Probabilistic limit theorems via the operator perturbation method, under optimal moment assumptions

Pène Françoise

PROBABILISTIC LIMIT THEOREMS VIA THE OPERATOR PERTURBATION METHOD, UNDER OPTIMAL MOMENT ASSUMPTIONS

FRANÇOISE PÈNE

Contents

Date: May 28, 2021.

2000 *Mathematics Subject Classification.* 60J05-60F05.

Key words and phrases. Markov chains, Markov random walks, central limit theorems, local limit theorems, Edgeworth expansion, spectral method.

ANR project MALIN: Projet-ANR-16-CE93-0003.

Abstract. The Nagaev-Guivarc'h operator perturbation method is well known to provide various probabilistic limit theorems for Markov random walks. A natural conjecture is that this method should provide these limit theorems under the same moment assumptions as the optimal ones in the case of sums of independent and identically distributed random variables. In the past decades, assumptions have been aweaken, without achieving fully this purpose (achieving it either with the help of an extra proof of the central limit theorem, or with an additional ε in the moment assumptions). The aim of this article is to give a positive answer to this conjecture via the Keller-Liverani theorem. We present here an approach allowing the establishment of limit theorems (including higher order ones) under optimal moment assumptions. Our method is based on Taylor expansions obtained via the perturbation operator method, combined with a new weak compacity argument without the use of any other extra tool (such as Martingale decomposition method, etc.).

1. Introduction

Let $(X_n)_{n\geq 0}$ be a Markov chain with values in Ω, with transition opertor P and with stationary measure μ and $f : \Omega\times\Omega\times E \to \mathbb{R}$ be a measurable function. Let ν be the distribution of X_0 (i.e. the initial distribution of the Markov chain). We set $\mathcal{P}_\nu$ for the Markov distribution initial probability measure ν. We are interested in the study of the Markov random walk $(S_n)_{n\geq 1}$ given by[1]

$$S_n := \sum_{k=1}^{n} Y_k \quad \text{with } Y_k := f(X_{k-1}X_k, Z_k)\,,$$

where Z_i are independent and identically distributed (i.i.d.) random variables independent of $(X_k)_k$ and with common distribution $\mathbf{P}$. We assume moreover throughout this article that Y_1 is centered with respect to $\mathcal{P}_\mu\otimes\mathbf{P}$. Our goal is to establish probabilistic limit theorems for $(S_n)_{n\geq 1}$ under moment assumptions known to be optimal in the case of sums of independent and identically distributed (i.i.d.) random variables. Recall that if $(Y_k)_{k\geq 1}$ were a sequence of centered i.i.d. random variables:

- If $Y_1\in\mathbb{L}^2(\mu)$, then the usual central limit theorem (CLT) holds true : $(S_n/\sqrt{n})_{n\geq 1}$ converges in distribution to a centered Gaussian random variable $\mathcal{W}$ with variance $\mathbb{E}[Y_1^2]$, with density $h_{\mathcal{W}}$.
- If $Y_1\in\mathbb{L}^2(\mu)$ is $\mathbb{Z}$-valued and satisfies some non-lattice condition, then the usual local limit theorem (LLT) holds true: $\mathbb{P}(S_n=k)\sim h_{\mathcal{W}}(k/\sqrt{n})n^{-\frac12}$, uniformly in $k\in\mathbb{Z}$.
- If $Y_1\in\mathbb{L}^3(\mu)$ and satisfies some non-lattice condition, then there is a first order Edgeworth expansion: $\mathbb{P}\left(\frac{S_n}{\sqrt{n}}\leq x\right)=\mathbb{P}\left(\mathcal{W}\leq x\right)+\frac{\mathfrak{B}_1(x)}{n^{\frac12}}h_{\mathcal{W}}(x)+o(n^{-\frac12})$, uniformly in $x\in\mathbb{R}$, where $\mathfrak{B}_1$ is a polynomial function.
- If $Y_1\in\mathbb{L}^{r+2}(\mu)$ is $\mathbb{Z}$-valued and satisfies some non-lattice condition, then there is an expansion of order r in the LLT: $\mathbb{P}\left(S_n=k\right)=h_{\mathcal{W}}(0)n^{-\frac12}+\sum_{j=1}^{\lfloor r/2\rfloor}\frac{A_j}{n^{\frac12+j}}+o(n^{-\frac{1+r}{2}})$.
- If $Y_1\in\mathbb{L}^{r+2}(\mu)$ satisfies some non-lattice condition as well as some diophantine condition of the form $\mathbb{E}[e^{isY_1}]<e^{-\widehat{C}|s|^{-\alpha}}$ for all s large enough (see [8]), then there is an Edgeworth expansion of order r: $\mathbb{P}\left(\frac{S_n}{\sqrt{n}}\leq x\right)=\mathbb{P}\left(\mathcal{W}\leq x\right)+h_{\mathcal{W}}(x)\sum_{j=1}^{r}\frac{\mathfrak{B}_j(x)}{n^{\frac{j}{2}}}+o(n^{-\frac{r}{2}})$, uniformly in $x\in\mathbb{R}$, for some polynomial functions $\mathfrak{B}_j$.

[1]We explain in appendix that this notion includes the discrete-time Markov additive processes considered in [24].

We will establish such results for Markov random walks. We will also investigate other results such as

- convergence to a stable distribution in the multi-dimensional setting,
- local limit theorem for observables with values in $\mathbb{Z}^d$, including the case of convergence to stable distributions,
- expansions in the LLT for non $\mathbb{Z}$-valued random variables.

We will state general results in the context of geometrically ergodic Markov chains and will illustrate all of them on a toy model of Knudsen gas. Let us recall that the Markov chain $(X_n)_{n\geq 0}$ (or equivalently its transition operator P) is said to be geometrically ergodic on some complex Banach space of functions $\mathcal{B}_1$ if its transition operator P satisfies

$$\exists\vartheta\in]0,1[,\quad \forall n\in\mathbb{N},\quad \|P^n-\mathbb{E}_\mu[\cdot]\|_{\mathcal{L}(\mathcal{B})}=\mathcal{O}(\vartheta^n)\,. \tag{1}$$

As a consequence of our general results, we will prove limit theorems under optimal moment assumptions. We illustrate our results on classical families of Markov random walks (for ρ-mixing, V-geometrically ergodic Markov chains or Lipschitz iterative Markov chains) and obtain in particular the following result.

Theorem 1.1. *Let $m\geq 2$ and $\kappa>0$. Assume one of the following conditions holds true:*

- *either P is ρ-mixing, $\mu=\nu$ and $Y_1\in\mathbb{L}^m(\mathcal{P}_\mu\otimes\mathbf{P})$ centered;*
- *or there exists $\vartheta\in]0,1[$, $C>0$ and an unbounded continuous function $V:\Omega\to[1,+\infty[$ such that $\mathbb{E}_\mu[V]+\mathbb{E}_\nu[V^{\frac{\kappa}{m}}]<\infty$ and $\left\|\frac{P^n(\cdot)-\mathbb{E}_\mu[\cdot]}{V}\right\|_\infty\leq C\vartheta^n\|\cdot/V\|_\infty$. Assume $\sup_{(x,y)\in\Omega^2}\mathbf{E}\left[|f(x,y,Z_1)|^m\right]/(V(x)+V(y))<\infty$, Y_1 is centered with respect to $\mathcal{P}_\mu\otimes\mathbf{P}$.*
- *or (Ω,d) is a non-compact metric space, $P(g)=\mathbb{E}[g(F(x,\theta))]$ with θ a random variable and[2] with $F(\cdot,\theta):\Omega\to\Omega$ strictly contracting, $f:\Omega\to\mathbb{R}$ is Lipschitz continuous (we consider here $Y_n=f(X_n)$) and $\mathbb{E}\left[d\left(x_0,F(x_0,\theta)\right)^{(r+1)}\right]+\mathbb{E}_\nu[d(x_0,\cdot)^\kappa]<\infty$ (for some fix $x_0\in\Omega$).*

Then $(S_n/\sqrt{n})_{n\geq1}$ converges in distribution to a centered Gaussian random variable with variance matrix $\sum_{n\in\mathbb{Z}}\mathcal{C}ov_{\mu\otimes\mathbf{P}^{\otimes\mathbb{N}}}\left(Y_1,Y_{|n|+1}\right)$.[3] Assume moreover that f is non-lattice (either in $\mathbb{Z}^d$ if we precise that f takes its values in $\mathbb{Z}^d$ or in $\mathbb{R}^d$ otherwise). Then

(i) *If f is $\mathbb{Z}^d$-valued, then $(S_n)_{n\geq1}$ satisfies the local limit theorem (LLT).*
(ii) *if $d=1$, $\kappa\geq m-1$, and $m\geq 3$, then $(S_n)_{n\geq1}$ satisfies a first order Edgeworth expansion.*
(iii) *if $d=1$, $\kappa\geq m-1$, $m\geq r+2$ and if f is $\mathbb{Z}$-valued, then $(S_n)_{n\geq1}$ satisfies a LLT with expansion of order r.*
(iv) *if $d=1$, $\kappa\geq m-1$ and $m\geq r+2$, and if some diophantine condition is satisfied, then there is an Edgeworth expansion of order r and also an expansion of order r in the LLT.*

A simple case in which the diophantine condition holds true is if $\mathbb{E}[e^{isY_1}|X_0,X_1]<e^{-\widehat{C}|s|^{-\alpha}}$ for all s large enough and for some $\alpha>0$ (with the additional assumption that $r<\alpha^{-1}+\frac{1}{2}$ for r-order Edgeworth expansion). Let us indicate that other examples in compact situations are given in [9].

This result will appear as an application of general results for Markov random walks, that are consequences of Taylor expansions for eigenprojectors and for the dominating eigenvalue of

[2] Actually, we state a much more general result (Theorem 7.8) under weaker assumptions on F and f.

[3] In this manuscript, given two d-dimensional square integrable random variables $A=(A_1,...,1_d)$ and $B=(B_1,...,B_d)$, we write $\mathcal{C}ov(A,B)$ for the symmetric matrix $\left(\frac{\mathcal{C}ov(A_i,B_j)+\mathcal{C}ov(A_j,B_i)}{2}\right)_{i,j=1,...,d}$.

the operators P_t obtained from the transition operator P by Fourier perturbation (P and P_t acting on some complex Banach space $\mathcal{B}_1$). The use of operator perturbation techniques to prove probabilistic limit theorem is usually called the Nagaev-Guivarc'h method in reference to the seminal works by these two mathematicians [35, 36, 16] (see also [30] and [19]). This method was first implemented in the case of nice bounded observables so that $t \mapsto P_t \in \mathcal{L}(\mathcal{B}_1)$ is smooth (continuous, $\mathcal{C}^k$, analytic) implying the smoothness of the eigenprojectors. The Keller-Liverani theorem [31] strenghen this approach making possible the study of the case of unbounded observables for which $t \mapsto P_t \in \mathcal{L}(\mathcal{B}_1)$ is not continuous (see also [11] for a presentation of this method in french). The idea consists in considering two Banach spaces $\mathcal{B}_1 \subset \mathcal{B}_2$ (with continuous inclusion) and then in using the continuity of $t \mapsto P_t \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ to prove the continuity of the eigenprojectors as elements of $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$.

In [22, 23], Hervé proved a local limit theorem and a Berry-Esséen estimate (i.e. an edgeworth expansion with error in $\mathcal{O}(n^{-\frac{1}{2}})$) for geometrically ergodic Markov chains (including V-geometrically ergodic) under the optimal assumption by proceeding in two steps: he first establishes the central limit theorem (CLT) using another method (using a martingale approximation, *à la* Gordin [13]) and then deduces from this result an expansion for the dominating eigenvalue, the continuity of the eigenprojectors being ensured by the Keller and Liverani theorem. This argument was reused in [26]. This method relies on the fact that we both have the continuity of the eigenprojectors and a proof of the CLT by another argument. Note also that it can only work for limit theorems using only the first non null derivative of the dominating eigenvalue λ_t of P_t, that is the dominating term of $\lambda_t - 1$ as t goes to 0 (remind that the geometrically ergodicity implies that $\lambda_0 = 1$ is the single dominating eigenvalue of P and that it is simple with only constant eigenvectors). As soon as we need more derivatives, we have to find another way.

The idea that continuity of the eigenprojectors and the first order term of $\lambda_t - 1$ as t goes to 0 are enough to prove convergence in distribution as well as local limit theorems has been implemented to prove convergence to stable distribution or gaussian distribution with non standard normalization in [1, 39] in the context of dynamical systems (chaotic billiards) and in [17] in the context affine random walks.

In [26], motivated by the establishment of further probabilistic limit theorems in Markovian context under the weakest possible moment assumptions, with Hervé, we extended the continuity statement of Keller and Liverani in a $\mathcal{C}^r$-smoothness result. This approach enabled us to prove some limit theorems under suboptimal assumptions, with an additional ε in the moment-type assumptions. In particular, we proved the first order Edgeworth expansion under the suboptimal moment assumption $m > 3$. The general $\mathcal{C}^r$-perturbation theorem of [26] was later used again in [24] and [25] in the context of Markov random walks and M-estimators, respectively.

