0$. To lighten the notations, we keep the notation u_i for $\rho_T(u_i)$ and e for $\tilde\rho_T(e)$. Fixing $t \in]0,T[$

and setting $R=N_2^{\frac{3}{4}}N_3N_4^{\frac{3}{4}}$, we then split I_t as

$$\begin{aligned}I_t(e,u_2,u_3,u_4)&=I_\infty(e,1_{t,R}^{high}u_2,1_t\,u_3,1_t\,u_4)+I_\infty(e,1_{t,R}^{low}u_2,1_{t,R}^{high}u_3,1_t\,u_4)\\&\quad+I_\infty(e,1_{t,R}^{low}u_2,1_{t,R}^{low}u_3,1_{t,R}^{high}u_4)+I_\infty(e,1_{t,R}^{low}u_2,1_{t,R}^{low}u_3,1_{t,R}^{low}u_4)\\&:=I_t^{high,1}+I_t^{high,2}+I_t^{high,3}+I_t^{low},\end{aligned}\tag{4.4}$$

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where $I_\infty(e,u_2,u_3,u_4)=\int_{\mathbb{R}^2}P_{N_4}(P_{N_1}e\,P_{N_2}u_2\partial_xP_{N_3}u_3)P_{N_4}u_4$. The contribution of $I_t^{high,1}$ is estimated thanks to Lemma 2.2 and Hölder and Bernstein inequalities by

$$\begin{aligned}I_t^{high,1}&\lesssim N_3\|1_{t,R}^{high}\|_{L^1}\|e\|_{L^\infty_{tx}}\|P_{N_2}u_2\|_{L^\infty_tL^4_x}\|P_{N_3}u_3\|_{L^\infty_tL^2_x}\|P_{N_4}u_4\|_{L^\infty_tL^4_x}\\&\lesssim T^{1/4}(N_2^{\frac{3}{4}}N_3N_4^{\frac{3}{4}})^{-\frac{3}{4}}N_3(N_2N_4)^{\frac{1}{4}}\|e\|_{L^\infty_{tx}}\prod_{i=2}^4\|P_{\sim N_i}u_i\|_{L^\infty_tL^2_x}\\&\lesssim T^{1/4}(N_2\vee N_3)^{-\frac{1}{16}}\|e\|_{L^\infty_{tx}}\prod_{i=2}^4\|P_{N_i}u_i\|_{L^\infty_tL^2_x}\end{aligned}\tag{4.5}$$

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where we used that the frequency projectors ensure that $N_2\vee N_4\sim N_2\vee N_3$. The contribution of $I_t^{high,2}$ and $I_t^{high,3}$ can be estimated in exactly the same way, using that$\|1_{t,R}^{low}\|_{L^\infty_t}\lesssim 1$ thanks to (2.9). To evaluate the contribution I_t^{low} we use the following decomposition :

$$\begin{aligned}I_\infty(e,1_{t,R}^{low}\,u_2,u_3,u_4)&=I_\infty(e,Q_{\gtrsim N_2N_3N_4}(1_{t,R}^{low}u_2),1_{t,R}^{low}u_3,1_{t,R}^{low}u_4)\\&+I_\infty(e,Q_{\ll N_2N_3N_4}(1_{t,R}^{low}u_2),Q_{\gtrsim N_2N_3N_4}(1_{t,R}^{low}u_3),1_{t,R}^{low}u_4)\\&+I_\infty(e,Q_{\ll N_2N_3N_4}(1_{t,R}^{low}u_2),Q_{\ll N_2N_3N_4}(1_{t,R}^{low}u_3),Q_{\gtrsim N_2N_3N_4}(1_{t,R}^{low}u_4))\\&+I_\infty(e,Q_{\ll N_2N_3N_4}(1_{t,R}^{low}u_2),Q_{\ll N_2N_3N_4}(1_{t,R}^{low}u_3),Q_{\ll N_2N_3N_4}(1_{t,R}^{low}u_4))\\&=I_t^{2,low}+I_t^{3,low}+I_t^{4,low}+I_t^{1,low}\ ,\end{aligned}\tag{4.6}$$

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To evaluate the contribution $I_t^{1,low}$ we notice that since $N_1^3\ll N_1N_2N_3$, Lemma 4.1 ensures that

$$I_t^{1,low}=I_\infty(R_{\sim N_2N_3N_4}e,Q_{\ll N_2N_3N_4}(1_{t,R}^{low}u_2),Q_{\ll N_2N_3N_4}(1_{t,R}^{low}u_3),Q_{\ll N_2N_3N_4}(1_{t,R}^{low}u_4))$$

where R_K is the projection on the time Fourier variable (see (2.2)). Therefore, by Bernstein inequality and Lemma 2.1 we get

$$\begin{aligned}|I_t^{1,low}|&\lesssim T(N_2N_3N_4)^{-1}\|e_t\|_{L^\infty_{tx}}\|P_{N_2}u_2\|_{L^\infty_{tx}}N_3\|P_{N_3}u_3\|_{L^\infty_tL^2_x}\|P_{N_4}u_4\|_{L^\infty_tL^2_x}\\&\lesssim T(N_2\vee N_3)^{-\frac{1}{2}}\|e_t\|_{L^\infty_{tx}}\|P_{N_2}u_2\|_{L^\infty_tL^2_x}\prod_{i=3}^4\|P_{N_i}u_i\|_{L^\infty_tL^2_x}\end{aligned}\tag{4.7}$$

Now, to evaluate the other contributions in (4.6) we have to separate different cases. For the future use of Lemma 2.3, it is worth noticing that since $N_2,N_4\gg 1$, $R=N_2^{\frac{3}{4}}N_3N_4^{\frac{3}{4}}\ll N_2N_3N_4$.

Case 1 : $N_4\sim N_3\gtrsim N_2$. Then $I_t^{2,low}$ can be easily estimated thanks to Lemma 2.3 and (2.9) by

$$\begin{aligned}|I_t^{2,low}|&\lesssim\|e\|_{L^\infty_{tx}}\|Q_{\gtrsim N_2N_3N_4}P_{N_2}(1_{t,R}^{low}u_2)\|_{L^2_{tx}}N_3\|1_{t,R}^{low}P_{N_3}u_3\|_{L^2_{tx}}\|1_{t,R}^{low}P_{N_4}u_4\|_{L^\infty_{tx}}\\&\lesssim T^{1/2}(N_2N_3N_4)^{-1}N_2N_3N_4^{\frac{1}{2}}\|e\|_{L^\infty_{tx}}\|P_{N_2}u_2\|_{X^{-1,1}}\|P_{N_3}u_3\|_{L^\infty_tL^2_x}\|P_{N_4}u_4\|_{L^\infty_tL^2_x}\\&\lesssim T^{\frac{1}{2}}(N_2\vee N_3)^{-1/2}\|e\|_{L^\infty_{tx}}\|u_2\|_{X^{-1,1}}\prod_{i=3}^4\|P_{N_i}u_i\|_{L^\infty_tL^2_x}\end{aligned}\tag{4.8}$$

To estimate the contribution of $I_t^{3,low}$ we notice that Lemma 2.2 together with the fact that $R \geq N_2 \vee N_3$ ensure that for any $w \in L^\infty_t L^2_x$

$$\|1_{t,R}^{low} w\|_{L^2_{tx}} \leq \|1_t w\|_{L^2_{tx}} + \|1_{t,R}^{high} w\|_{L^2_{tx}} \lesssim \|w\|_{L^2_T L^2_x} + T^{1/4}(N_2 \vee N_3)^{-1/4} \|w\|_{L^\infty_T L^2_x} .$$

Therefore Lemmas 2.1 and 2.3 lead to

$$\begin{aligned} |I_t^{3,low}| \lesssim & (N_2N_3N_4)^{-1} N_3^2 \|e\|_{L^\infty_{tx}} \|P_{N_2} u_2\|_{L^\infty_{tx}} \|P_{N_3} u_3\|_{X^{-1,1}} \|1_{t,R}^{low} P_{N_4} u_4\|_{L^2_{tx}} \\ \lesssim & N_2^{-1/2} \|e\|_{L^\infty_{tx}} \|P_{N_2} u_2\|_{L^\infty_t L^2_x} \Big(\|P_{N_3} u_3\|_{X^{-1,1}} \|P_{N_4} u_4\|_{L^2_T L^2_x} \\ & + T^{1/4} (N_2 \vee N_3)^{-1/4} \|P_{N_3} u_3\|_{X^{-1,1}} \|P_{N_4} u_4\|_{L^\infty_T L^2_x} \Big) \end{aligned} \tag{4.9}$$

and $I_t^{4,low}$ can be estimated in exactly the same way by exchanging the role of u_3 and u_4 to get

$$\begin{aligned} |I_t^{4,low}| \lesssim & N_2^{-1/2} \|e\|_{L^\infty_{tx}} \|P_{N_2} u_2\|_{L^\infty_t L^2_x} \Big(\|P_{N_3} u_4\|_{X^{-1,1}} \|P_{N_4} u_3\|_{L^2_T L^2_x} \\ & + T^{1/4} (N_2 \vee N_3)^{-1/4} \|P_{N_3} u_4\|_{X^{-1,1}} \|P_{N_4} u_3\|_{L^\infty_T L^2_x} \Big) \end{aligned} \tag{4.10}$$

Gathering (4.4)-(4.10), we obtain (4.3) whenever $N_4 \sim N_3 \gtrsim N_2$.
Case 2 : $N_2 \sim N_3 \gtrsim N_4$. Then we get exactly the same type of estimates just by exchanging the role of u_2 and u_4 with respect to the preceding case.
Case 3: $N_2 \sim N_4 \gtrsim N_3$. This case can be treated as the first ones and is even simplest since the derivative falls on the smallest frequency. We thus omit the details. □

4.2. **A priori estimates in** $H^s(\mathbb{R})$**.** For an initial data in $H^s(\mathbb{R})$, with $s > 1/2$, we will construct a solution to (1.2) in Y^s_T whereas the estimate of difference of two solutions emanating from initial data belonging to $H^s(\mathbb{R})$ will take place in Y^{s-1}_T.

estYs **Lemma 4.3.** *Let* $s > 1/2$*,* $0 < T < 1$ *and* $u \in L^\infty_T H^s \cap L^2_{[b]}(]0,T[; H^{s+1})$ *be a solution to* (1.2)*. Then* $u \in Y^s_T$ *and the following inequality holds*

$$\|u\|_{Y^s_T} \lesssim C\Big(\|u\|_{L^2_{[b]}(]0,T[;H^{s+1})} + (1 + \|u\|_{L^\infty_T H^{\frac12+}}) \|u\|_{L^\infty_T H^s} \Big) . \tag{4.11}$$ esta1