In the present paper, using carefully a new weak compacity argument, we obtain the limit theorems under the optimal moment assumptions without requiring an extra probabilistic argument (such as martingale). In Section 2 we present the key ideas with a focus on our new argument. We state in Sections 3 and 4 general results on quasi-compactness and Taylor expansions (in t) of the resolvant and so of eigenprojectors for general families $(P_t)_t$ of continuous linear operators. The last sections are devoted to the general context of Markov random walks. In Section 5, we establish Taylor expansions for the dominating eigenvalue. Probabilistic limit theorems are then established in Section 6 (CLT, LLT, higher order expansions in the Berry-Esseén theorem as well as in the local limit theorem). In Section 7, we state probabilistic limit theorems in a general context of arkov chains and apply it to examples, proving in particular Theorem 1.1. Actually, Section 7 contains more general results (especially for Lipschitz iterative models). We end this article with an appendix, in which we explain how the Markov additive processes studied in e.g. [24] are in the scope of the present work.

2. Overview of the key ideas of the proofs

The key idea of the Nagaev-Guivarc'h method [35, 36, 16] consists in

- noticing that
$$\mathbb{E}_{\mathcal{P}_\mu\otimes\mathbf{P}^{\otimes\mathbb{N}}}[e^{itS_n}]=\mathbb{E}_\mu[P_t^n(\mathbf{1})]\,,$$
and more generally that
$$\mathbb{E}_{\mathcal{P}_\nu\otimes\mathbf{P}^{\otimes\mathbb{N}}}\left[g(X_0)\,e^{itS_n}\,h(X_n)\right]=\mathbb{E}_\nu\left[g\,P_t^n(h)\right]\,,$$
with
$$P_t(h)(x)=\int_E P(e^{itf(x,\cdot,\omega)}h(\cdot))(x)\,d\mathbf{P}(\omega)=\mathbb{E}\left[e^{itY_1}\Big|\,X_0=x\right]\,,$$
- using the fact that the geometric ergodicity of the Markov chain (see (1)), i.e. the quasi-compactness with single simple dominating eigenvalue 1 of P on some Banach space $\mathcal{B}_1$:
$$\exists\vartheta\in]0,1[,\quad\forall n\ge 1,\quad\|P^n-\mathbb{E}_\mu[\cdot]\|_{\mathcal{L}(\mathcal{B}_1)}=\mathcal{O}(\vartheta^n)$$
will imply the one of P_t, for small $|t|$ with a uniform bound:
$$\exists\vartheta_1\in]0,1[,\quad\forall n\ge 1,\quad\|P_t^n-\lambda_t^n\Pi_t\|_{\mathcal{L}(\mathcal{B}_1)}=\mathcal{O}(\vartheta_1^n)\,,\tag{2}$$
with $\lambda_t\in\mathbb{C}$ the dominating eigenvalue of P_t and with $\Pi_t\in\mathcal{L}(\mathcal{B}_1)$ the corresponding eigenprojector,
- in proving the smoothness of $t\mapsto\lambda_t$ and of $t\mapsto\Pi_t$,
- infering the probabilistic limit theorems using characteristic functions as in the case of sums of i.i.d. random variables.

In this whole paper, we will work with different Banach spaces satisfying some continuous embedding property that we introduce now. For two Banach spaces $(\mathcal{B}_j,\|\cdot\|_{(j)})$, $j\in\{1,2\}$, the notation $\mathcal{B}_1\hookrightarrow\mathcal{B}_2$ will mean that $\mathcal{B}_1\subset\mathcal{B}_2$ and $\|\cdot\|_{(2)}\le\|\cdot\|_{(1)}$.
In [31], Keller and Liverani proved that when $t\mapsto P_t\in\mathcal{L}(\mathcal{B}_1)$ is not continuous but only $t\mapsto P_t\in\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)$ is continuous with $\mathcal{B}_1\hookrightarrow\mathcal{B}_2$, it may still be possible to implement this method to get (2) and the continuity of $t\mapsto\Pi_t\in\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)$.
This idea has been extended in [26] to prove the $\mathcal{C}^r$-smoothness of $t\mapsto\Pi_t\in\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)$ and of $t\mapsto\lambda_t\in\mathbb{C}$. To this end, assuming $Y_1\in\mathbb{L}^{r+1}$ and exploiting actually only the fact that $Y_1\in\mathbb{L}^{r+\varepsilon}$, we worked with a double chain of Banach spaces:
$$\mathcal{B}_0\hookrightarrow\widetilde{\mathcal{B}}_0\hookrightarrow\mathcal{B}_1\hookrightarrow\widetilde{\mathcal{B}}_1\hookrightarrow...\hookrightarrow\mathcal{B}_r\hookrightarrow\widetilde{\mathcal{B}}_r\hookrightarrow\mathbb{L}^1$$
such that $t\mapsto P_t\in\mathcal{L}(\mathcal{B}_j,\widetilde{\mathcal{B}}_j)$ is continuous and such that $t\mapsto P_t\in\mathcal{L}(\widetilde{\mathcal{B}}_j,\mathcal{B}_{j+m})$ is $\mathcal{C}^m$ with P_t acting quasi-compactly on each $\mathcal{B}_j$ and $\widetilde{\mathcal{B}}_j$. It is proved in [26] that the resolvant and so the eigenprojectors are $\mathcal{C}^r$ as elements of $\mathcal{L}(\mathcal{B}_j,\widetilde{\mathcal{B}}_{j+m})$ and that $t\mapsto\lambda_t$ is also $\mathcal{C}^r$.
This way of proceeding allowed us to prove limit theorems under suboptimal hypotheses (typically $Y_1\in\mathbb{L}^{r+\varepsilon}$ when the optimal condition in the i.i.d. case was $Y_1\in\mathbb{L}^r$). This was already a great improvement, but was not totally satisfactory because of the additional ε in the moment assumptions.
We present here an approach that allows to obtain the optimal moment assumptions. Before entering deaplier in the presentation of the operator method used here, we explain the key ideas

making this adaptation possible (in particular a new key weak compacity argument in $\mathbb{L}^p$ combined with tailored adaptations of the chain of Banach spaces). Assuming $Y_1 \in \mathbb{L}^{r+1}$, we work with a single chain of Banach spaces:

$$\mathcal{B}_0 \hookrightarrow \mathcal{B}_1 \hookrightarrow \mathcal{B}_2 \hookrightarrow ... \hookrightarrow \mathcal{B}_{r+1} \hookrightarrow \mathbb{L}^1$$

such that the operators P_t are quasi-compact on $\mathcal{B}_1, ..., \mathcal{B}_r$ and such that $t \mapsto P_t$ admits a Taylor expansion with error in $\mathcal{O}(t^m)$ in $\mathcal{L}(\mathcal{B}_{r-m}, \mathcal{B}_r)$, and with an error in $o(t^m)$ in $\mathcal{L}(\mathcal{B}_{r-m}, \mathcal{B}_{r+1})$. Compared to [26], we establish (see Proposition 4.1, Theorem 5.5 and Corollary 5.6):

- Taylor expansions for eigenprojectors
  - take the first space $\mathcal{B}_0$ to be the space of constant functions which is preserved just by P, we do not assume that P_t acts on $\mathcal{B}_0$ (gain of space at the begining of the chain),
  - replace $\mathcal{C}^r$-smoothness of $t \mapsto P_t$ and $t \mapsto \Pi_t$ by Taylor expansions with error in $\mathcal{O}(t^r)$ in $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_r)$, and with an error in $o(t^r)$ in $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_{r+1})$ (gain of space all along the chain, gain of space at the end of the chain for the estimate in $\mathcal{O}(t^r)$), [4]
  - choose the spaces $\mathcal{B}_j$ so that $\mathcal{B}_j \hookrightarrow \mathbb{L}^{\frac{r+1}{j}}$,
- Taylor expansion for the dominating eigenvalue by a key weak compacity argument
  - The $o(r+1)$-Taylor expansion of the dominating eigenvalue will follow from an $o(r)$-Taylor expansion of $\mathbb{E}_\mu[\Pi_t(\mathbf{1})]$ and from $o(r+1)$ Taylor expansions of $\mathbb{E}_\mu[(e^{itf}-1)]$ and of $\mathbb{E}_\mu[(e^{itf}-1)(\Pi_t - \Pi_0)(\mathbf{1})]$ (Fact 3.2). The main issue is to prove this last expansion. But, to this end, only r-order Taylor expansions of $e^{itf}-1$ and $(\Pi_t - \Pi_0)(\mathbf{1})$ will be needed.
  - $o(t^{r+1})$-Taylor expansion of $\mathbb{E}_\mu[(e^{itf}-1)(\Pi_t - \Pi_0)(\mathbf{1})]$: make an $o(r)$-Taylor expansion of $(e^{itf}-1)$ and study individually each term $\mathbb{E}_\mu[\frac{(itf)^k}{k!}(\Pi_t - \Pi_0)(\mathbf{1})]$, for $k = 1, ..., r$.
  - $o(t^{r+1})$-Taylor expansion of $\mathbb{E}_\mu[\frac{(itf)^k}{k!}(\Pi_t - \Pi_0)(\mathbf{1})]$: first use an $\mathcal{O}(t^{r+1-k})$-Taylor expansion of $(\Pi_t - \Pi_0)(\mathbf{1}) \in \mathcal{B}_{r+1-k}$ to prove weak compacity in the reflexive space $\mathbb{L}^{\frac{r+1}{r+1-k}}$, second use $o(t^{r+1-k})$-Taylor expansions in $\mathcal{B}_{r+1} \hookrightarrow \mathbb{L}^1(\mu)$ to identify the weak limit. Conclude.
  - gather the terms and conclude the desired $o(t^{r+1})$-Taylor expansion of $\mathbb{E}_\mu[(e^{itf}-1)(\Pi_t - \Pi_0)(\mathbf{1})]$.
  - The $o(r+1)$-Taylor expansion of $\mathbb{E}_\mu[(e^{itf}-1)]$ and the $o(r)$-Taylor expansion of $\mathbb{E}_\mu[\Pi_t(\mathbf{1})]$ follow directly from the corresponding Taylor expansions of respectively $(e^{itf}-1)$ in $\mathbb{L}^1(\mu)$ and $\Pi_t(\mathbf{1})$ in $\mathcal{B}_{r+1} \hookrightarrow \mathbb{L}^1(\mu)$.

3. Quasi-compactness and perturbation

Let $\mathcal{B}_1$ be complex Banach space. We write $\mathcal{L}(\mathcal{B}_1)$ for the set of continuous linear operators on $\mathcal{B}_1$. For any continuous linear operator $P \in \mathcal{B}_1$, we write $\rho(P)$ for its spectral radius and $\rho_{ess}(P)$ for its essential spectral radius. Recall that the operator P is said to be quasi-compact if $\rho_{ess}(P) < \rho(P)$.

Theorem 3.1 (Browder [4])**.** *Let $P \in \mathcal{L}(\mathcal{B}_1)$ be quasi-compact. Let $r \in]\rho_{ess}(P), \rho(P)[$. Then the spectrum of P outside $B(0,r)$ consists in a finite number of eigenvalues $\lambda_1, ..., \lambda_s$ (isolated*

[4] Let us indicate for completness that, in our examples, it has been proved in [26] that $t \mapsto P_t \in \mathcal{L}(\mathcal{B}_0, \mathcal{B}_{r+1})$ is $\mathcal{C}^r$. Indeed, in these situations, there is a continuum of Banach spaces and so the space between $\mathcal{B}_r$ and $\mathcal{B}_{r+1}$ (to get $o(t^r)$) can be spread along the chain to add spaces $\widetilde{\mathcal{B}_j}$, up to slightly moving the spaces $\mathcal{B}_0, ..., \mathcal{B}_r$. Nevertheless we do not need this smoothness but just the Taylor expansion with error in $o(t^r)$.