Moreover, for any couple $(u,v) \in L^\infty_T H^s$ *of solutions to* (1.2) *associated with a couple of initial data* $(u_0,v_0) \in (H^s(\mathbb{R}))^2$*, it holds*

$$\|u-v\|_{Y^{s-1}_T} \lesssim C\Big(\|u-v\|_{L^2_{[b]}(]0,T[;H^s)} + (1 + \|u+v\|_{L^\infty_T H^s}) \|u-v\|_{L^\infty_T H^{s-1}} \Big) , \tag{4.12}$$ estdiffXregular

where

$$C = C\Big(s, \|b\|_{L^\infty_T C_*^{((s+1)\vee 2)+}}, \|c\|_{L^\infty_T C_*^{s+}}, \|d\|_{L^\infty_T C_*^{s+}}, \|e\|_{L^\infty_T C_*^{(s\vee 1)+}}, \|f\|_{L^\infty_T C_*^{s+}} \Big) .$$

Proof. According to the extension Lemma 2.6 it suffices to establish estimates on the Bourgain's norms of u and $u-v$. Standard linear estimates in Bourgain's spaces lead to

$$\begin{aligned} \|u\|_{X^{s-1,1}_T} \lesssim & \|u_0\|_{H^{s-1}} + \|1_T(\partial_t - \partial_x^3) u\|_{X^{s-1,0}} \\ \lesssim & \|u_0\|_{H^{s-1}} + \|b u_x\|_{L^2_T H^s} + \|b_x u_x\|_{L^2_T H^{s-1}} + \|cu\|_{L^2_T H^s} \\ & + \|(-c_x + d) u\|_{L^2_T H^{s-1}} + \frac12 \|e\, u^2\|_{L^2_T H^s} + \|(-e_x/2 + f) u^2\|_{L^2_T H^{s-1}} . \end{aligned}$$

According to Lemma 2.4, using that $s > 1/2$, it holds

$$\|b_x u_x\|_{L^2_T H^{s-1}} \lesssim \|b_x\|_{L^\infty_T C_*^{|s-1|+}} \|u_x\|_{L^\infty_T H^{s-1}}$$

$$\|cu\|_{L^2_T H^s} + \|c_x u\|_{L^2_T H^{s-1}} \lesssim \|c\|_{L^\infty_T C_*^{s+}} \|u\|_{L^\infty_T H^s}$$

$$\|eu^2\|_{L^2_T H^s} + \|e_x u^2\|_{L^2_T H^{s-1}} \lesssim \|e\|_{L^\infty_T C_*^{s+}} \|u\|_{L^\infty_T H^{\frac12+}} \|u\|_{L^\infty_T H^s}$$

$\|du\|_{L^2_T H^{s-1}} \lesssim \|d\|_{L^\infty_T C_*^{|s-1|+}} \|u\|_{L^\infty_T H^s}$ and $\|fu^2\|_{L^2_T H^{s-1}} \lesssim \|f\|_{L^\infty_T C_*^{|s-1|+}} \|u\|^2_{L^\infty_T H^s}$
Therefore, we get

$$\|u\|_{X_T^{s-1,1}} \lesssim \|u\|_{L^\infty_T H^{s-1}} + C_1(1+\|u\|_{L^\infty_T H^{\frac{1}{2}+}})\|u\|_{L^\infty_T H^s} + \|bu_x\|_{L^2_T H^s} ,$$

where $C_1 = C_1(\|b_x\|_{L^\infty_T C_*^{|s-1|+}}, \|c\|_{L^\infty_T C_*^{s+}}, \|d\|_{L^\infty_T C_*^{|s-1|+}}, \|e\|_{L^\infty_T C_*^{s+}}, \|f\|_{L^\infty_T C_*^{|s-1|+}})$.
Now, noticing that Lemma 2.4 also leads for $s > 1/2$ to

$$\|b_x w_x\|_{L^2_T H^{s-2}} \lesssim \|b_x\|_{L^\infty_T C_*^{|s-2|+}} \|w_x\|_{L^\infty_T H^{s-2}}$$

$$\|cw\|_{L^2_T H^{s-1}} + \|c_x w\|_{L^2_T H^{s-2}} \lesssim \|c\|_{L^\infty_T C_*^{s+}} \|w\|_{L^\infty_T H^{s-1}}$$

$$\|euw\|_{L^2_T H^{s-1}} + \|e_x uw\|_{L^2_T H^{s-2}} \lesssim \|e\|_{L^\infty_T C_*^{(s\vee(2-s))+}} \|u\|_{L^\infty_T H^s} \|w\|_{L^\infty_T H^{s-1}}$$

$\|dw\|_{L^2_T H^{s-2}} \lesssim \|d\|_{L^\infty_T C_*^{s+}} \|w\|_{L^\infty_T H^{s-1}}$ and $\|fu^2\|_{L^2_T H^{s-2}} \lesssim \|f\|_{L^\infty_T C_*^{s+}} \|w\|^2_{L^\infty_T H^{s-1}}$,
we also get

$$\begin{aligned}\|u-v\|_{X_T^{s-1,1}} \lesssim & \|u_0 - v_0\|_{H^{s-1}} + C_2\,(1+\|u+v\|_{L^\infty_T H^s})\|u-v\|_{L^\infty_T H^{s-1}} \\ & + \|b\,\partial_x(u-v)\|_{L^2_T H^{s-1}}\end{aligned}$$

with $C_2 = C_2(\|b_x\|_{L^\infty_T C_*^{|s-2|+}}, \|c\|_{L^\infty_T C_*^{s+}}, \|d\|_{L^\infty_T C_*^{s+}} + \|e\|_{L^\infty_T C_*^{(s\vee(2-s))+}}, \|f\|_{L^\infty_T C_*^{s+}})$.
It just remains to get an estimate on $\|\partial_x(bv_x)\|_{L^2_T H^{\theta-1}}$ for $b \in L^\infty_T C_*^{(s\vee(3-s))+}$ and $v \in L^\infty_T H^\theta$ with $\theta > -1/2$. By using a non homogeneous dyadic decomposition it holds

$$\|\partial_x(bv_x)\|^2_{L^2_T H^{\theta-1}} \sim \|\partial_x P_{\lesssim 1} 1(bv_x)\|^2_{L^2_T L^2_x} + \sum_{N\gg 1} N^{2\theta} \|P_N(bv_x)\|^2_{L^2_T L^2_x} .$$

The first term of the above right-hand side is easily estimated as above by :

$$\|\partial_x P_{\lesssim 1}(bv_x)\|_{L^2_T L^2_x} \lesssim \|bv_x\|_{H^{-2}} \lesssim \|b\|_{L^\infty_T C_*^{\frac{3}{2}+}} \|v\|_{L^\infty_T H^{-\frac{1}{2}+}}$$

Now, for $N \gg 1$ we rewrite $P_N(bu_x)$ as

$$\begin{aligned}P_N(bu_x) &= P_N(P_{\gtrsim N} b\, u_x) + P_{\ll N} b P_N u_x + [P_N, P_{\ll N} b] u_x \\ &= A_N + B_N + C_N.\end{aligned}$$

We have

$$\begin{aligned}\sum_{N\gg 1} N^{2\theta} \|A_N\|^2_{L^2_{Tx}} &\lesssim \sum_{N\gg 1} N^{2\theta} \|P_{\gtrsim N} b P_{\ll N} u_x\|^2_{L^2_{Tx}} + \sum_{N\gg 1} N^{2\theta} \sum_{N_1 \gtrsim N} \|P_{N_1} b P_{\sim N_1} u_x\|^2_{L^2_{Tx}} \\ &\lesssim \sum_{N\gg 1} N^{2\theta} \|P_{\gtrsim N} b\|_{L^\infty_{Tx}} N^{(2-2\theta)\vee 0} \|P_{\ll N} u_x\|^2_{L^2_T H^{\theta-1}} \\ &\quad + \sum_{N\gg 1} N^{2\theta} \sum_{N_1\gtrsim N} \|P_{N_1} b\|_{L^\infty_{Tx}} N_1^{2-2\theta} \|P_{\sim N_1} u_x\|^2_{L^2_T H^{\theta-1}} \\ &\lesssim \Big(\|b\|^2_{L^\infty_T C_*^{(1\vee\theta)+}} + \|b\|^2_{L^\infty_T C_*^{(1\vee(1-\theta))+}} \Big) \|u\|^2_{L^\infty_T H^\theta} . \end{aligned} \tag{4.13}$$

To bound the contribution of B_N we observe that

$$\begin{aligned}\sum_{N\gg 1} N^{2\theta} \|B_N\|^2_{L^2_T L^2_x} &\le \sum_{N\gg 1} N^{2\theta} \Big(\|b\partial_x u_N\|_{L^2_T L^2_x} + \|P_{\gtrsim N} b \partial_x u_N\|_{L^2_T L^2_x} \Big)^2 \\ &\lesssim \sum_{N\gg 1} N^{2\theta} \int_0^T \int_{\mathbb{R}} b^2 (\partial_x u_N)^2 + \sum_{N\gg 1} N^2 \|P_{\gtrsim N} b\|^2_{L^\infty_{Tx}} \|u_N\|^2_{L^2_T H^\theta} \\ &\le \|u\|^2_{(L^2_T H^{\theta+1})_b} + \|b_x\|^2_{L^\infty_{Tx}} \|u\|^2_{L^\infty_T H^\theta} . \end{aligned} \tag{4.14}$$

Finally to bound the contribution of C_N we use (2.13) of Lemma 2.5 to get

$$\sum_{N\gg 1} N^{2\theta}\|C_N\|^2_{L^2_T L^2_x} \lesssim \|b_x\|^2_{L^\infty_{Tx}} \sum_{N\gg 1} N^{2\theta}\|\tilde{P}_N u\|^2_{L^2_T L^2_x} \lesssim \|b_x\|^2_{L^\infty_{Tx}}\|u\|^2_{L^\infty_T H^\theta} \quad . \tag{4.15}$$