in the spectrum of P) and there exist positive integers $m_1, ..., m_s$ such that

$$\mathcal{B}_1 = \oplus_{j=0}^{s}\mathcal{E}_j, \quad \text{with } P(\mathcal{E}_j) \subset \mathcal{E}_j \quad \text{and} \quad \|P^n_{|\mathcal{E}_0}\| = \mathcal{O}(r^n)\,,$$

with, for all $j = 1, ..., s$, $\mathcal{E}_j := \ker(P - \lambda_j \operatorname{Id})^{m_j}$ and $\dim(\mathcal{E}_j) < \infty$. For every $j = 0, ..., s$, there exists a continuous linear projection $\Pi_{[j]} : \mathcal{B}_1 \to \mathcal{E}_j$ such that

$$\sum_{j=0}^{s} \Pi_{[j]} = \operatorname{Id}, \quad P\Pi_{[j]} = \Pi_{[j]}P, \quad \Pi_{[j]}\Pi_{[\ell]} = \delta_{j,\ell}\Pi_{[j]}$$

and

$$\forall n \geq 0, \quad P^n\Pi_{[j]} = \frac{1}{2i\pi}\int_{\Gamma_j} z^n(z\operatorname{Id} - P)^{-1}\,dz\,, \tag{3}$$

with Γ_0 an oriented circle $\mathcal{C}(0, r_0)$ containing no λ_j and with $r_0 < r$, and with, for $j = 1, ..., s$, Γ_j an oriented circle $\mathcal{C}(\lambda_i, r_i) \subset \mathbb{C}$ such that λ_j is the only spectral value of P contained in the closed disk $\mathcal{D}(\lambda_i, r_i]$.
If moreover $m_j = 1$ for every $j = 1, ..., s$, then

$$P^n = \sum_{j=1}^{s} \lambda_j^n \Pi_{[j]} + P^n\Pi_{[0]}, \quad \text{with } \|P^n\Pi_{[0]}\| = \mathcal{O}(r^n)\,.$$

We consider now a quasi-compact operator $P \in \mathcal{L}(B)$ with simple peripherical spectrum and a family of quasi-compact operators $(P_t)_{|t|<\delta}$ such that $P_0 = P$ and admitting the same type of decomposition as P:

$$P_t^n = \sum_{j=1}^{s} \lambda_{j,t}^n \Pi_{[j],t} + N_t^n, \quad \text{with } \|N_t^n\| = \mathcal{O}(r^n)\,,$$

with $\lambda_{j,t}$ contained in the open disk $\mathcal{D}(\lambda_j, r_j[$. We will use the Keller-Liverani perturbation theorem recalled at the end of this section to prove that the family of operators we are considering satisfies this property.
Due to Theorem 3.1, the regularity in t of the eigenelements $\Pi_{[j],t}$ of P_t will follow from the regularity in t of the resolvant $(z\operatorname{Id} - P_t)^{-1}$ uniformly on $z \in \bigcup_{j=0}^{s}\Gamma_j$. Such results will be stated in Proposition 4.1 thanks to Theorem 3.3 (Keller-Liverani perturbation theorem).

As explained in Section 2, we will deduce from this an higher order Taylor expansion for the dominating eigenvalues due to the following key formula.

Fact 3.2 (see [1], or [21])**.** *In this context (assuming $m_j = 1$), if $v_j \in \ker(P - \lambda_j \operatorname{Id})$ and $\varphi_j \in \ker(P^* - \lambda_j \operatorname{Id})$ are such that $\varphi_j(v_j) = 1$ and that $t \mapsto \varphi_j \circ \Pi_{[j],t}(v_j)$ is continuous, then $P_t(\Pi_{[j],t}(v_j)) = \lambda_{j,t}(\Pi_{[j],t}(v_j))$ and, for t small enough, $\varphi_j(\Pi_{[j],t}(v_j)) \neq 0$, and so*

$$\lambda_{j,t} - \lambda_j = \frac{\varphi_j\left((P_t - P)(\Pi_{[j],t}(v_j))\right)}{\varphi_j\left(\Pi_{[j],t}(v_j)\right)}\,,$$

where we used the fact that $\varphi_j \circ P = \lambda_j\varphi_j$. Since $\Pi_{[j]} = \varphi_j(\cdot)v_j$, it follows also that

$$\begin{aligned}\lambda_{j,t} - \lambda_j &= \varphi_j\left((P_t - P)v_j\right) + \frac{\varphi_j\left((P_t - P)\left(\Pi_{[j],t}(v_j) - \Pi_{[j]}(\Pi_{[j],t}(v_j))\right)\right)}{\varphi_j\left(\Pi_{[j],t}(v_j)\right)}\\ &= \varphi_j\left((P_t - P)v_j\right) + \frac{\varphi_j\left((P_t - P)(\operatorname{Id} - \Pi_{[j]})(\Pi_{[j],t} - \Pi_{[j]})(v_j)\right)}{\varphi_j\left(\Pi_{[j],t}(v_j)\right)}\,.\end{aligned}$$

In Theorem 3.3 below, an auxiliary space $\mathcal{B}_2$ is used to study the spectral properties of a family of continuous linear operators acting on $\mathcal{B}_1$.

Theorem 3.3 (Keller-Liverani perturbation theorem, see [31] and [11])**.** *Let V be a neighbourhood of 0 in $\mathbb{R}^d$. Let $(P_t)_{t\in V}$ be a family of continuous linear operators on a Banach space $(\mathcal{B}_1,\|\cdot\|_{(1)})$. Let $(\mathcal{B}_2,\|\cdot\|_{(2)})$ be a Banach space such that $\mathcal{B}_1\hookrightarrow\mathcal{B}_2$. Assume that there exist two positive real numbers $R>\rho(P_0)$ and $r\in]\rho_{ess}(P_0),R[$ such that $(P_t)_{t\in V}$ satisfies the following uniform Doeblin-Fortet type inequality* [5]

$$\forall t\in V,\ \forall f\in\mathcal{B}_1,\ \forall n\geq 0\quad \|P_t^nf\|_{(1)}\leq r^n\|f\|_{(1)}+R^n\|f\|_{(2)}\,. \tag{4}$$

Assume moreover that $P_0\in\mathcal{L}(\mathcal{B}_2)$. Let $\varepsilon\in]0,R-r/2[$. Assume $P_0\in\mathcal{L}(\mathcal{B}_1)$ has eigenvalues $\lambda_{[1]},...,\lambda_{[m]}$ of modulus strictly larger than $r+2\varepsilon$ (here the eigenvalues are repeted with their multiplicities), and with no other eigenvalues of modulus larger than or equal to $r+\varepsilon$. Assume moreover that $t\mapsto P_t\in\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)$ is continuous at 0. Then there exists a neighbourhood U of 0 contained in V such that, for any $t\in U$, $P_t\in\mathcal{L}(\mathcal{B}_1)$ is quasi-compact with $\rho_{ess}(P_t)<r+\varepsilon$ and with eigenvalues $\lambda_{[1],t},...,\lambda_{[m],t}$ of modulus strictly larger that $r+\varepsilon$. Furthermore

$$\sup_{t\in U,\ z\in\mathbb{C}:\,r+\varepsilon<|z|<R+\varepsilon,\ \inf_j|z-\lambda_{[j]}|>\varepsilon}\|(z\,\mathrm{Id}-P_t)^{-1}\|_{\mathcal{L}(\mathcal{B}_1)}<+\infty\,,$$

$$\lim_{t\to 0}\sup_{z\in\mathbb{C}:\,r+\varepsilon<|z|<R+\varepsilon,\ \inf_j|z-\lambda_{[j]}|>\varepsilon}\left\|(z\,\mathrm{Id}-P_t)^{-1}-(z\,\mathrm{Id}-P_0)^{-1}\right\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}=0\,,$$

Moreover, for all j=1,...,m,

$$\lim_{t\to 0}\lambda_{[j],t}=\lambda_j\,,$$

$$\forall t\in U,\quad \dim\sum_{i:\lambda_i=\lambda_j}\bigcup_{k\geq 0}\ker(P_t-\lambda_{i,t}\,\mathrm{Id})^k=\dim\bigcup_{k\geq 0}\ker(P_t-\lambda_{[j]}\,\mathrm{Id})^k\,,$$

$$\lim_{t\to 0}\left\|\Pi_{[0]}-\Pi_{[0],t}\right\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}=0\quad\text{and}\quad\lim_{t\to 0}\left\|\Pi_{[j]}-\sum_{i\,:\,\lambda_i=\lambda_j}\Pi_{[i],t}\right\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}=0\,,$$

where $\Pi_{[i],t}$ are the projectors associated to $P_t\in\mathcal{L}(\mathcal{B}_1)$ and $\lambda_{i,t}$ as considered in Theorem 3.1.

This theorem ensures that

$$P_t^n=\left(\sum_{j=1}^m P_t^n\Pi_{[j],t}\right)+N_t^n\,,$$

with $N_t:=P_t\circ\Pi_0$ satisfies $\sup_t\|N_t^n\|_{\mathcal{L}(\mathcal{B}_1)}=\mathcal{O}\left((r+\varepsilon)^n\right)$. It will be crucial to notice that in the particular case where all the characteristic spaces $\bigcup_{k\geq 0}\ker(P_t-\lambda_{i,t}\,\mathrm{Id})^k$ consist only of eigenvectors, then this decomposition can be simplified in

$$P_t^n=\left(\sum_{j=1}^m \lambda_{[j],t}^n\Pi_{[j],t}\right)+N_t^n\,.$$

[5]We call it Doeblin-Fortet type inequality in reference to [6].

4. Taylor expansions for the resolvant and eigenprojectors

The continuity in $t\in\mathbb{R}^d$ of the eigenprojectors stated in Theorem 3.3 appears as a consequence of the continuity in t of the Resolvant $R_{z,t}:=(z\,\mathrm{Id}-P_t)^{-1}$ of P_t, due to Formula (3) (taken with $n=0$) providing an expression of $\Pi_{[j],t}$ as an integral of $(z\,\mathrm{Id}-P_t)^{-1}$. In the present section, we establish higher order Taylor expansions for the Resolvant $t\mapsto R_{z,t}=(z\,\mathrm{Id}-P_t)^{-1}$, that will imply immediately the corresponding Taylor expansions for $t\mapsto \Pi_{[j],t}$ thanks to (3). This section is devoted to the proof of the next result providing Taylor expansions of the resolvant. This result contains estimates of orders 0 and 1 (useful to establish the convergence in distribution to stable distributions for non square integrable observables) but also higher order Taylor expansions for the resolvant. Since our result holds true in multi-dimension, we need to introduce some different notions related to the multilinear forms appearing in multi-dimensional Taylor expansions.
We write $\mathcal{B}_1^d=(\mathcal{B}_1)^d$ for the set of d-dimensional vectors with entries in $\mathcal{B}_1$, and more generally $\mathcal{B}_\ell^{d^{\otimes k}}$ for the set of k-linear maps on $(\mathbb{R}^d)^k$ with values in $\mathcal{B}_\ell$, and we identify its elements $\mathcal{H}$ with $\mathcal{H}=(h_{i_1,...,i_k})_{i_1,...,i_k=1,...,d}$. We write $t^{\otimes m}$ for $(t_{i_1}...t_{i_m})_{i_1,...,i_m=1,...,d}$ and . for the scalar product. Finally, for any $\mathcal{H}^{(1)}\in\mathcal{L}(\mathcal{B}_i,\mathcal{B}_j^{d^{\otimes m}})$ and $\mathcal{H}^{(2)}\in\mathcal{L}(\mathcal{B}_j,\mathcal{B}_\ell^{d^{\otimes k}})$, we write

$$\mathcal{H}^{(2)}\otimes\mathcal{H}^{(1)}:=\left(\frac{1}{(m+k)!}\sum_{\sigma\in\mathfrak{S}_{m+k}}\mathcal{H}^{(2)}_{i_{\sigma(1)},...,i_{\sigma(m)}}H^{(1)}_{i_{\sigma(m+1)},...,i_{\sigma(m+k)}}\right)_{i_1,...,i_{m+k}=1,...,d},$$

where we write as usual $\mathfrak{S}_{m+k}$ for the set of permutations of $\{1,...,m+k\}$. Note that, with these notations, $\mathcal{H}^{(2)}\otimes\mathcal{H}^{(1)}\in\mathcal{L}(\mathcal{B}_i,\mathcal{B}_\ell^{d^{\otimes m+k}})$ and that $(\mathcal{H}^{(2)}\otimes\mathcal{H}^{(1)}).t^{\otimes(k+m)}=(\mathcal{H}^{(2)}.t^{\otimes k})(H^{(1)}.t^{\otimes m})$. In dimension 1 (when $d=1$), $\otimes$ as well as . both correspond to the usual product, $t^{\otimes m}$ simply to t^m and $\mathcal{B}_\ell^{d^{\otimes k}}$ to $\mathcal{B}_\ell$.