Gathering the above estimates we observe that it is enough to have $b \in L^\infty_T C_*^{(3-s)+}$ for $1/2 < s < 3/2$ and $b \in L^\infty_T C_*^{s+}$ for $s \geq 3/2$. and completes the proof of the lemma. ☐

prou **Proposition 4.1.** *Let $0 < T < 2$ and $u \in Y^s_T$ with $s > 1/2$ be a solution to* (1.2) *associated with an initial datum $u_0 \in H^s(\mathbb{R})$. Then it holds*

$$\|u\|^2_{L^\infty_T H^s} + \|u\|^2_{L^2_{[b]}(]0,T[;H^{s+1})} \leq \|u_0\|^2_{H^s} + C\, T^{\frac{1}{16}}(1 + \|u\|_{Y_T^{\frac{1}{2}+}})\|u\|^2_{Y^s_T} . \tag{4.16}$$ estHsregular

where

$$C = C\Big(s, \|b\|_{L^\infty_T C_*^{((s+1)\vee 2)+}}, \|c\|_{L^\infty_T C_*^{(s\vee 1)+}}, \|d\|_{L^\infty_T C_*^{s+}}, \|e\|_{L^\infty_T C_*^{s+\frac{1}{2}+}}, \|f\|_{L^\infty_T C_*^{s+}}, \|e_t\|_{L^\infty_{Tx}}\Big) \tag{4.17}$$ const

Proof. We apply the operator P_N with $N \in 2^{\mathbb{N}}$ dyadic to equation (1.2). On account of Remark 1.1, it is clear that $P_N u \in C([0,T]; H^\infty)$ with $\partial_t u_N \in L^\infty(0,T;H^\infty)$. Therefore , taking the L^2_x-scalar product of the resulting equation with $P_N u$, multiplying by $\langle N\rangle^{2s}$ and integrating on $]0,t[$ with $0 < t < T$ we obtain

$$\begin{aligned}\langle N\rangle^{2s}\|P_N u(t)\|^2_{L^2} = \langle N\rangle^{2s}\|P_N u_0\|^2_{L^2} + \langle N\rangle^{2s}\int_0^t\int_{\mathbb{R}} P_N\Big(-b_x u_x - c u_x - du + f u^2\Big)P_N u \\ + \langle N\rangle^{2s}\Big(\int_0^t\int_{\mathbb{R}} P_N(e\, u u_x)P_N u + \int_0^t\int_{\mathbb{R}} \partial_x P_N(b u_x)P_N u\Big) .\end{aligned} \tag{4.18}$$ estenergy

Now we are going to estimate successively all the terms of the right-hand of (4.18). Note that, even if $s > 1/2$, we will give estimates of the linear terms (in u) valid for $s > -1/2$ that will be directly usable in Proposition 4.2 when estimating the difference of two solutions in $H^{s-1}(\mathbb{R})$.

• *Contribution of $P_N(du)$.*

Making use of Sobolev inequalities, this contribution is easily estimated by :

$$\begin{aligned}\langle N\rangle^{2s}\Big|\int_{]0,t[\times\mathbb{R}} P_N(du)P_N u\Big| &\lesssim \langle N\rangle^{2s}\|P_N(du)\|_{L^2_T L^2_x}\|P_N u\|_{L^2_T L^2_x} \\ &\lesssim T\delta_N\|du\|_{L^\infty_T H^s}\|u\|_{L^\infty_T H^s} \\ &\lesssim T\delta_N\,\|d\|_{L^\infty_T C_*^{|s|+}}\|u\|^2_{L^\infty_T H^s}\end{aligned} \tag{4.19}$$

with $\|(\delta_{2^j})_{j\geq 0}\|_{l^1} \leq 1$. In the sequel, we denote by $(\delta_q)_{q\geq 1}$ any sequence of real numbers such that $\|(\delta_{2^j})_{j\geq 0}\|_{l^1} \leq 1$.

• *Contribution of $P_N(fu^2)$.*

This term is only estimated for $s > 1/2$. Proceding exactly as above we get

$$\begin{aligned}\langle N\rangle^{2s}\Big|\int_{]0,t[\times\mathbb{R}} P_N(fu^2)\Big)P_N u\Big| &\lesssim T\delta_N\|fu^2\|_{L^\infty_T H^s}\|u\|_{L^\infty_T H^s} \\ &\lesssim T\delta_N\,\|f\|_{L^\infty_T C_*^{|s|+}}(1 + \|u\|_{L^\infty_{Tx}})\|u\|^2_{L^\infty_T H^s} .\end{aligned} \tag{4.20}$$

• *Contribution of* $P_N((b_x+c)u_x)$.
For $1\le N\lesssim 1$, (2.12) leads to

$$\langle N\rangle^{2s}\Big|\int_0^t\int_{\mathbb{R}}\Big(P_N((b_x+c)u_x)\Big)P_N u\Big| \lesssim \int_0^t \|(b_x+c)u_x\|_{H^{s-1}}\|u\|_{H^s}$$
$$\lesssim \Big(\|b_x\|_{L^\infty_T C_*^{|s-1|+}}+\|c\|_{L^\infty_T C_*^{|s-1|+}}\Big)\|u\|^2_{H^s} \; .$$

For $N\gg 1$, We first notice that

$$\begin{aligned}
N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} & P_N\Big(P_{\gtrsim N}(b_x+c)u_x\Big)P_N u\Big| \\
&\lesssim N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} P_N\Big(P_{\gtrsim N}(b_x+c)P_{\ll N}u_x\Big)P_N u\Big| \\
&\quad + N^{2s}\sum_{N_1\gtrsim N}\Big|\int_{]0,t[\times\mathbb{R}} P_N\Big(P_{N_1}(b_x+c)P_{\sim N_1}u_x\Big)P_N u\Big| \\
&\lesssim \int_0^t N^s\|P_{\gtrsim N}(b_x+c)\|_{L^\infty_x}N^{(1-s)\vee 0}\|P_{\ll N}u_x\|_{H^{s-1}}\|u\|_{H^s} \\
&\quad + \int_0^t N^s\sum_{N_1\gtrsim N}\|P_{N_1}(b_x+c)\|_{L^\infty_x}N_1^{1-s}\|P_{\sim N_1}u_x\|_{H^{s-1}}\|u\|_{H^s} \\
&\lesssim T\delta_N\,(\|b_x\|_{L^\infty_T C_*^{(1\vee s\vee 1-s)+}}+\|c\|_{L^\infty_T C_*^{(1\vee s\vee 1-s)+}})\|u\|^2_{L^\infty_T H^s} \qquad (4.21)
\end{aligned}$$

Then we use the commutator estimate (2.13) and integration by parts to get

$$\begin{aligned}
N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} P_N\Big(P_{\ll N}(b_x+c)u_x\Big)P_N u\Big| &= N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} P_{\ll N}(b_{xx}+c_x)(P_N u)^2\Big| \\
&\quad + N^{2s}\Big|\int_{]0,t[\times\mathbb{R}}[P_N,P_{\ll N}(b_x+c)]u_x P_N u\Big| \\
&\lesssim N^{2s}\|b_{xx}+c_x\|_{L^\infty_{Tx}}\|\tilde{P}_N u\|^2_{L^2_T L^2_x} \\
&\lesssim T\delta_N\,(\|b_{xx}\|_{L^\infty_{Tx}}+\|c_x\|_{L^\infty_{Tx}})\|u\|^2_{L^\infty_T H^s} \qquad (4.22)
\end{aligned}$$

with $\|(\delta_{2^j})_{j\ge 0}\|_{l^1}\le 1$.

• *Contribution of* $P_N(euu_x)$.
This term is only estimated for $s>1/2$. For $1\le N\lesssim 1$, we write $e\,\partial_x(u^2)=\frac{1}{2}\partial_x(eu^2)-\frac{1}{2}e_x u^2$ to get

$$\begin{aligned}
\langle N\rangle^{2s}\Big|\int_{]0,t[\times\mathbb{R}} P_N\Big(e\partial_x(u^2)\Big)P_N u\Big| &\lesssim T\|u\|_{L^\infty_T L^2_x}(\|e_x u^2\|_{L^\infty_T L^2_T}+\|eu^2\|_{L^\infty_T L^2_x}) \\
&\lesssim T\|e\|_{L^\infty_T W^{1,\infty}}\|u\|^2_{L^\infty_T H^{\frac14}}\|u\|_{L^\infty_T L^2_x} \qquad (4.23)
\end{aligned}$$

It thus remains to consider $N \gg 1$. We first separate two contributions.
1. The contribution of $P_N(P_{\gtrsim N} e\, uu_x)$. This contribution is easily estimated by

$$
\begin{aligned}
N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} & P_N\Big(P_{\gtrsim N} e\,\partial_x(u^2)\Big)P_N u\Big| \\
&= N^{2s}\sum_{N_1\ll N}\Big|\int_{]0,t[\times\mathbb{R}} P_N\Big(P_{\sim N} e\, P_{N_1}\partial_x(u^2)\Big)P_N u\Big| \\
&+ N^{2s}\sum_{N_1\gtrsim N}\Big|\int_{]0,t[\times\mathbb{R}} P_N\Big(P_{\sim N_1} e\, P_{N_1}\partial_x(u^2)\Big)P_N u\Big| \\
&\lesssim N^{2s}\int_0^t \|P_{\sim N} e\|_{L^\infty_x}\|P_N u\|_{L^2_x}\sum_{N_1\ll N} N_1^{1/2}\|D_x^{1/2}(u^2)\|_{L^2_x} \\
&+ N^{2s}\int_0^t \|P_N u\|_{L^2_x}\sum_{N_1\gtrsim N}\|P_{\sim N_1} e\|_{L^\infty_x} N_1^{1/2}\|D_x^{1/2}(u^2)\|_{L^2_x} \\
&\lesssim \delta_N\, T\|e\|_{L^\infty_T C^{s+1/2}_*}\|u\|^2_{L^\infty_T H^{\frac12+}}\|u\|_{L^\infty_T H^s} \qquad (4.24)
\end{aligned}
$$

with $\|(\delta_{2^j})_{j\ge 0}\|_{l^1}\le 1$.

2. The contribution of $P_N(P_{\ll N} e\, uu_x)$. We rewrite this term as

$$
\begin{aligned}
P_N(P_{\ll N} e\, uu_x) =& P_N\Big(P_{\ll N} e\, P_{\lesssim 1} u\, \tilde P_N(u_x)\Big) \\
&+ \sum_{1\ll N_2\ll N} P_N\Big(P_{\ll N} e\, u_{N_2}\tilde P_N u_x\Big) \\
&+ P_N\Big(P_{\ll N} e\, \tilde P_N u\, P_{\lesssim 1} u_x\Big) \\
&+ \sum_{1\ll N_3\lesssim N_1\ll N} P_N\Big(e_{N_1}\tilde P_N u\,\partial_x u_{N_3}\Big) \\
&+ \sum_{1\ll N_3\lesssim N_2} P_N\Big(P_{\ll N_3\wedge N} e\, u_{N_2}\partial_x u_{N_3}\Big) \\
=& A+B+C+D+E. \qquad (4.25)
\end{aligned}
$$

First, the contribution of C is easily estimated by

$$
\begin{aligned}
N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} C P_N u\Big| &\lesssim \int_0^t \|u_N\|_{H^s}\|u_{\sim N}\|_{H^s}\|e\|_{L^\infty_x}\|u\|_{L^\infty_T L^2_x} \\
&\lesssim \delta_N\, T\|e\|_{L^\infty_{Tx}}\|u\|_{L^\infty_T L^2_x}\|u\|^2_{L^\infty_T H^s} \qquad (4.26)
\end{aligned}
$$

td1

with $\|(\delta_{2^j})_{j\ge 0}\|_{l^1}\le 1$.
The contribution of D is estimated in the following way :

$$
\begin{aligned}
N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} D P_N u\Big| &= N^{2s}\Big|\sum_{1\ll N_3\lesssim N_1\ll N}\int_{]0,t[\times\mathbb{R}} P_N\Big(e_{N_1} u_{\sim N}\partial_x u_{N_3}\Big)P_N u\Big| \\
&\lesssim \int_0^t \|u_N\|_{H^s}\|u_{\sim N}\|_{H^s}\sum_{1\le N_1\ll N}\|e_{N_1}\|_{L^\infty_x}\sum_{N_3\lesssim N_1} N_3 N_3^{0-}\|u_{N_3}\|_{H^{\frac12+}} \\
&\lesssim \|u_N\|^2_{L^2_T H^s}\|e\|_{L^\infty_T C^1_*}\|u\|_{L^\infty_T H^{\frac12+}} \\
&\lesssim \delta_N\, T^{1/2}\|e\|_{L^\infty_T C^1_*}\|u\|_{L^\infty_T H^{\frac12+}}\|u\|^2_{L^\infty_T H^s} \qquad (4.27)
\end{aligned}
$$

d1

with $\|(\delta_{2^j})_{j\ge 0}\|_{l^1}\le 1$.

To bound the contribution of A we use the commutator estimate (2.13) and integration by parts to get

$$\begin{aligned} N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} AP_N u\Big| &\lesssim N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} \partial_x(P_{\ll N}eP_{\lesssim 1}u)(P_Nu)^2\Big| \\ &\quad + N^{2s}\sum_{N_1\ll N, N_2\lesssim 1}\Big|\int_{]0,t[\times\mathbb{R}}[P_N, P_{\ll N}eP_{\lesssim 1}u]\tilde{P}_N u_x P_N u\Big| \\ &\lesssim TN^{2s}\|\partial_x(P_{\ll N}eP_{\lesssim 1}u)\|_{L^\infty_{Tx}}\|\tilde{P}_N u\|^2_{L^\infty_T L^2_x} \\ &\lesssim \delta_N\, T\|e\|_{L^\infty_T C^{1+}_*}\|u\|_{L^\infty_T L^2_x}\|u\|^2_{L^\infty_T H^s} \qquad (4.28) \end{aligned}$$

cco

with $\|(\delta_{2^j})_{j\ge 0}\|_{l^1}\le 1$.

To bound the contribution of E , we notice that the integral is of the form (4.2) so that we can use Lemma 4.2. We separate the contribution E_1 of the sum over $N_2\sim N_3\gtrsim N$ and the contribution E_2 of the sum over $N_2\sim N\gg N_3$. For the first contribution, Lemma 4.2 leads to

$$\begin{aligned} &N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} E_1 P_N u\Big| \\ &\lesssim \sum_{N_2\gtrsim N}(\|e\|_{L^\infty_{Tx}}+\|e_t\|_{L^\infty_{Tx}})\Big[\|P_Nu\|_{L^\infty_T L^2_x}\|P_{\sim N_2}u\|_{L^2_T H^s}\|P_{\sim N_2}u\|_{X^{s-1,1}_T} \\ &\quad + T^{\frac{1}{16}}N_2^{-\frac14}\Big(\|P_Nu\|_{X^{-1,1}_T}+\|P_Nu\|_{L^\infty_T L^2_x}\Big)\|P_{\sim N_2}u\|^2_{L^\infty_T H^s} \\ &\quad + T^{\frac{1}{16}}N_2^{-\frac14}\Big(\|P_{\sim N_2}u\|_{X^{s-1,1}_T}+\|P_{\sim N_2}u\|_{L^\infty_T H^s}\Big)\|P_{\sim N_2}u\|_{L^\infty_T H^s}\|P_Nu\|_{L^\infty_T L^2_x}\Big] \\ &\lesssim N^{-(0+)}(\|e\|_{L^\infty_{Tx}}+\|e_t\|_{L^\infty_{Tx}})\Big(\|u\|_{L^\infty_T H^{0+}}\|u\|_{X^{s-1,1}_T}\|u\|_{L^2_T H^s} \\ &\qquad\qquad + T^{\frac{1}{16}}\|u\|_{Y^0_T}\|u\|^2_{Y^s_T}\Big) \\ &\lesssim T^{\frac{1}{16}}N^{-(0+)}(\|e\|_{L^\infty_{Tx}}+\|e_t\|_{L^\infty_{Tx}})\|u\|_{Y^{0+}_T}\|u\|^2_{Y^s_T}\,. \qquad (4.29) \end{aligned}$$

E1

In the same way Lemma 4.2 leads to

$$\begin{aligned} &N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} E_2 P_N u\Big| \\ &\lesssim \sum_{1\ll N_3\ll N}(\|e\|_{L^\infty_{Tx}}+\|e_t\|_{L^\infty_{Tx}})\Big[\|P_{N_3}u\|_{L^\infty_T L^2_x}\|P_{\sim N}u\|_{L^2_T H^s}\|P_{\sim N}u\|_{X^{s-1,1}_T} \\ &\qquad + T^{\frac{1}{16}}N^{-\frac14}\|u\|_{Y^0_T}\|u\|^2_{Y^s_T}\Big] \\ &\lesssim \delta_N\,(\|e\|_{L^\infty_{Tx}}+\|e_t\|_{L^\infty_{Tx}})\|u\|_{Y^{0+}_T}\Big(\|u\|_{X^{s-1,1}_T}\|u\|_{L^2_T H^s}+T^{\frac{1}{16}}\|u\|^2_{Y^s_T}\Big) \\ &\lesssim T^{\frac{1}{16}}\delta_N\,(\|e\|_{L^\infty_{Tx}}+\|e_t\|_{L^\infty_{Tx}})\|u\|_{Y^{0+}_T}\|u\|^2_{Y^s_T}\,, \qquad (4.30) \end{aligned}$$

E2

with $\|(\delta_{2^j})_{j\ge 0}\|_{l^1}\le 1$.

Finally we rewrite B as

$$\begin{aligned} B &= \sum_{1\ll N_2\ll N} P_N\Big(P_{\ll N_2}e\,u_{N_2}\tilde{P}_N(u_x)\Big) + \sum_{1\ll N_2\ll N} P_N\Big(P_{\ll N}P_{\gtrsim N_2}e\,u_{N_2}\tilde{P}_N(u_x)\Big) \\ &= B_1 + B_2\,. \qquad (4.31) \end{aligned}$$

We notice that the integral in the contribution of B_1 is of the form of (4.2) with $N_3\sim N_4\gtrsim N_2$ and thus using again Lemma 4.2, we get exactly the same estimate as for D_2.

To bound the contribution of B_2 we use integration by parts and the commutator estimate (2.13) and proceed as in (4.28) to get

$$\begin{aligned}
N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} B_2 P_N u\Big| &\lesssim N^{2s}\sum_{1\ll N_2\ll N}\Big|\int_{]0,t[\times\mathbb{R}}\partial_x\Big(P_{\ll N}P_{\gtrsim N_2}e\,u_{N_2}\Big)(P_N u)^2\Big| \\
&\quad + N^{2s}\sum_{1\ll N_2\ll N}\Big|\int_{]0,t[\times\mathbb{R}}[P_N, P_{\ll N}P_{\gtrsim N_2}e\,u_{N_2}]\tilde{P}_N u_x P_N u\Big| \\
&\lesssim T\sum_{1\ll N_2\ll N} N^{2s}\|\partial_x(P_{\ll N}P_{\gtrsim N_2}e\,u_{N_2})\|_{L^\infty_{Tx}}\|\tilde{P}_N u\|^2_{L^\infty_T L^2_x} \\
&\lesssim \delta_N\, T\|e\|_{L^\infty_T C^{1+}_*}\|u\|_{L^\infty_T H^{\frac12+}}\|u\|^2_{L^\infty_T H^s} \qquad (4.32)
\end{aligned}$$

cco2

with $\|(\delta_{2^j})_{j\ge 0}\|_{l^1}\le 1$.

• *Contribution of $\partial_x P_N(bu_x)$.* This term being linear, we will give an estimate for $s>-1/2$. Integrating by parts, the contribution of this term can be rewritten as :

$$\langle N\rangle^{2s}\int_{]0,t[\times\mathbb{R}}\partial_x P_N(bu_x)P_N u = -\langle N\rangle^{2s}\int_{]0,t[\times\mathbb{R}} P_N(bu_x)P_N u_x$$