Proposition 4.1. *Let $\delta>0$, $\Gamma\subset\mathbb{C}$ and r be a nonnegative integer. Let $(\mathcal{B}_j,\|\cdot\|_{(j)})$, $j\in\{0,...,r+1\}$ be a chain of $(r+2)$ Banach spaces, increasing in the sense that $\mathcal{B}_j\hookrightarrow\mathcal{B}_{j+1}$ for all $j=0,...,r$. Let $(P_t)_t$ be a family of linear operators acting continuously on $\mathcal{B}_1,...,\mathcal{B}_{r+1}$ (for all $t\in\mathbb{R}^d$ such that $|t|<\delta$). Assume $P=P_0$ acts continuously on $\mathcal{B}_0$ and that*

$$K_0:=\sup_{j=1,...,r}\sup_{|t|<\delta}\sup_{z\in\Gamma}\left(\|(z\,\mathrm{Id}-P_t)^{-1}\|_{\mathcal{L}(\mathcal{B}_j)}+\|(z\,\mathrm{Id}-P)^{-1}\|_{\mathcal{L}(\mathcal{B}_0)}\right)<\infty\,. \tag{5}$$

Let $(P_0^{(k)})_{k=0,...,r}$ be a family of operators such that, for all $k=0,...,r$, $P_0^{(k)}\in\bigcap_{j=0}^{r-k}\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+k}^{d^{\otimes k}})$ and such that $P_0^{(0)}=P_0$. Then

(A) *For all $j=0,...,r$,*

$$\left\|(z\,\mathrm{Id}-P_t)^{-1}-(z\,\mathrm{Id}-P)^{-1}\right\|_{\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+1})}\le K_0^2\,\|P_t-P\|_{\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+1})}\,. \tag{6}$$

(B) *If $r=1$, then, setting $P_0'=P_0^{(1)}$,*

$$\begin{aligned}&\left\|(z\,\mathrm{Id}-P_t)^{-1}-(z\,\mathrm{Id}-P)^{-1}-(z\,\mathrm{Id}-P)^{-1}(P_0'.t)(z\,\mathrm{Id}-P)^{-1}\right\|_{\mathcal{L}(\mathcal{B}_0,\mathcal{B}_2)}\\&\le K_0^2\,\|P_t-P-P_0'.t\|_{\mathcal{L}(\mathcal{B}_0,\mathcal{B}_2)}+|t|\,K_0^3\|P_t-P\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}\|P_0'\|_{\mathcal{L}(\mathcal{B}_0,\mathcal{B}_1^d)}\,.\end{aligned} \tag{7}$$

(C) *If, for all $j=0,...,r$,*[6]

$$\left\|P_t-\sum_{k=0}^{r-j}\frac{P_0^{(k)}}{k!}.t^{\otimes k}\right\|_{\mathcal{L}(\mathcal{B}_j,\mathcal{B}_r)}\le K_1|t|^{r-j}\,,$$

[6]Note that $\mathcal{B}_{r+1}$ does not play any role in this result.

then there exists a constant $\widetilde{K_r}$ which is given by a polynomial expression in K_0, K_1 such that

$$\left\|(z\,\mathrm{Id}-P_t)^{-1}-(z\,\mathrm{Id}-P)^{-1}-\sum_{j=1}^{r}R^{(j)}_{z,0}.t^{\otimes j}\right\|_{\mathcal{L}(\mathcal{B}_0,\mathcal{B}_r)}\le \widetilde{K_r}|t|^r\,,$$

for all $|t|<\delta$ and all $z\in\Gamma$, with

$$R^{(j)}_{z,0}:=(z\,\mathrm{Id}-P)^{-1}\sum_{\ell\ge1,k_1,\dots,k_\ell\ge1\,:\,k_1+\dots+k_\ell=j}\mathcal{A}_{k_\ell}\otimes\cdots\otimes\mathcal{A}_{k_1}\,,$$

and $\mathcal{A}_\ell:=\frac{P_0^{(\ell)}}{\ell!}(z\,\mathrm{Id}-P)^{-1}\in\bigcap_{j=0}^{r-\ell}\mathcal{L}(\mathcal{B}_j,\mathcal{B}^{d^{\otimes\ell}}_{j+\ell})$.

(D) *If for all $j=0,...,r$, $\left\|P_t-\sum_{k=0}^{r-j}\frac{P_0^{(k)}}{k!}.t^{\otimes k}\right\|_{\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{r+1})}=o(|t|^{r-j})$, then, on $\mathcal{L}(\mathcal{B}_0,\mathcal{B}_{r+1})$,*

$$(z\,\mathrm{Id}-P_t)^{-1}-(z\,\mathrm{Id}-P)^{-1}-\sum_{j=1}^{r}R^{(j)}_{z,0}.t^{\otimes j}=o(|t|^r)\,,$$

uniformly in $|t|<\delta$ and $z\in\Gamma$, with the same notations as in the previous item.

It may be worthwhile to note that

$$R^{(j)}_{z,0}.t^{\otimes j}=(z\,\mathrm{Id}-P)^{-1}\sum_{\ell\ge1,k_1,\dots,k_\ell\ge1\,:\,k_1+\dots+k_\ell=j}(\mathcal{A}_{k_\ell}.t^{\otimes k_\ell})\cdots(\mathcal{A}_{k_1}.t^{\otimes k_1})\,.$$

Since, in our examples, P_t will have the form $P(e^{it.f}\cdot)$, the Taylor expansions of $t\mapsto P_t\in\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+m})$ will be proved using Taylor expansions in t of $t\mapsto(e^{it.f}\cdot)\in\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+m})$ and the operators $P_0^{(\ell)}$ will have the form $P_0^{(\ell)}=P((if)^{\otimes\ell}\cdot)$.

Remark 4.2. *In practice we will prove the first part of Assumption* (5) *by applying Theorem 3.3 with Banach spaces $\mathcal{B}_j\hookrightarrow\mathcal{B}_{j+1}$ for all $j=1,...,r+1$ and thus we will need a $(r+2)$-th Banach space $\mathcal{B}_{r+2}$. For the second part of Assumption* (5)*, a useful idea (used several times in applications) will be to take the eigenspace associated to the dominating eigenvalue (in our applications, it will be the space of constant functions).*

We could have stated the previous lemma in a much more general way by replacing $P_0^{(k)}.t^{\otimes k}$ by $a_{k,t}$ and $\mathcal{A}_k.t^{\otimes k}$ by $\frac{a_{k,t}}{k!}(z\,\mathrm{Id}-P)^{-1}\in\bigcap_{j=0}^{r-\ell}\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+\ell})$. We have not chosen this presentation since we do not have application in mind, except maybe the case of convergence to stable distribution, but for which in practice Item (A) is enough (see Proposition 5.11 and Example 5.12).

Proof of Proposition 4.1. We will use the following key identity

$$(z\,\mathrm{Id}-P_t)^{-1}-(z\,\mathrm{Id}-P)^{-1}=(z\,\mathrm{Id}-P_t)^{-1}(P_t-P)(z\,\mathrm{Id}-P)^{-1}\,.$$

Observe that Item (A) is a direct consequence of this identity. Analogously

$$\begin{aligned}&(z\,\mathrm{Id}-P_t)^{-1}-(z\,\mathrm{Id}-P)^{-1}-(z\,\mathrm{Id}-P)^{-1}(P_0'.t)(z\,\mathrm{Id}-P)^{-1}\\&=(z\,\mathrm{Id}-P_t)^{-1}(P_t-P)(z\,\mathrm{Id}-P)^{-1}-(z\,\mathrm{Id}-P)^{-1}(P_0'.t)(z\,\mathrm{Id}-P)^{-1}\\&=(z\,\mathrm{Id}-P_t)^{-1}(P_t-P-P_0'.t)(z\,\mathrm{Id}-P)^{-1}+\left[(z\,\mathrm{Id}-P_t)^{-1}-(z\,\mathrm{Id}-P)^{-1}\right](P_0'.t)(z\,\mathrm{Id}-P)^{-1}\end{aligned}$$

The first term has norm less than $K_0^2\,\|P_t-P-P_0'.t\|_{\mathcal{L}(\mathcal{B}_0,\mathcal{B}_1)}$ in $\mathcal{L}(\mathcal{B}_0,\mathcal{B}_1)$ and the second one can be rewritten

$$(z\,\mathrm{Id}-P_t)^{-1}(P_t-P)(z\,\mathrm{Id}-P)^{-1}(P_0'.t)(z\,\mathrm{Id}-P)^{-1}\,,$$

which ends the proof of (B).
To establish (C), we prove by induction on $m = 1, ..., r$ that

$$\left\| R_{z,t} - R_{z,0} - \sum_{j=1}^{m-1} R^{(j)}_{z,0}.t^{\otimes j} \right\|_{\mathcal{L}(\mathcal{B}_{r-m},\mathcal{B}_r)} \leq \widetilde{K}_m |t|^m \,, \tag{8}$$

for all $|t| < \delta$ and $z \in \Gamma$. Due to Item (A) applied with $j = r - 1$, (8) holds true for $m = 1$ with $\widetilde{K}_1 = K_1 K_0^2$. for all $|t| < \delta$ and $z \in \Gamma$.

Let $N = 2, ..., r$. Assume (8) holds true for all $m = 0, ..., N - 1$, and let us prove it holds also true for $m = N$. Observe that

$$\begin{aligned} R_{z,t} - R_{z,0} &= R_{z,t}(P_t - P)R_{z,0} \\ &= R_{z,t} \sum_{k=1}^{N-1} \frac{P_0^{(k)}}{k!}.t^{\otimes k} R_{z,0} + \mathcal{O}(|t|^N) \\ &= \sum_{k=1}^{N-1} (R_{z,t} - R_{z,0})\, \mathcal{A}_k.t^{\otimes k} + R_{z,0} \sum_{k=1}^{N-1} \mathcal{A}_k.t^{\otimes k} + \mathcal{O}(|t|^N) \,, \end{aligned}$$

in $\mathcal{L}(\mathcal{B}_{r-N}, \mathcal{B}_r)$, with $\mathcal{O}(|t|^N)$ bounded by $K_0^2 K_1$ (uniformly in t, z). Recall that $\mathcal{A}_k \in \mathcal{L}(\mathcal{B}_{r-N}, \mathcal{B}_{r-N+k})$. It follows from the inductive hypothesis that

$$R_{z,t} - R_{z,0} = \sum_{\ell\geq 1, k_1,...,k_\ell \geq 1 \,:\, k_1+...+k_\ell \leq N-k-1} R_{z,0}(\mathcal{A}_{k_\ell}.t^{\otimes k_\ell}) \cdots (\mathcal{A}_{k_1}.t^{\otimes k_1}) + \mathcal{O}(|t|^{N-k})$$

in $\mathcal{L}(\mathcal{B}_{r-N+k}, \mathcal{B}_r)$ uniformly in t, z. This ends the proof of Item (C).
It remains finally to prove Item (D). To this end, it is enough to prove by induction on $m = 0, ..., r$ that, on $\mathcal{L}(\mathcal{B}_{r-m}, \mathcal{B}_{r+1})$,

$$R_{z,t} - R_{z,0} - \sum_{\ell\geq 1, k_1,...,k_\ell \geq 1 \,:\, k_1+...+k_\ell \leq m} (z \operatorname{Id} - P)^{-1} (\mathcal{A}_{k_\ell}.t^{\otimes k_\ell}) \cdots (\mathcal{A}_{k_1} t^{\otimes k_1}) = o(|t|^m) \tag{9}$$

uniformly in $|t| < \delta$ and $z \in \Gamma$. This is true for $m = 0$ since

$$R_{z,t} - R_{z,0} = R_{z,t}(P_t - P)R_{z,0} = o(1) \,,$$

on $\mathcal{L}(\mathcal{B}_r, \mathcal{B}_{r+1})$ uniformly in t, z. Fix $N = 2, ..., r$. Assume (9) for all $m = 0, ..., N - 1$ and let us prove it holds also true for $m = N$. Observe that

$$\begin{aligned} R_{z,t} - R_{z,0} &= R_{z,t}(P_t - P)R_{z,0} \\ &= R_{z,t} \sum_{k=1}^{N} \frac{P_0^{(k)}}{k!}.t^{\otimes k} R_{z,0} + o\left(|t|^N\right) \\ &= \sum_{k=1}^{N} (R_{z,t} - R_{z,0})\, \mathcal{A}_k.t^{\otimes k} + R_{z,0} \sum_{k=1}^{N} \mathcal{A}_k.t^{\otimes k} + o\left(|t|^N\right) \,, \end{aligned}$$

in $\mathcal{L}(\mathcal{B}_{r-N}, \mathcal{B}_{r+1})$ uniformly in t, z. Recall that $\mathcal{A}_k \in \mathcal{L}(\mathcal{B}_{r-N}, \mathcal{B}^{d^{\otimes k}}_{r-N+k})$. It follows from the inductive hypothesis that

$$R_{z,t} - R_{z,0} = \sum_{\ell\geq 1, k_1,...,k_\ell \geq 1 \,:\, k_1+...+k_\ell \leq N-k} R_{z,0}(\mathcal{A}_{k_\ell}.t^{\otimes k_\ell}) \cdots (\mathcal{A}_{k_1}.t^{\otimes k_1}) + o(|t|^{N-k})$$

in $\mathcal{L}(\mathcal{B}_{r-N+k}, \mathcal{B}_{r+1})$ uniformly in t, z. This ends the proof of Item (D) and so of Proposition 4.1. □

To make easier the comparison with previous works, let us recall the result of [26, Appendix A] about $\mathcal{C}^r$-smoothness.