For $1\le N\lesssim 1$, it then holds

$$\begin{aligned}
\langle N\rangle^{2s}\Big|\int_{]0,t[\times\mathbb{R}} P_N(bu_x)P_N u_x\Big| \\
&\lesssim \langle N\rangle^{2s}\Big|\int_{]0,t[\times\mathbb{R}} P_N(\tilde{P}_N b\,\partial_x u_{\ll N})P_N u_x\Big| \\
&\quad + \langle N\rangle^{2s}\Big|\sum_{N_1\gtrsim N}\int_{]0,t[\times\mathbb{R}}\tilde{P}_{N_1} b\,\partial_x u_{N_1})P_N u_x\Big| \\
&\lesssim T\|b\|_{L^\infty_T C^0_*}\|u\|^2_{L^\infty_T L^2_x} + \|u\|_{L^\infty_T L^2_x}\int_0^t\sum_{N_1\gtrsim N} N_1\|b_{N_1}\|_{L^\infty_x}\|u_{N_1}\|_{L^2_x} \\
&\lesssim T\|b\|_{L^\infty_T C^1_*}\|u\|^2_{L^\infty_T L^2_x} \qquad (4.33)
\end{aligned}$$

which is acceptable. For $N\gg 1$, we decompose this term as

$$\begin{aligned}
\langle N\rangle^{2s}&\int_{]0,t[\times\mathbb{R}}\partial_x P_N(bu_x)P_N u \\
&= -\langle N\rangle^{2s}\int_{]0,t[\times\mathbb{R}} b\,(P_N u_x)^2 - \langle N\rangle^{2s}\int_{]0,t[\times\mathbb{R}}[P_N, b]u_x P_N u_x \qquad (4.34)
\end{aligned}$$

bb

The first term of the right-hand side is non positive and will give us an estimate on the $L^2_{[b]}(0,T;H^s)$-semi norm of u. Note that the contribution of the low frequency part of u, $N\lesssim 1$, to this semi norm is easily estimated by

$$\sum_{1\le N\lesssim 1}\langle N\rangle^{2s}\int_{]0,t[\times\mathbb{R}} b(P_N u_x)^2 \lesssim \|b\|_{L^\infty_{Tx}}\|u\|^2_{L^\infty_T H^s}\ . \qquad (4.35)$$

bba

To control the second term of the right-hand side, we perform a frequency decomposition of b in the following way :

$$\begin{aligned}
N^{2s}\int_{]0,t[\times\mathbb{R}}[P_N, b]u_x P_N u_x &= N^{2s}\int_{]0,t[\times\mathbb{R}}[P_N, b_{\gtrsim N}]u_x P_N u_x \\
&\quad + N^{2s}\int_{]0,t[\times\mathbb{R}}[P_N, b_{\ll N}]u_x P_N u_x \\
&= A + B\ . \qquad (4.36)
\end{aligned}$$

A is easily estimated by

$$
\begin{aligned}
|A| &\le N^{2s}\sum_{N_1\sim N}\Big|\int_{]0,t[\times\mathbb{R}}[P_N,b_{N_1}]P_{\lesssim N}u_xP_Nu_x\Big| \\
&\quad + N^{2s}\sum_{N_1\gg N}\Big|\int_{]0,t[\times\mathbb{R}}P_N(b_{N_1}\tilde{P}_{N_1}u_x)P_Nu_x\Big| \\
&\lesssim TN^{s+1}N^{0\vee 1-s}\|b_{\sim N}\|_{L^\infty_{Tx}}\|u_x\|^2_{L^\infty_T H^{s-1}} \\
&\quad + N^{s+1}\|u_x\|_{L^2_T H^{s-1}}\sum_{N_1\gg N}N_1^{-s-1}\|P_{N_1}b_x\|_{L^\infty_T C^1_*}\|\tilde{P}_{N_1}u_x\|_{L^2_T H^{s-1}} \\
&\lesssim \delta_N T\|b_x\|_{L^\infty_T C^{s\vee 1}_*}\|u\|^2_{L^\infty_T H^s} \qquad (4.37)
\end{aligned}
$$

that is acceptable. Finally applying (2.15) and (2.14) we easily obtain

$$
|B|\lesssim T\|b_{xx}\|_{L^\infty_{Tx}}\|\tilde{P}_Nu\|^2_{L^2_T H^s}\lesssim \delta_N T\|b_{xx}\|_{L^\infty_{Tx}}\|u\|^2_{L^\infty_T H^s} \qquad (4.38)
$$ estfin

Gathering (4.18)-(4.38), (4.16) follows. □

4.3. Estimate in $H^{s-1}(\mathbb{R})$ on the difference of two solutions.

prodif **Proposition 4.2.** *Let $0<T<1$ and $u,v\in Y^s_T$ with $s>1/2$ be two solutions to (1.2) associated with two initial data $u_0,v_0\in H^s(\mathbb{R})$. Then it holds*

$$
\|u-v\|^2_{L^\infty_T H^{s-1}}+\|u-v\|^2_{L^2_{[b]}(]0,T[;H^s)}\lesssim \|u_0-v_0\|^2_{H^{s-1}}+CT^{\frac{1}{16}}\|u+v\|_{Y^s_T}\|u-v\|^2_{Y^{s-1}_T}\,. \qquad (4.39)
$$ estdiffHsregular

with

$$
C=C\Big(s,\|b\|_{L^\infty_T C^{2\vee s}_*},\|c\|_{L^\infty_T C^{(1\vee(s-1)\vee(2-s))+}_*},\|d\|_{L^\infty_T C^{|s-1|+}_*},\|e\|_{L^\infty_T C^{(\frac{3}{2}\vee(s+\frac{1}{2}))+}_*}\Big)
$$

Proof. The difference $w=u-v$ satisfies

$$
w_t+w_{3x}-bw_{2x}+cw_x+dw=\frac{1}{2}e\partial_x(zw)+fzw \qquad (4.40)
$$ eq-diff

where $z=u+v$. We proceed as in the proof of the preceding proposition by applying the operator P_N, with $N\in 2^{\mathbb{N}}$, to the above equation, taking the L^2_x scalar product with P_Nw, multiplying by $\langle N\rangle^{2(s-1)}$ and integrating on $]0,t[$ with $0<t<T$. Clearly the terms coming from the linear part of (1.2) (i.e. the term where z is not involves) may be treated by the estimates established in the proof of the preceding proposition. They lead to

$$
\langle N\rangle^{2(s-1)}\Big|\int_{]0,t[\times\mathbb{R}}P_N(dw)P_Nw\Big|\lesssim T\delta_N\,\|d\|_{L^\infty_T C^{|s-1|+}_*}\|w\|^2_{L^\infty_T H^{s-1}} \qquad (4.41)
$$

$$
\begin{aligned}
&\langle N\rangle^{2(s-1)}\Big|\int_0^t\int_{\mathbb{R}}\Big(P_N((b_x+c)w_x)\Big)P_Nw\Big| \\
&\qquad\lesssim T\delta_N\left(\|b_x\|_{L^\infty_T C^{(1\vee s-1\vee 2-s)+}_*}+\|c\|_{L^\infty_T C^{(1\vee s-1\vee 2-s)+}_*}\right)\|w\|^2_{L^\infty_T H^{s-1}} \qquad (4.42)
\end{aligned}
$$

$$
\langle N\rangle^{2(s-1)}\int_{]0,t[\times\mathbb{R}}\partial_xP_N(bw_x)P_Nw\lesssim \|b\|_{L^\infty_T C^2_*}\|w\|^2_{L^\infty_T H^{s-1}} \qquad (4.43)
$$

Therefore, proceeding as in the proof of the preceding proposition, we infer that for $N\ge 1$,

$$
\begin{aligned}
\|P_Nw\|^2_{L^\infty_T H^{s-1}} &\lesssim \|P_Nw_0\|^2_{H^{s-1}}+\delta_N T\tilde{C}\|w\|^2_{L^\infty_T H^{s-1}} \\
&\quad + \sup_{t\in]0,T[}\langle N\rangle^{2(s-1)}\Big|\int_0^t\int_{\mathbb{R}}P_N\Big(e\partial_x(zw)+fzw\Big)P_Nw\Big| \qquad (4.44)
\end{aligned}
$$

with

$$\tilde{C}=\tilde{C}\Big(s,\|b\|_{L^\infty_T C^{2\vee s}_*},\|c\|_{L^\infty_T C^{(1\vee(s-1)\vee(2-s))+}_*},\|d\|_{L^\infty_T C^{|s-1|+}_*}\Big)$$

To control the contribution of $P_N(fzw)$ we use Lemma 2.4 to get

$$\begin{aligned}\langle N\rangle^{2(s-1)}\Big|\int_0^t\int_{\mathbb{R}}P_N(fzw)P_Nw\Big| &\lesssim \delta_N T\|fzw\|_{L^\infty_T H^{s-1}}\|w\|_{L^\infty_T H^{s-1}}\\ &\lesssim \delta_N T\|fz\|_{L^\infty_T H^{s}}\|w\|^2_{L^\infty_T H^{s-1}}\\ &\lesssim \delta_N T\|f\|_{L^\infty_T C^{|s|+}_*}\|z\|_{L^\infty_T H^s}\|w\|^2_{L^\infty_T H^{s-1}} \qquad (4.45)\end{aligned}$$

It remains to tackle the contribution of $P_N\Big(e\partial_x(zw)\Big)$. For $1\le N\lesssim 1$, we write $e\,\partial_x(zw)=\frac12\partial_x(ezw)-\frac12 e_x zw$ to get

$$\begin{aligned}&N^{2(s-1)}\Big|\int_{]0,t[\times\mathbb{R}}P_N\Big(e\partial_x(zw)\Big)P_Nu\Big|\\ &\qquad\lesssim T\|w\|_{L^\infty_T H^{s-1}}(\|e_xzw\|_{L^\infty_T H^{-1}}+\|ezw\|_{L^\infty_T H^{-1}})\\ &\qquad\lesssim T\|e\|_{L^\infty_T C^{\frac32+}_*}\|z\|_{L^\infty_T H^{\frac12+}}\|w\|^2_{L^\infty_T H^{-\frac12+}} \qquad (4.46)\end{aligned}$$

since $s+s-1>0$.

It thus remains to consider $N\gg 1$. Because of the lack of symmetry with respect to the estimate on u, we consider this time three different contributions.

1. *The contribution of* $P_N(P_{\gtrsim N}e\,\partial_x(zw))$. This contribution is easily estimated by

$$\begin{aligned}&N^{2(s-1)}\Big|\int_{]0,t[\times\mathbb{R}}P_N\Big(P_{\gtrsim N}e\,\partial_x(zw)\Big)P_Nw\Big|\\ &\quad= N^{2(s-1)}\Big|\int_{]0,t[\times\mathbb{R}}P_N\Big(P_{\sim N}e\,P_{\ll N}\partial_x(zw)\Big)P_Nw\Big|\\ &\quad+N^{2(s-1)}\sum_{N_1\gtrsim N}\Big|\int_{]0,t[\times\mathbb{R}}P_N\Big(P_{\sim N_1}e\,P_{N_1}\partial_x(zw)\Big)P_Nw\Big|\\ &\quad\lesssim N^{2(s-1)}\int_0^t\|P_{\sim N}e\|_{L^\infty_x}\|P_Nw\|_{L^2_x}N^{3/2}\|zw\|_{H^{-1/2}}\\ &\quad+N^{2(s-1)}\int_0^t\|P_Nw\|_{L^2_x}\sum_{N_1\gtrsim N}\|P_{\sim N_1}e\|_{L^\infty_x}N_1^{2-s}\|\partial_x(zw)\|_{H^{s-2}}\\ &\quad\lesssim \delta_N\,T\|e\|_{L^\infty_T C^{((2-s)\vee(s+\frac12))+}_*}\|z\|_{L^\infty_T H^s}\|w\|^2_{L^\infty_T H^{s-1}} \qquad (4.47)\end{aligned}$$

since for $s>1/2$, $((2-s)\vee 1\vee(s+1/2)=(2-s)\vee(s+\frac12)$. .