Proposition 4.3 ([26]). *Assume there exists a double chain of Banach spaces*

$$\mathcal{B}_0 \hookrightarrow \widetilde{\mathcal{B}}_0 \hookrightarrow \mathcal{B}_1 \hookrightarrow \widetilde{\mathcal{B}}_1 \hookrightarrow \dots \hookrightarrow \mathcal{B}_r \hookrightarrow \widetilde{\mathcal{B}}_r \,.$$

Assume $(P_t)_{t\in U}$ is a family of linear operators acting continuously on all these Banach spaces (with U an open subset of $\mathbb{R}^d$) such that

$$\sup_{j=0,\dots,r}\ \sup_{|t|<\delta}\sup_{z\in\Gamma}\left(\|(z\,\mathrm{Id}-P_t)^{-1}\|_{\mathcal{L}(\mathcal{B}_j)}+\|(z\,\mathrm{Id}-P_t)^{-1}\|_{\mathcal{L}(\widetilde{\mathcal{B}}_j)}\right)<\infty\,,$$

and such that $t\mapsto P_t\in\bigcap_{j=0}^r\mathcal{L}\left(\mathcal{B}_j,\widetilde{\mathcal{B}}_j\right)$ is continuous on $U\subset\mathbb{R}^d$ and such that $t\mapsto P_t\in\bigcap_{j=0}^{r-m}\mathcal{L}(\widetilde{\mathcal{B}}_j,\mathcal{B}_{j+m})$ is $\mathcal{C}^m$. Then $t\mapsto(z\,\mathrm{Id}-P_t)^{-1}\in\mathcal{L}(\mathcal{B}_0,\widetilde{\mathcal{B}}_r)$ is $\mathcal{C}^r$, with derivatives at 0 given by the Taylor expansion established in Proposition 4.1.

Proof. This result follows directly [26, Proposition A] applied with $I=\bigcup_{j=1}^m\{\mathcal{B}_j,\widetilde{\mathcal{B}}_j\}$, $T_0(\mathcal{B}_j)=\widetilde{\mathcal{B}}_j$, $T_1(\widetilde{\mathcal{B}}_j)=\mathcal{B}_{j+1}$ (up to identify I with a subset of $\mathbb{R}$). □

5. Expansions of Fourier eigenprojectors and eigenvalues in Markovian or dynamical contexts

In this section, we will see how the general results of Section 4 can be implemented to study dynamical or markovian random walks $(S_n)_{n\ge1}$ defined as follows.

Hypothesis 5.1. *Let $(\Omega,\mathcal{F},\mu)$ and $(E,\mathcal{T},\mathbf{P})$ be two probability spaces.*

(I) *either $X_n=T^n$ where $T:\Omega\to\Omega$ is a μ-preserving transformation, with transfer operator P and $f:\Omega\times E\to\mathbb{R}^d$ is a measurable $\mu\otimes\mathbf{P}$-centered function. We consider ν a probability measure on Ω absolutely continuous with density h with respect to μ. To unify notations with the markovian setting, we also set $\mathcal{P}_\mu=\mu$ and $f(x,y,\omega):=f(y,\omega)$.*

(II) *or $(X_n)_{n\ge0}$ is a Markov chain (identified with the canonical Markov chain) with values in Ω and with stationary measure μ and $f:\Omega\times\Omega\times E\to\mathbb{R}^d$ is a measurable function. Let ν be the distribution of X_0 (i.e. the initial distribution of the Markov chain). We set $\mathcal{P}_\nu$ for the Markov distribution with transition operator P and initial probability measure ν. We assume that $((x_k)_k,\omega)\mapsto f(x_0,x_1,\omega)$ is $\mathcal{P}_\mu\otimes\mathbf{P}$-centered.*

We set

$$P_t(h)(x)=\int_E P\left(e^{it.f(x,\cdot,\omega)}h(\cdot)\right)(x)\,d\mathbf{P}(\omega)\,,$$

and $S_n:=\sum_{k=1}^n Y_k$ with $Y_k:=f(X_{k-1}X_k,Z_k)$ where Z_i are i.i.d. random variables independent of $(X_k)_{k\ge0}$ and with common distribution $\mathbf{P}$. [7]

In the dynamical setting (I), identifying $H(x,\omega)$ with $H(x)$:

$$\mathbb{E}_\mu\left[g\,P_t^n(hG)\right]=\mathbb{E}_{\mu\otimes\mathbf{P}^{\otimes\mathbb{N}}}\left[(g\circ T^n)e^{it.S_n}.hG\right]=\mathbb{E}_{\nu\otimes\mathbf{P}^{\otimes\mathbb{N}}}\left[(g\circ T^n)e^{it.S_n}G\right]\,. \tag{10}$$

In the Markovian setting (II), $P_t(h)(x)=\mathbb{E}\left[e^{it.Y_1}h\,\middle|\,X_0=x\right]$, and so

$$\mathbb{E}_\nu\left[g\,P_t^n(h)\right]=\mathbb{E}_{\mathcal{P}_\nu\otimes\mathbf{P}^{\otimes\mathbb{N}}}\left[g(X_0)e^{it.S_n}h(X_n)\right]\,. \tag{11}$$

Before considering applications seen in Theorem 1.1, we will state results in the general dynamical or markovian context and will illustrate them on the following toy model of Knudsen

[7] Let us indicate for completness that Markov random walks are usually defined as a Markov chain $(X_n,S_n)_{n\ge0}$ satisfying

$$\mathbb{E}\left[h(X_n,\tilde S_n)\,\middle|\,X_{n-1}=x,\tilde S_{n-1}=y\right]=\mathbb{E}\left[h(X_n,y+\tilde S_n)\,\middle|\,X_{n-1}=x,\tilde S_{n-1}=0\right]\,.$$

and that we aprove in appendix that these two definitions are equivalent.

gas considered in [3], which is one of the simplest example of uniformly geometrically ergodic Markov chains (i.e. satisfying (1) with $\mathcal{B} = L^\infty$ the set of uniformly bounded complex-valued functions on Ω).

Example 5.2 (a Toy model of Knudsen gas)**.** *Let r be a nonnegative integer. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\alpha \in]0,1[$. Consider the Markov chain $(X_n)_n$ with transition operator given by*

$$P(h) = \alpha h + (1-\alpha)\mathbb{E}_\mu[h]\,, \quad i.e. \quad P(h - \mathbb{E}_\mu[h]) = \alpha(h - \mathbb{E}_\mu[h])\,.$$

This Markov chain describes the evolution of a process which, at each step, remains the same with probability α and change to an independent copy of distribution μ with probability $1-\alpha$. We consider also $Y_n = f(X_n)$, with $f : \Omega \to \mathbb{R}^d$ centered and admitting moments of order $(r+1)$ with respect both to the invariant distribution μ and to the initial distribution ν. Then P_t is given by

$$P_t(h)(x) = \alpha e^{itf(x)} h(x) + (1-\alpha)\mathbb{E}_\mu[e^{itf} h]\,.$$

More generally, we will consider the context described in the following:

Remark 5.3. *Assume Hypothesis 5.1 and that P is geometrically ergodic on some Banach space $\mathcal{B}_1 \hookrightarrow \mathbb{L}^1(\mu)$ containing the constant functions, so that $\|P^n - \mathbb{E}_\mu\|_{\mathcal{L}(\mathcal{B}_1)} = \mathcal{O}(\vartheta^n)$ (for some $\vartheta \in]0,1[$). Let $\vartheta_1 \in]\vartheta, 1[$. Assume moreover that the assumptions of Theorem 3.3 are satisfied for this choice of $(P_t, \mathcal{B}_1)$ and for some $\mathcal{B}_2$. Then there exists $\delta_0 > 0$ such that, in $\mathcal{L}(\mathcal{B}_1)$,*

$$\forall |t| < \delta_0, \quad P_t^n = \lambda_t^n \Pi_t + N_t^n\,, \quad with \sup_{|t|<\delta_0} \|N_t^n\|_{\mathcal{B}_1} = \mathcal{O}(\vartheta_1^n)\,, \tag{12}$$

with $t \mapsto \Pi_t \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ and $t \mapsto \lambda_t \in \mathbb{C}$ continuous and $\Pi_0 = \mathbb{E}_\mu[\cdot]\mathbf{1}$ and $\lambda_0 = 1$. Moreover Theorem 3.3 ensures that Π_t and N_t are given by the following formulas

$$\Pi_t := \frac{1}{2i\pi} \int_{\Gamma_1} (z \operatorname{Id} - P_t)^{-1}\, dz, \quad N_t^n := \frac{1}{2i\pi} \int_{\Gamma_0} z^n (z \operatorname{Id} - P_t)^{-1}\, dz\,, \tag{13}$$

with Γ_1 the oriented circle $\mathcal{C}(1,\delta)$ and Γ_0 the oriented circle $\mathcal{C}(0,a)$, with $\vartheta_1 < a < a + \delta < 1$ and that

$$\sup_{|t|<\delta,\ z \in \Gamma_1 \cup \Gamma_0} \|(z \operatorname{Id} - P_t)^{-1}\|_{\mathcal{L}(\mathcal{B}_1)} < \infty\,,$$

with $t \mapsto R_{z,t} = (z \operatorname{Id} - P_t)^{-1} \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ continuous on $\{t \in \mathbb{R}^d : |t| < \delta_0\}$ (uniformly in $z \in \Gamma_1 \cup \Gamma_0$).

Example 5.4 (Knudsen gas)**.** *We consider again the Knudsen gas introduced in Example 5.2. Note that*

$$\forall p \in [1, +\infty], \quad \|P^n(h) - \mathbb{E}_\mu[h]\|_{\mathbb{L}^p(\mu)} = \alpha^n \|h - \mathbb{E}_\mu[h]\|_{\mathbb{L}^p(\mu)}\,,$$

and if $\nu \neq \mu$, it is worthwhile to notice that we also have

$$\forall \gamma \in [0, +\infty[, \quad \left\| \frac{P^n(h) - \mathbb{E}_\mu[h]}{(1+|f|)^\gamma} \right\|_\infty = \alpha^n \left\| \frac{h - \mathbb{E}_\mu[h]}{(1+|f|)^\gamma} \right\|_\infty \leq (1 + \mathbb{E}_\mu[(1+|f|)^\gamma])\, \alpha^n \left\| \frac{h}{(1+|f|)^\gamma} \right\|_\infty .$$

Thus, since $|P_t^n(h)| \leq P^n(|h|)$, Theorem 3.3 applies with

- *$\mathcal{B}_i := \mathbb{L}^{p_i}(\mu)$ for all p_1, p_2 such that $1 \leq p_2 < p_1 \leq +\infty$,*
- *and also (useful when $\nu \neq \mu$) with $\mathcal{B}_i = (1+|f|)^{\gamma_i} L^\infty$ for all γ_1, γ_2 such that $0 \leq \gamma_1 < \gamma_2 < \infty$ and $\mathbb{E}\left[|f|^{\gamma_2}\right] < \infty$, where we write again L^∞ for the set of bounded measurable complex valued functions on Ω and $\mathcal{B}_i$ being endowed with the norm $\left\| \frac{\cdot}{(1+|f|)^{\gamma_i}} \right\|_\infty$.*

In the general context of Remark 5.3, Item (C) (resp. Item (D)) of Proposition 4.1 applied to this context will provide the following expansions for Π and N^n:

$$\left\|\Pi_t - \sum_{k=0}^{r-1} \frac{\Pi_0^{(k)}}{k!}.t^{\otimes k}\right\|_{\mathcal{L}(\mathcal{B}_0,\mathcal{B}_r)} = \mathcal{O}(|t|^r) \quad \text{and} \quad \left\|N_t^n - \sum_{k=0}^{r-1} \frac{(N^n)_0^{(k)}}{k!}.t^{\otimes k}\right\|_{\mathcal{L}(\mathcal{B}_0,\mathcal{B}_r)} \le \mathcal{O}(a^n)\mathcal{O}(|t|^r)\,,$$

(resp. with $(\mathcal{O}(t^r),\mathcal{B}_r)$ being replaced by $(o(t^r),\mathcal{B}_{r+1})$) with $\Pi_0^{(k)},(N^n)_0^{(k)} \in \bigcap_{j=0}^{r-k}\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+k}^{d^{\otimes k}})$ given by

$$\frac{\Pi_0^{(k)}}{k!} := \frac{1}{2i\pi}\int_{\Gamma_1} \sum_{\ell\ge 1,k_1,\dots,k_\ell\ge 1\,:\,k_1+\dots+k_\ell=k} (z\,\mathrm{Id}-P)^{-1}\mathcal{A}_{k_\ell}\otimes\cdots\otimes A_{k_1}\,dz \tag{14}$$

and

$$\frac{(N^n)_0^{(k)}}{k!} := \frac{1}{2i\pi}\int_{\Gamma_0} z^n \sum_{\ell\ge 1,k_1,\dots,k_\ell\ge 1\,:\,k_1+\dots+k_\ell=k} (z\,\mathrm{Id}-P)^{-1}\mathcal{A}_{k_\ell}\otimes\cdots\otimes \mathcal{A}_{k_1}\,dz\,, \tag{15}$$

with $\mathcal{A}_\ell = \frac{P^{(\ell)}}{\ell!}(z\,\mathrm{Id}-P)^{-1} \in \bigcap_{j=0}^{r-\ell}\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+\ell}^{d^{\otimes \ell}})$.

Theorem 5.5 (Main result)**.** *Let $\delta_0>0$. Assume Hypothesis 5.1. Let r be a nonnegative integer and $Y_1\in\mathbb{L}^{r+1}(\mathcal{P}_\mu\otimes\mathbf{P})$. Let $(\mathcal{B}_j,\|\cdot\|_{(j)})$, $j=0,\dots,r+1$ be a chain of $(r+2)$ Banach spaces such that, for all $j=1,\dots,r+1$, $\mathcal{B}_{j-1}\hookrightarrow\mathcal{B}_j$, $\mathcal{B}_j\hookrightarrow\mathbb{L}^{\frac{r+1}{j}}(\mu)$. Assume that P_t (for $|t|<\delta_0$) acts continuously on $\mathcal{B}_1,\dots,\mathcal{B}_{r+1}$ and that P_0 acts continuously on $\mathcal{B}_0$. Assume moreover that $h,\mathbf{1}\in\mathcal{B}_0$ and that $g:\Omega\to\mathbb{R}$ is such that $\mathbb{E}_\nu[g\cdot]$ defines a continuous linear form on $\mathcal{B}_r$. We also assume that* (12) *with* (13) *hold true on $\mathcal{B}_1$ and that*

$$\sup_{j=1,\dots,r+1}\ \sup_{|t|<\delta_0}\ \sup_{z\in\Gamma_0\cup\Gamma_1}\left(\|(z\,\mathrm{Id}-P_t)^{-1}\|_{\mathcal{B}_j}+\|(z\,\mathrm{Id}-P)^{-1}\|_{\mathcal{B}_0}\right)<\infty\,, \tag{16}$$

and that, for all $m=0,\dots,r$, $P_t-\sum_{k=0}^m\frac{P_0^{(k)}}{k!}.t^{\otimes k}$ is both in $\mathcal{O}(t^m)$ in $\bigcap_{j=0}^{r-m}\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+m})$ and in $o(t^m)$ in $\mathcal{L}(\mathcal{B}_{r-m},\mathcal{B}_{r+1})$, with

$$P_0^{(k)}(h)(x) = i^k\mathcal{Q}_k(h) := \int_E P\left((if(x,\cdot,\omega))^{\otimes k}h(\cdot)\right)(x)\,d\mathbf{P}(\omega)\in\bigcap_{j=0}^{r-k}\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+k}^{d^{\otimes k}})\,.$$

Then

$$\mathbb{E}_\nu\left[g\,P_t^n(h)\right] = \lambda_t^n\left(\sum_{\ell=0}^r\frac{t^{\otimes\ell}}{\ell!}.\mathbb{E}_\nu\left[g\Pi_0^{(\ell)}(h)\right]+(t^r)\right)+\sum_{\ell=0}^r\frac{t^{\otimes\ell}}{\ell!}.\mathbb{E}_\nu\left[g\,(N^n)_0^{(\ell)}(h)\right]+\mathcal{O}(a^n|t|^r)\,, \tag{17}$$

with $\Pi_0^{(\ell)}$ and $(N^n)_0^{(\ell)}$ given by (14) *given by* (15)*.*
If moreover $\mathbb{E}_\nu[g\cdot]$ defines a continuous linear form on $\mathcal{B}_{r+1}$, then

$$\mathbb{E}_\nu\left[g\,P_t^n(h)\right] = \lambda_t^n\left(\sum_{\ell=0}^r\frac{t^{\otimes\ell}}{\ell!}.\mathbb{E}_\nu\left[g\Pi_0^{(\ell)}(h)\right]+o(t^r)\right)+\sum_{\ell=0}^r\frac{t^{\otimes\ell}}{\ell!}.\mathbb{E}_\nu\left[g\,(N^n)_0^{(\ell)}(h)\right]+\mathcal{O}(a^n o(|t|^r))\,. \tag{18}$$

Moreover

$$\lambda_t-1=\sum_{k=2}^{r+1}\frac{\mathbb{E}_{\mathcal{P}_\mu\otimes\mathbf{P}}[(it.Y_1)^k]}{k!}+\frac{\sum_{k=1}^r\sum_{\ell=1}^{r+1-k}\frac{t^{\otimes(k+\ell)}}{k!\ell!}\mathbb{E}_\mu[P_0^{(k)}(\Pi_0^{(\ell)}(\mathbf{1})-\mathbb{E}_\mu[\Pi_0^{(\ell)}(\mathbf{1})])]}{1+\sum_{\ell=1}^{r-1}\frac{t^{\otimes\ell}}{\ell!}\mathbb{E}_\mu[\Pi_0^{(\ell)}(\mathbf{1})]}+o(|t|^{r+1})\,.$$

Note that, in the Markovian context,

$$t^{\otimes(k+\ell)}.\mathbb{E}_\mu[P_0^{(k)}(\Pi_0^{(\ell)}(g)-\mathbb{E}_\mu[\Pi_0^{(\ell)}(g)])] = \mathcal{C}ov_{\mathcal{P}_\mu\otimes\mathbf{P}}\left((it.Y_1)^k,\Pi_0^{(\ell)}(\mathbf{1})(X_1).t^{\otimes\ell}\right)\,.$$

In particular, if $f(x,y,z)=f(y)\in\mathbb{R}$, then

$$\mathbb{E}_\mu[P_0^{(k)}(\Pi_0^{(\ell)}(g)-\mathbb{E}_\mu[\Pi_0^{(\ell)}(g)])]=i^k\mathcal{C}ov_\mu\left(f^{\otimes k},\Pi_0^{(\ell)}(\mathbf{1})\right).$$

Before proving Theorem 5.5, we state a corollary and apply it on our Knudsen gas model.

Corollary 5.6. *Let $\delta_0>0$, $\vartheta\in]0,1[$, $R>0$ and r be a nonnegative integer. Assume Hypothesis 5.1 with $Y_1\in\mathbb{L}^{r+1}(\mathcal{P}_\mu\otimes\mathbf{P})$. Let $(\mathcal{B}_j,\|\cdot\|_{(j)})_{j=0,\dots,r+2}$ be a chain of $(r+3)$ Banach spaces such that:*

- *for all $j=0,\dots,r+1$, $\mathcal{B}_j\hookrightarrow\mathcal{B}_{j+1}$, and for all $j=1,\dots,r+1$, $\mathcal{B}_j\hookrightarrow\mathbb{L}^{\frac{r+1}{j}}(\mu)$,*
- *for all $j=1,\dots,r+1$, P is geometrically ergodic on $\mathcal{B}_j$: $\|P^n-\mathbb{E}_\mu\|_{\mathcal{L}(\mathcal{B}_1)}=\mathcal{O}(\vartheta^n)$,*
- *$\mathbf{1}\in\mathcal{B}_0$, $P\in\mathcal{L}(\mathcal{B}_0)$ and $\sup_{z\in\Gamma_0\cup\Gamma_1}\|(z\,\mathrm{Id}-P)^{-1}\|_{\mathcal{B}_0}<\infty$,*
- *for all $m=0,\dots,r$, $P_t-\sum_{k=0}^m\frac{P_0^{(k)}}{k!}.t^{\otimes k}$ is both in $\mathcal{O}(t^m)$ in $\bigcap_{j=0}^{r-m}\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+m})$ and in $o(t^m)$ in $\mathcal{L}(\mathcal{B}_{r-m},\mathcal{B}_{r+1})$, with*

$$P_0^{(k)}(h)(x)=i^k\mathcal{Q}_k(h):=\int_E P\left((if(x,\cdot,\omega))^{\otimes k}h(\cdot)\right)(x)\,d\mathbf{P}(\omega)\in\bigcap_{j=0}^{r-k}\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+k}^{d^{\otimes k}}),$$

- *$t\mapsto P_t\in\mathcal{L}(\mathcal{B}_{r+1},\mathcal{B}_{r+2})$ is continuous at 0, $P\in\mathcal{L}(\mathcal{B}_{r+2})$,*
- *for all $|t|<\delta_0$ and all $j=1,\dots,r+1$,*

$$\forall f\in\mathcal{B}_j,\ \forall n\ge0\quad \|P_t^nf\|_{(j)}\le\vartheta^n\|f\|_{(j)}+R^n\|f\|_{(j+1)}.\tag{19}$$

Assume that $h\in\mathcal{B}_0$ and that $g:\Omega\to\mathbb{R}$ is such that $\mathbb{E}_\nu[g\cdot]$ defines a continuous linear form on $\mathcal{B}_r$. Then the assumptions of Theorem 5.5 are satisfied (except maybe the fact that $\mathbb{E}_\nu[g\cdot]$ is a linear form on $\mathcal{B}_{r+1}$) and

$$\mathbb{E}_\nu\left[g\,P_t^n(h)\right]=\lambda_t^n\left(\sum_{\ell=0}^r\frac{t^{\otimes\ell}}{\ell!}.\mathbb{E}_\nu\left[g\Pi_0^{(\ell)}(h)\right]+(t^r)\right)+\sum_{\ell=0}^r\frac{t^{\otimes\ell}}{\ell!}.\mathbb{E}_\nu\left[g\,(N^n)_0^{(\ell)}(h)\right]+\mathcal{O}(a^n|t|^r),\tag{20}$$

with $\Pi_0^{(\ell)}$ and $(N^n)_0^{(\ell)}$ given by (14) given by (15) and

$$\lambda_t-1=\sum_{k=2}^{r+1}\frac{\mathbb{E}_{\mathcal{P}_\mu\otimes\mathbf{P}}[(it.Y_1)^k]}{k!}+\frac{\sum_{k=1}^r\sum_{\ell=1}^{r+1-k}\frac{t^{\otimes(k+\ell)}}{k!\ell!}\mathbb{E}_\mu[P_0^{(k)}(\Pi_0^{(\ell)}(g)-\mathbb{E}_\mu[\Pi_0^{(\ell)}(g)])]}{1+\sum_{\ell=1}^{r-1}\frac{t^{\otimes\ell}}{\ell!}\mathbb{E}_\mu[\Pi_0^{(\ell)}(\mathbf{1})]}+o(|t|^{r+1}).$$

Proof. Observe that, given $\vartheta_1\in]\vartheta,1[$, up to reduce the value of δ_0, Theorem 3.3 applied with $\mathcal{B}_j,\mathcal{B}_{j+1}$ for $j=1,\dots,r+1$ ensures that (12), (13) and the first part of (16). We conclude by Theorem 5.5. □

Example 5.7 (Knudsen gas)**.** *Let $Q\in]r,r+1[$. Since $Y_1\in\mathbb{L}^{r+1}(\mathcal{P}_\mu\otimes\mathbf{P})$ and since $|P_t(H)|\le P(|H|)$, Example 5.2 satisfies the assumptions of Corollary 5.6 for $g\in\mathcal{V}_1$ for (17) and for $g\in\mathcal{V}_{r+1-Q}$ for (18) with $\mathcal{B}_j=\mathcal{V}_j$ for $j=0,\dots,r$, $\mathcal{B}_{r+1}=\mathcal{V}_Q$ and $\mathcal{B}_{r+2}=\mathcal{V}_{j+1}$, with*

- *either, if $\nu=\mu$, $\mathcal{V}_j:=\mathbb{L}^{\frac{r+1}{j}}(\mu)$ (with convention $\frac{r+1}{0}=\infty$),*
- *or, in the general case, $\mathcal{V}_j=(1+|f|)^jL^\infty$.*

This follows from Example 5.4 and from the two following facts

$$\left\|P_t-\sum_{k=0}^m\frac{P((it.f)^k\cdot)}{k!}\right\|_{\mathcal{L}(\mathcal{V}_j,\mathcal{V}_{j+m})}\le\|P\|_{\mathcal{V}_{j+m}}\left\|e^{it.f}-\sum_{k=0}^m\frac{(it.f)^k}{k!}\right\|_{\mathcal{V}_m}\le\|P\|_{\mathcal{V}_{j+m}}\left\|\frac{|tf|^m}{m!}\right\|_{\mathcal{V}_m}=\mathcal{O}(|t|^m),$$

and analogously

$$\left\| P_t - \sum_{k=0}^{r-j} \frac{P((it.f)^k \cdot)}{k!} \right\|_{\mathcal{L}(\mathcal{V}_j,\mathcal{V}_Q)} \leq \|P\|_{\mathcal{V}_Q} \left\| e^{itf} - \sum_{k=0}^{r-j} \frac{(itf)^k}{k!} \right\|_{\mathcal{V}_{Q-j}} \leq \|P\|_{\mathcal{V}_Q} \left\| |tf|^{Q-j} \right\|_{\mathcal{V}_{Q-j}} = \mathcal{O}(t^{Q-j}) \, .$$

Proof of Theorem 5.5. Proposition 4.1 applies so that $N_t^n - \sum_{\ell=0}^r \frac{t^{\otimes \ell}}{\ell!} .(N^n)_0^{(\ell)} = \mathcal{O}(a^n o(|t|^r))$ and $\Pi_t - \sum_{\ell=0}^r \frac{t^{\otimes \ell}}{\ell!} .\Pi_0^{(\ell)} = o(t^r)$ in $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_{r+1})$, with $\Pi_0^{(\ell)}, (N^n)_0^{(\ell)} \in \bigcap_{j=0}^{r-\ell} \mathcal{L}(\mathcal{B}_j, \mathcal{B}_{j+\ell})$ given by (14). This ends the proof of (17) and (18).