2. *The contribution of* $P_N(P_{\ll N}e\,z_xw)$. We rewrite this term as

$$\begin{aligned}P_N(P_{\ll N}e\,z_xw)=&P_N\Big(P_{\ll N}e\,P_{\lesssim 1}w\,\tilde{P}_Nz_x\Big)+P_N\Big(P_{\ll N}e\,\tilde{P}_Nw\,P_{\lesssim 1}z_x\Big)\\ &+\sum_{1\ll N_3,N_2}P_N\Big(P_{\ll N_2\wedge N_3}P_{\ll N}e\,w_{N_2}\partial_xz_{N_3}\Big)\\ &+\sum_{1\ll N_3,N_2}P_N\Big(P_{\gtrsim N_2\wedge N_3}P_{\ll N}e\,w_{N_2}\partial_xz_{N_3}\Big)\\ =&A+B+C+D. \qquad (4.48)\end{aligned}$$

decou

Proceeding as in the proof of (4.26), it is not too difficult to check that the contributions of A and B can be bounded by

$$N^{2(s-1)}\Big|\int_{]0,t[\times\mathbb{R}}(A+B)P_Nw\Big|\lesssim T\delta_N\|e\|_{L^\infty_{Tx}}\|z\|_{L^\infty_T H^s}\|w\|^2_{L^\infty_T H^{s-1}}\,. \qquad (4.49)$$

To bound the contribution of C , we notice that the integral is of the form (4.2) so that we can use Lemma 4.2. Proceeding as in (4.29)-(4.30) we get

$$N^{2(s-1)}\Big|\int_{]0,t[\times\mathbb{R}} CP_N w\Big| \lesssim T^{\frac{1}{16}}\delta_N\ (\|e\|_{L^\infty_{Tx}}+\|e_t\|_{L^\infty_{Tx}})\|z\|_{Y^s_T}\|w\|^2_{Y^{s-1}_T} \qquad (4.50)$$

Finally we rewrite D as

$$D=\sum_{N_2\gg 1} P_N\Big(P_{\gtrsim N_2}P_{\ll N} e\, w_{N_2}\tilde{P}_N z_x\Big)+\sum_{N_3\gg 1} P_N\Big(P_{\gtrsim N_3}P_{\ll N} e\,\tilde{P}_N w\partial_x z_{N_3}\Big)$$

Proceeding as in (4.27) we easily get

$$N^{2(s-1)}\Big|\int_{]0,t[\times\mathbb{R}} DP_N w\Big| \lesssim T\delta_N\|e\|_{L^\infty_T C^1_*}\|z\|_{L^\infty_T H^s}\|w\|_{L^\infty_T H^{s-1}} \qquad (4.51)$$

3. *The contribution of* $P_N(P_{\ll N} e\,(zw_x))$. We rewrite this term as

$$\begin{aligned}P_N(P_{\ll N} e\, zw_x) =& P_N\Big(P_{\ll N} e\,\tilde{P}_N z P_{\lesssim 1} w_x\Big)+P_N\Big(P_{\ll N} e\, P_{\lesssim 1} z\, w_x\Big)\\ &+\sum_{1\ll N_3,N_2} P_N\Big(P_{\ll N_2\wedge N_3}P_{\ll N} e\, z_{N_2}\partial_x w_{N_3}\Big)\\ &+\sum_{1\ll N_3,N_2} P_N\Big(P_{\gtrsim N_2\wedge N_3}P_{\ll N} e\, z_{N_2}\partial_x w_{N_3}\Big)\\ =&\tilde{A}+\tilde{B}+\tilde{C}+\tilde{D}. \end{aligned} \qquad (4.52)$$

Proceeding as in (4.26), we easily get

$$N^{2(s-1)}\Big|\int_{]0,t[\times\mathbb{R}} \tilde{A}P_N w\Big| \lesssim T\delta_N\|e\|_{L^\infty_{Tx}}\|z\|_{L^\infty_T H^s}\|w\|^2_{L^\infty_T H^{s-1}}\ . \qquad (4.53)$$

To bound the contribution of $\tilde{B}$ we proceed as in (4.28), integrating by parts and using the commutor estimate (2.13) to get

$$N^{2(s-1)}\Big|\int_{]0,t[\times\mathbb{R}} \tilde{B}P_N u\Big| \lesssim \delta_N\, T\|e\|_{L^\infty_T C^{1+}_*}\,\|z\|_{L^\infty_T L^2_x}\|w\|^2_{L^\infty_T H^{s-1}} \qquad (4.54)$$

Finally the contributions of $\tilde{C}$ and $\tilde{D}$ can be estimated exactly as the ones of C and D. □

Remark 4.1. *Gathering Lemma 4.3 and Propositions 4.1-4.2 we observe that sufficient hypotheses for these statements to hold are*

$$\begin{gathered} b\in L^\infty_T C^{((s+1)\vee 2)+}_*,\quad c\in L^\infty_T C^{((2-s)\vee s)+}_*,\quad d\in L^\infty_T C^{|s|+}_*\\ e\in L^\infty_T C^{((s+\frac{1}{2})\vee\frac{3}{2})+}_*,\quad e_t\in L^\infty_{Tx}\quad \text{and}\quad f\in L^\infty_T C^{|s|+}_* \end{gathered} \qquad (4.55)$$

5. Proof of Theorem 1.1

5.1. **Uniqueness.** Assume (4.55) are fulfilled and $u_0\in H^s(\mathbb{R})$ with $s>1/2$. Let u and v be two solutions of (1.2) emanating from u_0 that belong to $L^\infty_T H^s\cap L^2_{[b]}(]0,T[;H^{s+1})$ for some $T>0$. Then according to Lemma 4.3, u and v belong to Y^s_T and Proposition 4.2 together with (4.12) ensure that for any $0<T_0\le T\wedge 2$ it holds

$$\begin{aligned}&\|u-v\|^2_{L^\infty_{T_0}H^{s-1}}+\|u-v\|^2_{L^2_{[b]}(]0,T_0[;H^s)}\\ &\quad\lesssim T_0^{\frac{1}{16}}(1+\|u+v\|_{Y^s_T})^3\Big(\|u-v\|^2_{L^\infty_{T_0}H^{s-1}}+\|u-v\|^2_{L^2_{[b]}(]0,T_0[;H^s)}\Big)\ .\end{aligned}$$

This forces $u\equiv v$ on some time interval $]0,T_1[$ with $0<T_1\le T_0$. Taking now T_1 as initial time we can repeat the same argument to get that $u\equiv v$ on $]0,T\vee 2T_1[$ and a finite iteration of this argument leads to $u\equiv v$ on $]0,T[$. It is worth noticing

that in the case $b \equiv 0$, $L^\infty_T H^s \cap L^2_{[b]}(]0,T[;H^{s+1}) = L^\infty_T H^s$ and thus we get the unconditional uniqueness of (1.2) in $H^s(\mathbb{R})$ for $s > 1/2$.

5.2. **Existence.** We make use of the famous existence result of Craig-Kappeler-Strauss [7] for the general quasilinear KdV type equations :

$$u_t + F(\partial_x^3 u, \partial_x^2 u, \partial_x u, u, x, t) = 0\ . \tag{5.1}$$

eqCKS

In this paper, the following assumptions on F are made :
$F : \mathbb{R}^5 \times [0,T] \to \mathbb{R}$ is C^∞ in all its variables and satisfies

(A1) $\exists c > 0$ such that $\partial_1 F(y,x,t) \ge c > 0$ for all $y \in \mathbb{R}^4$, $x \in \mathbb{R}$ and $t \in [0,T]$.
(A2) $\partial_2 F(y,x,t) \le 0$.
(A3) All the derivatives of $F(y,x,t)$ are bounded for $x \in \mathbb{R}$, $t \in [0,T]$ and y in a bounded set.
(A4) $x^N \partial_x^j F(0,x,t)$ is bounded for all $N \ge 0$, $j \ge 0$, $x \in \mathbb{R}$ and $t \in (0,T]$.

Fixing F that satisfies (A1)-(A4), in [7] it is shown that for any $k \in \mathbb{N}$ with $k \ge 7$ and any $c_0 > 0$ there exists $T = T(c_0) > 0$ such that for any $u_0 \in H^k(\mathbb{R})$, with $\|u_0\|_{H^7} \le c_0$, the Cauchy problem associated with (5.1) has a unique local solution $u \in L^\infty(0,T;H^k(\mathbb{R}))$.

This implies that for any F satisfying (A1)-(A4) and any $u_0 \in H^k$ with $k \ge 7$, the unique solution u to (5.1) can be prolonged on a maximal time interval $[0,T^*[$ with either

$$T^* = +\infty \quad \text{or} \quad \limsup_{T \nearrow T^*} \|u\|_{L^\infty(0,T;H^7)} = +\infty\ . \tag{5.2}$$

alt

We notice that (1.2) corresponds to (5.1) with

$$F(y,x,t) = y_1 - b(t,x)y_2 + c(t,x)y_3 + d(t,x)y_4 - e(t,x)y_3y_4 - f(t,x)y_4^2$$

In particular, for any $y \in \mathbb{R}^4$, $x \in \mathbb{R}$ and $t \in [0,T]$ we have $\partial_1 F(y,x,t) = 1$ and $F(0,x,t) = 0$ which ensure that $(A1)$ and $(A4)$ are clearly fulfilled. Moreover, the hypothesis $b \ge 0$ ensures that $(A2)$ is also fulfilled. Therefore, since our coefficient functions are by hypothesis all bounded on $[0,T] \times \mathbb{R}$, it thus suffice to regularize them by convoluting in (t,x) with a smooth positive sequence of mollifiers to fulfill the assumptions (A1)-(A4).

So let the coefficient functions a, b, c, d, e, f satisfying the hypotheses of Theorem 1.1 and let $u_0 \in H^s(\mathbb{R})$ with $s > 1/2$. We first construct the solution emanating from u_0 to (1.2) with a, b, c, d, e replaced by their smooth regularizations. For this we regularize the initial datum by setting, for any $n \in \mathbb{N}^*$, $u_{0,n} = P_{\le n} u_0 \in H^\infty(\mathbb{R})$. According to the existence result of [7] there exists a sequence (T_n) with $0 < T_n < 1$ such that, for any $n \in \mathbb{N}^*$, (1.2) has a unique solution $u_n \in L^\infty(0,T_n;H^\infty(\mathbb{R}))$ emanating from $u_{0,n}$. Note that (1.2) then implies that actually $u_n \in C([0,T_n];H^\infty(\mathbb{R}))$. Now, applying (4.11) and (4.16) for u_n on $[0,T_n]$ we obtain that

$$\begin{aligned}\|u_n\|^2_{L^\infty_{T_n} H^{s_0}} &+ \|u_n\|^2_{L^2_{[b]}(]0,T_n[;H^{s_0+1})} \\ &\le \|u_0\|^2_{H^s} + C\, T_n^{\frac{1}{16}} \Big(1 + \|u\|^2_{L^\infty_{T_n} H^{s_0}} + \|u\|_{L^2_{[b]}(]0,T_n[;H^{s_0+1})}\Big)^6\end{aligned}$$

for $s_0 = \frac{1}{2}+ < s$. Using the continuity of $T \to \|u_n\|_{L^\infty(0,T;H^{s_0})} + \|u\|_{L^2_{[b]}(]0,T[;H^{s_0+1})}$ this ensures that there exists $0 < T_0 = T_0(\|u_0\|_{H^{s_0}}) < 2$ such that

$$\|u_n\|_{L^\infty(0,T_2;H^{s_0})} + \|u\|_{L^2_{[b]}(]0,T_2[;H^{s_0+1})} \le 4\|u_{0,n}\|_{H^{s_0}} \quad \text{for} \quad T_2 = T_n \wedge T_0\ .$$

Using again (4.11) and (4.16), we obtain that, for any fixed $n \ge 0$, u_n is bounded in $L^\infty_{T_2} H^7$. Therefore (5.2) ensures that u_n can be extended on $[0,T_0]$. Hence, it holds

$$\|u_n\|_{L^\infty(0,T_0;H^{s_0})} + \|u_n\|_{L^2_{[b]}(0,T_0;H^{s_0+1})} \le 4\|u_0\|_{H^{s_0}}\ .$$

Applying again (4.11) and (4.16) but at the H^s-regularity this forces

$$\|u_n\|_{L^\infty(0,T_0;H^s)} + \|u_n\|_{L^2_{[b]}(0,T_0;H^{s+1})} \lesssim \|u_0\|_{H^s} \; .$$

Note that Lemma 4.3 and Proposition 4.2 then ensure that (u_n) is a Cauchy sequence in $L^\infty(0,T_0;H^{s-1})$ and thus it is also a Cauchy sequence in $L^\infty(0,T_0;H^{\frac12+})$. Let u be the limit of u_n in $L^\infty(0,T_0;H^{\frac12+})$. From the above estimates we know that $u\in L^\infty(0,T_0;H^s)$ and it is immediat to check that u satisfies (1.2) at least in $L^\infty(0,T_0;H^{s-3})$.

Now we can pass to the limit on the coefficient functions. Since their regularizations are bounded in the function spaces appearing in Remark 4.1, we obtain the existence of a solution $u\in L^\infty(0,T_0;H^s)\cap L^2_{[b]}(0,T_0;H^{s+1})$ that is the unique one in this class on account of Subsection 5.1. Now the continuity of u with values in $H^s(\mathbb{R})$ as well as the continuity of the flow-map in $H^s(\mathbb{R})$ will follow from the Bona-Smith argument (see [6]). For any $\varphi\in H^s(\mathbb{R})$, any integer $n\ge 1$ and any $r\ge 0$, straightforward calculations in Fourier space lead to

$$\|P_{\le n}\varphi\|_{H^{s+r}_x} \lesssim n^r \|\varphi\|_{H^s_x} \quad \text{and} \quad \|\varphi - P_{\le n}\varphi\|_{H^{s-r}_x} \lesssim n^{-r}\|P_{>n}\varphi\|_{H^s_x} \; . \tag{5.3}$$ init

Let $u_0\in H^s$ with $s>1/2$ and let $T_0=T_0(\|u_0\|_{H^{\frac12+}})>0$ the associated minimum time of existence. We denote by $u_n\in L^\infty(0,T_0;H^s)$ the solution of (1.2) emanating from $u_{0,n}=P_{\le n}u_0$ and for $1\le n_1\le n_2$, we set

$$w := u_{n_1} - u_{n_2} \; .$$

Then, (4.39)-(4.12) lead to

$$\|w\|_{Y^{s-1}_{T_0}} \lesssim \|w(0)\|_{H^{s-1}} \lesssim n_1^{-1}\|P_{>n_1}u_0\|_{H^s} \; . \tag{5.4}$$ to1

Moreover, for any $r\ge 0$ and $s>1/2$ we have

$$\|u_{n_i}\|_{Y^{s+r}_{T_0}} \lesssim \|u_{0,n_i}\|_{H^{s+r}} \lesssim n_i^r \|u_0\|_{H^s} \; . \tag{5.5}$$ to2

Next, we observe that w solves the equation

$$w_t + w_{3x} - b w_{2x} + c w_x + d w = \frac12 e\partial_x(w^2) + e\partial_x(u_{n_1}w) + f w^2 + 2 f u_{n_1} w \, . \tag{5.6}$$ W

pro3 **Proposition 5.1.** *Let $0<T<1$ and $w\in Y^s_T$ with $s>1/2$ be a solution to* (5.6). *Then it holds*

$$\begin{aligned}\|w\|^2_{L^\infty_T H^s} \quad \lesssim \quad & \|w(0)\|^2_{H^s} + C\,T^{\frac{1}{16}}\Big((\|u_{n_1}\|_{Y^s_T} + \|u_{n_2}\|_{Y^s_T})\|w\|^2_{Y^s_T} \\ & + \|u_{n_1}\|_{L^\infty_T H^{s+1}}\|w\|_{L^\infty_T H^{s-1}}\|w\|_{L^\infty_T H^s}\Big) \; . \end{aligned} \tag{5.7}$$

Proof. It is a consequence of estimates derived in the proof of Propositions 4.1 and 4.2. Actually because of the loss of symetry we only have to take care of the contribution of $P_N(P_{\ll N} e\partial_x u_{N_1} w)$. We decompose this term as in (4.48) to get

$$\begin{aligned} P_N(P_{\ll N} e\,\partial_x u_{n_1} w) = & P_N\Big(P_{\ll N} e\, P_{\lesssim 1} w\, \tilde P_N \partial_x u_{n_1}\Big) + P_N\Big(P_{\ll N} e\, \tilde P_N w\, P_{\lesssim 1}\partial_x u_{n_1}\Big) \\ & + \sum_{1\ll N_3,N_2} P_N\Big(P_{\ll N_2\wedge N_3} P_{\ll N} e\, w_{N_2} P_{N_3}\partial_x u_{n_1}\Big) \\ & + \sum_{1\ll N_3,N_2} P_N\Big(P_{\gtrsim N_2\wedge N_3} P_{\ll N} e\, w_{N_2} P_{N_3}\partial_x u_{n_1}\Big) \\ = & A+B+C+D. \end{aligned} \tag{5.8}$$ decou2

The contribution of A and B can be easily estimated by

$$N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} A P_N w\Big| \lesssim T\delta_N \|e\|_{L^\infty_{Tx}} \|u_{n_1}\|_{L^\infty_T H^{s+1}} \|w\|_{L^\infty_T H^{-1/2}} \|w\|_{L^\infty_T H^s} \; . \tag{5.9}$$

and

$$N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} BP_N w\Big| \lesssim T\delta_N \|e\|_{L^\infty_{Tx}}\|u_{n_1}\|_{L^\infty_T L^2}\|w\|^2_{L^\infty_T H^s}\ . \tag{5.10}$$

To bound the contribution of C we use again Lemma 4.2 and proceed as in (4.29)-(4.30) to get

$$N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} CP_N w\Big| \lesssim T^{\frac{1}{16}}\delta_N\ (\|e\|_{L^\infty_{Tx}}+\|e_t\|_{L^\infty_{Tx}})\|u_{n_1}\|_{Y^s_T}\|w\|^2_{Y^s_T} \tag{5.11}$$

Finally we rewrite D as

$$\begin{aligned} D &= \sum_{N_2\gg 1} P_N\Big(P_{\gtrsim N_2}P_{\ll N}e\, w_{N_2}\tilde{P}_N\partial_x u_{n_1}\Big) + \sum_{N_3\gg 1} P_N\Big(P_{\gtrsim N_3}P_{\ll N}e\,\tilde{P}_N w\partial_x u_{n_1}\Big)\\ &= D_1 + D_2 \end{aligned}$$

We easily get

$$\begin{aligned} N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} D_1 P_N w\Big| &\lesssim \delta_N\int_0^t \sum_{N_2\gg 1}\|P_{\gtrsim N_2}e\|_{L^\infty_x}\|w_{N_2}\|_{L^\infty_x}\|u_{n_1}\|_{H^{s+1}}\|w\|_{H^s}\\ &\lesssim \delta_N\int_0^t\sum_{N_2\gg 1}\|P_{\gtrsim N_2}e\|_{L^\infty_x}N_2^{\frac{3}{2}-s}\|w_{N_2}\|_{H^{s-1}}\|u_{n_1}\|_{H^{s+1}}\|w\|_{H^s}\\ &\lesssim \delta_N T\|e\|_{L^\infty_x C^1_*}\|u_{n_1}\|_{L^\infty_T H^{s+1}}\|w\|_{L^\infty_T H^{s-1}}\|w\|_{L^\infty_T H^s} \end{aligned} \tag{5.12}$$

since $s>1/2$. In the same way we get

$$\begin{aligned} N^{2s}\Big|\int_{]0,t[\times\mathbb{R}} D_2 P_N w\Big| &\lesssim \delta_N\int_0^t\sum_{N_3\gg 1}\|P_{\gtrsim N_3}e\|_{L^\infty_x}\|\partial_x P_{N_3}u_{n_1}\|_{L^\infty_x}\|w\|^2_{H^s}\\ &\lesssim \delta_N\int_0^t\sum_{N_3\gg 1}\|P_{\gtrsim N_3}e\|_{L^\infty_x}N_3^{\frac{3}{2}-s}\|P_{N_3}u_{n_1}\|_{H^s}\|w\|^2_{H^s}\\ &\lesssim \delta_N T\|e\|_{L^\infty_x C^1_*}\|u\|_{L^\infty_T H^s}\|w\|^2_{L^\infty_T H^s} \end{aligned} \tag{5.13}$$

that completes the proof of the proposition. □

Combining (4.12) with (5.7) and (5.5) we get for $0<T<T_0$,

$$\begin{aligned} \|w\|^2_{Y^s_T} &\lesssim \|w(0)\|^2_{H^s} + T^{\frac{1}{16}}\Big[\|u_0\|_{H^s}\|w\|^2_{Y^s_T}\\ &\quad + n_1\|u_0\|_{H^s}\|w\|_{Y^s_T}\|w\|_{Y^{s-1}_T}\Big]\ . \end{aligned}$$