Due to Fact 3.2 with $v_0 = \mathbf{1}$ and $\varphi_0 = \mathbb{E}_\mu[\cdot]$,

$$\lambda_t - 1 = \mathbb{E}_\mu[(P_t - P)(\mathbf{1})] + \frac{\mathbb{E}_\mu\left[(P_t - P)(\mathrm{Id} - \Pi_0)(\Pi_t - \Pi_0)(\mathbf{1})\right]}{\mathbb{E}_\mu[\Pi_t(\mathbf{1})]} \, .$$

The first expectation of the above right hand side is in $\sum_{k=1}^{r+1} \frac{\mathbb{E}_{\mathcal{P}_\mu \otimes \mathbf{P}}[(it.Y_1)^k]}{k!} + o(|t|^{r+1})$ since Y_1 is $(r+1)$ times integrable and $\mathbb{E}_{\mathcal{P}_\mu \otimes \mathbf{P}}[Y_1] = 0$. Moreover, since $\mathbf{1} \in \mathcal{B}_0$ and $\mathcal{B}_{r+1} \hookrightarrow \mathbb{L}^1(\mu)$, we also know that

$$\mathbb{E}_\mu[\Pi_t] = \sum_{\ell=0}^r \frac{t^{\otimes \ell}}{\ell!} .\mathbb{E}_\mu[\Pi_0^{(\ell)}(\mathbf{1})] + o(t^r) \, .$$

It remains to study

$$\mathbb{E}_\mu\left[(P_t - P)(\Pi_t - \Pi_0)(\mathbf{1})\right] \, .$$

Due to the dominated convergence theorem, since Y_1 is $(r+1)$ times integrable,

$$\left\| e^{it.Y_1} - \sum_{k=0}^r \frac{(it.Y_1)^k}{k!} \right\|_{\mathbb{L}^{\frac{r+1}{r}}(\mathcal{P}_\mu \times \mathbf{P})} = o(t^r) \, .$$

Thus, since $\mathcal{B}_1 \hookrightarrow \mathbb{L}^{r+1}(\mathcal{P}_\mu \otimes \mathbf{P})$, in case (I):

$$\begin{aligned} \mathbb{E}_\mu[(P_t - P)(\Pi_t - \Pi_0)(\mathbf{1})] &= \mathbb{E}_{\mu \otimes \mathbf{P}}\left[(e^{it.f} - 1)(\Pi_t - \Pi_0)(\mathbf{1})\right] \\ &= \sum_{k=1}^r \frac{\mathbb{E}_{\mu \otimes \mathbf{P}}[(it.f)^k (\Pi_t - \Pi_0)(\mathbf{1})]}{k!} + o\left(|t|^r \|(\Pi_t - \Pi_0)(\mathbf{1})\|_{(1)}\right) \, . \end{aligned} \tag{21}$$

and, in case (II):

$$\begin{aligned} \mathbb{E}_\mu[(P_t - P)(\Pi_t - \Pi_0)(\mathbf{1})] &= \mathbb{E}_{\mathcal{P}_\mu \otimes \mathbf{P}}\left[(e^{it.Y_1} - 1)(\Pi_t - \Pi_0)(\mathbf{1})(X_1)\right] \\ &= \sum_{k=1}^r \frac{\mathbb{E}_{\mathcal{P}_\mu \otimes \mathbf{P}}[(it.Y_1)^k (\Pi_t - \Pi_0)(\mathbf{1})(X_1)]}{k!} + o\left(|t|^r \|(\Pi_t - \Pi_0)(\mathbf{1})\|_{(1)}\right) \, . \end{aligned} \tag{22}$$

Fix $k = 1, ..., r$. Note that $Y_1^k \in \mathbb{L}^{\frac{r+1}{k}}(\mathcal{P}_\mu \times \mathbf{P})$. Due to Proposition 4.1, we know that the quantity $\Psi_{r+1-k,t}(\mathbf{1}) := \Pi_t(\mathbf{1}) - \sum_{\ell=0}^{r+1-k} \frac{t^{\otimes \ell}}{\ell!} .\Pi_0^{(\ell)}(\mathbf{1})$ is in $\mathcal{O}(t^{r+1-k})$ in $\mathcal{B}_{r+1-k} \hookrightarrow \mathbb{L}^{\frac{r+1}{r+1-k}}(\mu)$ and is in $o(t^{r+1-k})$ in $\mathcal{B}_{r+1} \hookrightarrow \mathbb{L}^1(\mu)$. We will deduce that

$$\mathbb{E}_{\mu \otimes \mathbf{P}}\left[f^{\otimes k} .\Psi_{r+1-k,t}(\mathbf{1})\right] = o(|t|^{r+1-k}), \quad \text{resp. } \mathbb{E}_{\mathcal{P}_\mu \otimes \mathbf{P}}\left[Y_1^{\otimes k} .\Psi_{r+1-k,t}(\mathbf{1})(X_1)\right] = o(|t|^{r+1-k}) \, . \tag{23}$$

Indeed, since $\left(\Psi_{r+1-k,t}(\mathbf{1})/|t|^{r+1-k}\right)_t$ is bounded in $\mathbb{L}^{\frac{r+1}{r+1-k}}(\mu)$, it is contained in a relative compact set for the weak topology. Let h be one of its weak limits as $t \to 0$. Noticing that (23) holds true if $f^{\otimes k}$ (resp. $Y_1^{\otimes k}$) is replaced by any bounded measurable function H, we conclude that $h = 0$, and so that $\left(\Psi_{r+1-k,t}(\mathbf{1})/|t|^{r+1-k}\right)_t$ converges weakly in $\mathbb{L}^{\frac{r+1}{r+1-k}}(\mu)$ to 0. This ends the proof of (23) since $Y_1^k \in \mathbb{L}^{\frac{r+1}{k}}(\mathcal{P}_\mu \otimes \mathbf{P})$. We conclude by combining this with (21) and (22) respectively (and using the fact that $\|(\Pi_t - \Pi_0)(\mathbf{1})\|_{(1)} = \mathcal{O}(t)$ since $\mathbf{1} \in \mathcal{B}_0$). □

We now study the consequences on the smoothness of λ.

Proposition 5.8. *Assume Assumptions of Theorem 5.5 and $r \geq 1$. Then $\lambda_t = 1 - \frac{\mathfrak{a}}{2}.t^{\otimes 2} + o(|t|^2)$, with*

$$\mathfrak{a}.t^{\otimes 2} = \mathbb{E}_\mu[(t.Y_1)^2] + 2\sum_{n\geq 0} \mathbb{E}_\mu[(t.\mathcal{Q}_1)(P^n(t.\mathcal{Q}_1(\mathbf{1})))] = \sum_{n\in\mathbb{Z}} \mathbb{E}_{\mathcal{P}_\mu \otimes \mathbf{P}^{\otimes\mathbb{N}}}[(t.Y_1)(t.Y_{|n|+1})] \,.$$

Proof of Proposition 5.8. We write Π_0' for $\Pi_0^{(1)}$. We study the term of order t^2 of $\lambda_t - 1$. Observe that $\mathbb{E}_\mu[P_0'(\mathbf{1})] = 0$ since Y_1 is centered. Due to Proposition 4.1,

$$\begin{aligned}\mathbb{E}_\mu\left[\Pi_0'(\mathbf{1})\right] &= \frac{1}{2i\pi}\int_{\Gamma_1} \mathbb{E}_\mu\left[(z\operatorname{Id} - P)^{-1}P_0'(z\operatorname{Id} - P)^{-1}(\mathbf{1})\right]\,dz \\ &= \frac{1}{2i\pi}\int_{\Gamma_1} (z-1)^{-2}\mathbb{E}_\mu\left[P_0'(\mathbf{1})\right]\,dz = 0\,,\end{aligned}$$

since $\mathbb{E}_\mu[P(h)] = \mathbb{E}_\mu[h]$ and $P(\mathbf{1}) = \mathbf{1}$. Thus $\mathbb{E}_\mu[\Pi_0'(\mathbf{1})] = 0$. Therefore, writing $\mathbb{E}$ for $\mathbb{E}_{\mathcal{P}_\mu\otimes\mathbf{P}^{\otimes\mathbb{N}}}$, it follows from Theorem 5.5 that

$$\begin{aligned}\lambda_t - 1 &= -\frac{\mathbb{E}_{\mathcal{P}_\mu\otimes\mathbf{P}^{\otimes\mathbb{N}}}[(t.Y_1)^2]}{2} + t^{\otimes 2}.\mathbb{E}_\mu\left[P_0'(\Pi_0'(\mathbf{1}))\right] + o(|t|^2) \\ &= -\frac{\mathbb{E}_{\mathcal{P}_\mu\otimes\mathbf{P}^{\otimes\mathbb{N}}}[(t.Y_1)^2]}{2} + i\mathbb{E}_\mu\left[(t.\mathcal{Q}_1)(t.\Pi_0'(\mathbf{1}))\right] + o(|t|^2)\,.\end{aligned}$$

It follows from (14) combined with $(z\operatorname{Id} - P)^{-1}(\mathbf{1}) = (z-1)^{-1}\mathbf{1}$ that

$$\begin{aligned}\Pi_0'(\mathbf{1}) &= \frac{1}{2i\pi}\int_{\Gamma_1}(z\operatorname{Id} - P)^{-1}P_0'(z\operatorname{Id} - P)^{-1}(\mathbf{1})\,dz \\ &= \frac{i}{2i\pi}\int_{\Gamma_1}(z\operatorname{Id} - P)^{-1}\mathcal{Q}_1(\mathbf{1})(z-1)^{-1}\,dz \\ &= \frac{i}{2i\pi}\int_{\Gamma_1}(z-1)^{-1}\sum_{n\geq 0} z^{-n-1}P^n(\mathcal{Q}_1(\mathbf{1}))\,dz \\ &= i\sum_{n\geq 0} P^n(\mathcal{Q}_1(\mathbf{1}))\,,\end{aligned} \tag{24}$$

where we used the fact that $\|z^{-n}P^n(\mathcal{Q}_1(\mathbf{1}))\|_{\mathcal{L}(\mathcal{B}_1)} \leq (1-\delta)^{-n}a^n\|\mathcal{Q}_1(\mathbf{1})\|_{(1)}$ (since $\mathbb{E}_\mu[\mathcal{Q}_1(\mathbf{1})] = 0$ and recalling that $a < 1-\delta$). This ends the proof of the expression of $\mathfrak{a}$. □

Example 5.9 (Knudsen gas, normal case)**.** *Due to Example 5.7, Example 5.2 with $r \geq 1$ satisfies the assumptions of Proposition 5.8 and so $\lambda_t - 1 \sim -\frac{\mathfrak{a}}{2}.t^{\otimes 2}$, with*

$$\mathfrak{a}.t^{\otimes 2} = \sum_{n\in\mathbb{Z}} \mathbb{E}_{\mathcal{P}_\mu}\left[(t.f(X_1)).(t.f(X_{|n|+1}))\right] = \sum_{n\in\mathbb{Z}} \alpha^n \mathbb{E}_\mu[(t.f)^2] = \frac{1+\alpha}{1-\alpha}\mathbb{E}_\mu[(t.f)^2]\,.$$

If moreover $r \geq 2$, then the next proposition (Proposition 5.10) also applies, with

$$\mathfrak{b} = \mathbb{E}[f^{\otimes 3}]\left(1 + 6\sum_{n,m\geq 1}\alpha^{n+m} + 3\sum_{n\geq 1}2\alpha^n\right) = \frac{\alpha^2+4\alpha+1}{(1-\alpha)^2}\mathbb{E}[f^{\otimes 3}]\,.$$

Let us compute now the term of order 3 in the Taylor expansion of $t \mapsto \lambda_t$.

Proposition 5.10. *Assume the Assumptions of Theorem 5.5 with $r \geq 2$. Then $\lambda_t = 1 - \frac{\mathfrak{a}}{2}.t^{\otimes 2} - \frac{i\mathfrak{b}}{6}.t^{\otimes 3} + o(|t|^3)$, with $\mathfrak{a}$ as in Proposition 5.8 and with*

$$\mathfrak{b}.t^{\otimes 3} = \sum_{n,m\geq 0} a_{0,n,n+m}\mathbb{E}_{\mathcal{P}_\mu\otimes\mathbf{P}^{\otimes\mathbb{N}}}\left[(t.Y_1)(t.Y_{1+n})(t.Y_{1+n+m})\right] ,$$

with $a_{0,n,n+m} = \#\{(p,q,r)\in\mathbb{Z}^3 \,:\, \{p,q,r\} = \{0,n,n+m\}\}$.