Therefore, for $T>0$ small enough, (5.4) leads to

$$\begin{aligned} \|w\|^2_{Y^s_{T_0}} &\lesssim \|w(0)\|^2_{H^s} + n_1^2\|w\|^2_{Y^{s-1}_{T_0}}\\ &\lesssim \|P_{>n_1}u_0\|^2_{H^s}\to 0 \text{ as } n_1\to 0\,. \end{aligned} \tag{5.14}$$

This shows that $\{u_n\}$ is a Cauchy sequence in $C([0,T];H^s)$ and thus $\{u_n\}$ converges in $C([0,T];H^s)$ to a solution of (1.2) emanating from u_0.Then, the uniqueness result ensures that $u\in C([0,T];H^s)$. Repeating this argument with $u(T)$ as initial data we obtain that $u\in C([0,T_1];H^s)$ with $T_1=\max(2T,T_0)$. This leads to $u\in C([0,T_0];H^s)$ after finite number of repetitions.

cont **Continuity of the flow map.** Let now $\{u_0^k\}\subset H^s(\mathbb{R})$ be such that $u_0^k\to u_0$ in $H^s(\mathbb{R})$. We want to prove that the emanating solution u^k tends to u in $C([0,T_0];H^s)$. By the triangle inequality, for k large enough,

$$\|u-u^k\|_{L^\infty_{T_0}H^s}\le\|u-u_n\|_{L^\infty_{T_0}H^s}+\|u_n-u_n^k\|_{L^\infty_{T_0}H^s}+\|u_n^k-u^k\|_{L^\infty_{T_0}H^s}\quad .$$

Using the estimate (5.14) on the solution to (5.6) we first infer that

$$\|u-u_n\|_{Y^s_{T_0}}+\|u^k-u^k_n\|_{Y^s_{T_0}}\lesssim \|P_{>n}u_0\|_{H^s}+\|P_{>n}u^k_0\|_{H^s}$$

and thus

$$\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\Bigl(\|u-u^k\|_{L^\infty_{T_0}H^s}+\|u^k-u^k_n\|_{L^\infty_{T_0}H^s}\Bigr)=0\,. \tag{5.15}$$ kak1

Next, we notice that (4.39)-(4.12) ensure that

$$\|u_n-u^k_n\|_{Y^{s-1}_{T_0}}\lesssim \|u_{0,n}-u^k_{0,n}\|_{H^{s-1}}$$

and thus (5.14) and (5.4) lead to

$$\begin{aligned}\|u_n-u^k_n\|^2_{Y^s_{T_0}} &\lesssim \|u_{0,n}-u^k_{0,n}\|^2_{H^s}+n^2\|u_{0,n}-u^k_{0,n}\|^2_{H^{s-1}}\\ &\lesssim \|u_0-u^k_0\|^2_{H^s}(1+n^2)\,. \end{aligned}\tag{5.16}$$

Combining (5.15) and (5.16), we obtain the continuity of the flow map.

6. Appendix

6.1. **Proof of Lemma 2.5.** We start by proving (2.13). Let $N>0$. We follow [10]. By Plancherel and the mean-value theorem,

$$\begin{aligned}\Bigl|([P_N,P_{\ll N}f]g)(x)\Bigr| &= \Bigl|([P_N,P_{\ll N}f]\tilde{P}_Ng)(x)\Bigr|\\ &=\Bigl|\int_{\mathbb{R}}\mathcal{F}_x^{-1}(\varphi_N)(x-y)P_{\ll N}f(y)\tilde{P}_Ng(y)\,dy\\ &\qquad -\int_{\mathbb{R}}P_{\ll N}f(x)\mathcal{F}_x^{-1}(\varphi_N)(x-y)\tilde{P}_Ng(y)\,dy\Bigr|\\ &=\Bigl|\int_{\mathbb{R}}(P_{\ll N}f(y)-P_{\ll N}f(x))N\mathcal{F}_x^{-1}(\varphi)(N(x-y))\tilde{P}_Ng(y)\,dy\Bigr|\\ &\le \|P_{\ll N}f_x\|_{L^\infty_x}\int_{\mathbb{R}}N|x-y||\mathcal{F}_x^{-1}(\varphi)(N(x-y))||\tilde{P}_Ng(y)|\,dy\end{aligned}$$

Therefore, since $N|\cdot||\mathcal{F}_x^{-1}(\varphi)(N\cdot)|=|\mathcal{F}_x^{-1}(\varphi')(N\cdot)|$ we deduce from Young's convolution inequalities that

$$\Bigl\|[P_N,P_{\ll N}f]g)\|_{L^2}\lesssim N^{-1}\|P_{\ll N}f_x\|_{L^\infty_x}\|\tilde{P}_Ng\|_{L^2}\,.$$

To prove (2.14) we proceed in the same way. We first notice that

$$\begin{aligned}I_N(x)&=([P_N,[P_N,P_{\ll N}f]]g)(x)=([P_N,[P_N,P_{\ll N}f]]\tilde{P}_Ng)(x)\\ &=\int_{\mathbb{R}^2}\mathcal{F}_x^{-1}(\varphi_N)(x-y)\mathcal{F}_x^{-1}(\varphi_N)(y-z)\Bigl(P_{\ll N}f(z)-P_{\ll N}f(y)\Bigr)\tilde{P}_Ng(z)\,dy\,dz\\ &-\int_{\mathbb{R}^2}\mathcal{F}_x^{-1}(\varphi_N)(x-y)\mathcal{F}_x^{-1}(\varphi_N)(y-z)\Bigl(P_{\ll N}f(y)-P_{\ll N}f(x)\Bigr)\tilde{P}_Ng(z)\,dy\,dz\\ &=\int_{\mathbb{R}^2}\mathcal{F}_x^{-1}(\varphi_N)(x-y)\mathcal{F}_x^{-1}(\varphi_N)(y-z)(z-y)P_{\ll N}f_x(\alpha_{y,z})\tilde{P}_Ng(z)\,dy\,dz\\ &-\int_{\mathbb{R}^2}\mathcal{F}_x^{-1}(\varphi_N)(x-y)\mathcal{F}_x^{-1}(\varphi_N)(y-z)(y-x)P_{\ll N}f_x(\alpha_{y,x})\Bigr)\tilde{P}_Ng(z)\,dy\,dz\end{aligned}$$

with $\alpha_{y,z}\in[y,z]$ and $\alpha_{y,x}\in[y,x]$. Performing the change of variable $\theta=x+z-y$ in the last integral we get

$$\begin{aligned}I_N(x)=\int_{\mathbb{R}^2}\mathcal{F}_x^{-1}(\varphi_N)(x-y)\mathcal{F}_x^{-1}(\varphi_N)(y-z)(z-y)\Bigl((P_{\ll N}f_x(\alpha_{y,z})\\ -P_{\ll N}f_x(\alpha_{x,x+z-y})\Bigr)\tilde{P}_Ng(z)\,dy\,dz\end{aligned}$$

with $\alpha_{x,x+z-y} \in [x, x+z-y]$. Finally, noticing that

$$|\alpha_{y,z} - \alpha_{x,x+z-y}| \leq \max(|x-y|, |x-z|, x+z-2y|) \leq 2\max(|x-y|, |y-z|)$$

and using again the mean-value theorem we eventually obtain

$$|I_N| \leq 2\|P_{\ll N} f_{xx}\|_{L^\infty_x} \Big[\int_{\mathbb{R}^2} |z-y|^2 N^2 |\mathcal{F}_x^{-1}(\varphi)(N(z-y))||\mathcal{F}_x^{-1}(\varphi)(N(x-y))||\tilde{P}_N g(z)|\, dy\, dz + \int_{\mathbb{R}^2} |x-y| N |\mathcal{F}_x^{-1}(\varphi)(N(x-y))||z-y| N |\mathcal{F}_x^{-1}(\varphi)(N(z-y))||\tilde{P}_N g(z)|\, dy\, dz \Big]$$

which yields to the desired result for the same reasons as above.

Finally, to prove (2.15) we first use Parseval identity and the fact that g is real-valued to obtain

$$\int_{\mathbb{R}} [P_N, P_{\ll N} f] g \, P_N g = \int_{\mathbb{R}^2} (\varphi_N(\xi_1+\xi_2) - \varphi_N(\xi_2)) \widehat{P_{\ll N} f}(\xi_1) \hat{g}(\xi_2) \varphi_N(\xi_1+\xi_2) \hat{g}(-\xi_1-\xi_2)\, d\xi_1\, d\xi_2 \, .$$

Performing the change of variable $(\breve{\xi}_1, \breve{\xi}_2) = (\xi_1, -\xi_1-\xi_2)$ and recalling that φ_N is an even real valued function we then get

$$\begin{aligned} \int_{\mathbb{R}} [P_N, P_{\ll N} f] g \, P_N g &= \int_{\mathbb{R}^2} (\varphi_N(\breve{\xi}_2) - \varphi_N(\breve{\xi}_1+\breve{\xi}_2)) \widehat{P_{\ll N} f}(\breve{\xi}_1) \hat{g}(-\breve{\xi}_1-\breve{\xi}_2) \varphi_N(\breve{\xi}_2) \hat{g}(\breve{\xi}_2)\, d\breve{\xi}_1\, d\breve{\xi}_2 \\ &= -\int_{\mathbb{R}} [P_N, P_{\ll N} f] g \, P_N g \\ &\quad + \int_{\mathbb{R}^2} (\varphi_N(\breve{\xi}_2) - \varphi_N(\breve{\xi}_1+\breve{\xi}_2))^2 \widehat{P_{\ll N} f}(\breve{\xi}_1) \hat{g}(-\breve{\xi}_1-\breve{\xi}_2) \hat{g}(\breve{\xi}_2)\, d\breve{\xi}_1\, d\breve{\xi}_2 \\ &= -\int_{\mathbb{R}} [P_N, P_{\ll N} f] g \, P_N g + \int_{\mathbb{R}} \Big[P_N, [P_N, P_{\ll N} f]\Big] g \, g \, . \end{aligned}$$

This yields (2.15) by noticing that g can be replaced by $\tilde{P}_N g$ without changing the value of $\int_{\mathbb{R}} [P_N, P_{\ll N} f] g \, P_N g$.

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