Proof. Writing again simply $\mathbb{E}$ for $\mathbb{E}_{\mathcal{P}_\mu\otimes\mathbf{P}^{\otimes\mathbb{N}}}$, it follows that

$$\lambda_t - 1 = -\frac{\mathfrak{a}.t^{\otimes 2}}{2} - \frac{i}{6}\mathbb{E}_{\mathcal{P}_\mu\otimes\mathbf{P}}[(t.Y_1)^3] + \frac{it^{\otimes 3}}{2}.\mathbb{E}_\mu\left[\mathcal{Q}_1(\Pi_0''(\mathbf{1}))\right] - \frac{t^{\otimes 3}}{2}.\mathbb{E}_\mu\left[\mathcal{Q}_2(\Pi_0'(\mathbf{1}))\right] + o(|t|^3)\,.$$

Recall that, due to (24), $\Pi_0'(\mathbf{1}) = i\sum_{n\geq 0}P^n(\mathcal{Q}_1(\mathbf{1}))$. It follows moreover from (14) that

$$\begin{aligned}\Pi_0''(\mathbf{1}) &:= \frac{1}{2i\pi}\int_{\Gamma_1}(z\,\mathrm{Id}-P)^{-1}(P_0''+2P_0'(z\,\mathrm{Id}-P)^{-1}P_0')(z-1)^{-1}(\mathbf{1})\,dz\\ &= \frac{1}{2i\pi}\int_{\Gamma_1}(z-1)^{-1}(z\,\mathrm{Id}-P)^{-1}(-\mathcal{Q}_2+2i\mathcal{Q}_1(z\,\mathrm{Id}-P)^{-1}i\mathcal{Q}_1)(\mathbf{1})\,dz\\ &= -\frac{1}{2i\pi}\int_{\Gamma_1}I_1(z)+I_2(z)\,dz\,,\end{aligned}$$

with

$$I_1(z) := (z-1)^{-2}\left(\mathbb{E}_\mu[\mathcal{Q}_2(\mathbf{1})] + 2\mathbb{E}_\mu\left[\mathcal{Q}_1\left(\sum_{m\geq 0}z^{-m-1}P^m(\mathcal{Q}_1(\mathbf{1}))\right)\right]\right)$$

$$\begin{aligned}I_2(z) := (z-1)^{-1}\sum_{n\geq 0}z^{-n-1}P^n\Bigg(&\mathcal{Q}_2(\mathbf{1}) - \mathbb{E}_\mu[\mathcal{Q}_2(\mathbf{1})]\\ &+2\mathcal{Q}_1\left(\sum_{m\geq 0}z^{-m-1}P^m(\mathcal{Q}_1(\mathbf{1}))\right) - 2\mathbb{E}_\mu\left[\mathcal{Q}_1\left(\sum_{m\geq 0}z^{-m-1}P^m(\mathcal{Q}_1(\mathbf{1}))\right)\right]\Bigg)\,.\end{aligned}$$

It follows that

$$\begin{aligned}-\Pi_0''(\mathbf{1}) := -2\sum_{m\geq 0}(m+1)\mathbb{E}_\mu\left[\mathcal{Q}_1\left(P^m(\mathcal{Q}_1(\mathbf{1}))\right)\right] + \sum_{n\geq 0}\left(P^n(\mathcal{Q}_2(\mathbf{1})) - \mathbb{E}_\mu[\mathcal{Q}_2(\mathbf{1})]\right)\\ + 2\sum_{n\geq 0}\left(P^n\left(\mathcal{Q}_1\sum_{m\geq 1}(P^m\mathcal{Q}_1(\mathbf{1})) - \sum_{m\geq 0}\mathbb{E}_\mu\left[\mathcal{Q}_1\left(P^m(\mathcal{Q}_1(\mathbf{1}))\right)\right]\right)\right)\,.\end{aligned}$$

This ends the proof of the lemma. □

In view of establishing results of convergence to a stable random variable, we consider now a less smooth situation. If we assume that the distribution of Y_1 is in the standard domain of attraction of an α_0-stable distribution with $\alpha_0\in]1,2[$ (so that $\mathbb{P}(|Y|>s)\sim|s|^{-\alpha_0}$ as $s\to+\infty$), then we expect that $\lambda_t - 1 \sim -c|t|^{\alpha_0}$. But, unlike in Theorem 5.5, we cannot use an argument of weak convergence to conclude, since we do not have convergence of $\frac{P_t-P_0-tP_0'}{t^{\alpha_0}}$ and thus we cannot hope the convergence of $\frac{\Pi_t-\Pi_0-t\Pi_0'}{t^{\alpha_0}}$. The next general statement can be seen as a first step to convergence to stable random variables. We will apply it immediatly on our easy Knudsen gas model.

Proposition 5.11 ($d=1$)**.** *Let $\delta>0$. Assume Hypothesis 5.1 with $d=1$ and that there exist two Banach spaces $\mathcal{B}_1,\mathcal{B}_2$ such that $\mathbf{1}\in\mathcal{B}_1\hookrightarrow\mathcal{B}_2\hookrightarrow\mathbb{L}^1(\mu)$, and such that P_t (for any $|t|<\delta$) acts continuously on $\mathcal{B}_1$. Assume P is geometrically ergodic on $\mathcal{B}_1$ and $\mathcal{B}_2$: $\|P^n-\mathbb{E}_\mu\|_{\mathcal{L}(\mathcal{B}_j)} =$*

$\|\widetilde{N}_0^n\|_{\mathcal{L}(\mathcal{B}_j)} = \mathcal{O}(\vartheta^n)$. *Assume moreover that* $P_t - P_0$ *is in* $\mathcal{O}(t)$ *in* $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ *and in* $o(1)$ *in* $\mathcal{L}(\mathcal{B}_2, \mathbb{L}^1(\mu))$. *Then*

$$\lambda_t - 1 \sim \mathbb{E}_\mu\left[(P_t - P)(\mathrm{Id} - \widetilde{N}_t)^{-1}(\mathbf{1})\right] + o(|t|^2)$$
$$\sim \sum_{n\geq 0} \mathbb{E}_\mu\left[(P_t - P)\widetilde{N}_t^n(\mathbf{1})\right] + o(|t|^2)\,,$$

with $\widetilde{N}_t := (\mathrm{Id} - \Pi_0)P_t$ *(note that* $\widetilde{N}_t(h) = \mathbb{E}[e^{itY_1}h(X_1)|X_0] - \mathbb{E}[e^{itY_1}h(X_1)]$ *in the Markovian context) and* $\|\widetilde{N}_t^n\|_{\mathcal{L}(\mathcal{B}_1)} = \mathcal{O}(\vartheta_1^n)$ *(for all* t *small enough).*

Example 5.12 (Knudsen gas, stable case)**.** *Let* $\alpha_0 \in]1, 2]$ *and* $p > 1$ *such that* $\|f\|_{\mathbb{L}^p(\mu)} < \infty$. *Consider the Knudsen gas introduced in Example 5.2 with* $r = 0$ *and* $d = 1$. *Assume that the characteristic function* φ_f *of* f *with respect to* μ *satisfies*

$$\varphi_f(t) - 1 = \mathbb{E}_\mu[e^{itf}] - 1 = -|t|^{\alpha_0}(1 - i\beta\,\mathrm{sgn}(t))L_0(|t|^{-1})\,, \tag{25}$$

with $|\beta| < \tan(\alpha_0\pi/2)$ *and* $c > 0$ *and with* L_0 *slowly varying at infinity. Then*

$$\lambda_t - 1 \sim (\varphi_f(t) - 1)(1 - \alpha)^2 \sum_{n\geq 0} \alpha^n (n+1)^{\alpha_0}\,, \quad \textit{as } t \to 0\,.$$

Proof of Example 5.12. We apply Proposition 5.11 with $\mathcal{B}_1 = \mathbb{L}^\infty(\mu)$ and $\mathcal{B}_2 = \mathbb{L}^p(\mu)$. We have already seen in Example 5.4 that P is geometrically ergodic on $\mathcal{B}_1$ and on $\mathcal{B}_2$ and that P_t acts continuously on both these spaces. Moreover

$$\|P_t - P\|_{\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)} \leq \|e^{itf} - 1\|_{\mathbb{L}^p(\mu)} = \|tf\|_{\mathbb{L}^p(\mu)} = \mathcal{O}(|t|)\,,$$

since $f \in \mathbb{L}^p(\mu)$ and

$$\|P_t - P\|_{\mathcal{L}(\mathcal{B}_2, \mathbb{L}^1(\mu))} \leq \|e^{itf} - 1\|_{\mathbb{L}^q(\mu)} = o(1)\,,$$

with $q \in]1, +\infty[$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Moreover $\widetilde{N}_t = \alpha\widetilde{M}_t$ with $\widetilde{M}_t := (\mathrm{Id} - \Pi_0)(e^{itf}\cdot)$, and so

$$\sum_{n\geq 0} \mathbb{E}_\mu\left[(P_t - P)\widetilde{N}_t^n(\mathbf{1})\right] = \mathcal{O}\left(\alpha^{n_t} e^{n_t\varepsilon(t)}\right) + \sum_{n=0}^{n_t} \alpha^n \mathbb{E}_\mu\left[(e^{itf} - 1)\widetilde{M}_t^n(\mathbf{1})\right]$$
$$= \mathcal{O}\left(t^3\right) + \sum_{n=0}^{n_t} \alpha^n \mathbb{E}_\mu\left[(e^{itf} - 1)\widetilde{M}_t^n(\mathbf{1})\right]\,,$$

with $\lim_{t\to 0}\varepsilon(t) = 0$ and $n_t = \lfloor(\log(t))^2\rfloor$. Moreover, for $n \geq 1$,

$$\mathbb{E}_\mu\left[(e^{itf} - 1)\widetilde{M}_t^n(\mathbf{1})\right] = \mathbb{E}_\mu\left[(e^{itf} - 1)e^{itnf}\right] - \sum_{m=1}^{n} \mathbb{E}_\mu\left[(e^{itf} - 1)e^{it(m-1)f}\right]\mathbb{E}_\mu\left[e^{itf}\widetilde{M}_t^{n-m}(\mathbf{1})\right]$$
$$= \mathbb{E}_\mu\left[(e^{itf} - 1)e^{itnf}\right] - \mathbb{E}\left[(e^{itf} - 1)e^{it(n-1)f}\right]\mathbb{E}_\mu\left[e^{itf}\right]$$
$$- \sum_{m=1}^{n-1} \mathbb{E}_\mu\left[(e^{itf} - 1)e^{it(m-1)f}\right]\mathbb{E}_\mu\left[e^{itf}\widetilde{M}_t^{n-m-1}(e^{itf} - \mathbb{E}[e^{itf}])\right]$$
$$= \mathbb{E}_\mu\left[(e^{itf} - 1)(e^{itnf} - e^{it(n-1)f})\right] + \mathcal{O}\left(t^{1+r'} + \sum_{m=1}^{n-1} |mt|^{r'} e^{(n-m)\varepsilon(t)} 2\left\|e^{itf} - 1\right\|_{\mathbb{L}^{r'}(\mu)}\right)$$
$$= \mathbb{E}_\mu\left[e^{it(n+1)f} - 2e^{itnf} + e^{it(n-1)f}\right] + \mathcal{O}\left(|nt|^{1+r'} e^{n|\varepsilon(t)|}\right)\,.$$

It follows that

$$\sum_{n\geq 0} \mathbb{E}_\mu\left[(P_t - P)\widetilde{N}_t^n(\mathbf{1})\right] = \mathcal{O}\left(t^{1+r'}\right) + \mathbb{E}_\mu\left[e^{itf} - 1\right] + \sum_{n=1}^{n_t} \alpha^n \mathbb{E}_\mu\left[e^{it(n+1)f} - 2e^{itnf} + e^{it(n-1)f}\right]\,.$$

Recall that Karamata proved in [29] that there exist $u_0 > 0$ and two functions c, ε_0 such that $\lim_{+\infty} c > 0$ and $\lim_{+\infty} \varepsilon_0 = 0$ such that $L_0(u) = c(u)e^{\int_{u_0}^{u} \frac{\varepsilon_0(s)}{s}\, ds}$. Thus

$$\begin{aligned}
&\mathbb{E}_\mu\left[e^{itf}-1\right] + \sum_{n=1}^{n_t} \alpha^n \mathbb{E}\left[e^{it(n+1)f} - 2e^{itnf} + e^{it(n-1)f}\right] \\
&= \varphi_f(t) - 1 - \sum_{n=1}^{n_t} \alpha^n \left(\varphi_f((n+1)t) + \varphi_f((n-1)t) - 2\varphi_f(nt)\right) \\
&\sim (\varphi_f(t)-1)\left[1 + \sum_{n=1}^{n_t} \alpha^n \left((n+1)^{\alpha_0} - 2|n|^{\alpha_0} + (n-1)^{\alpha_0}\right)\right] \\
&\sim (\varphi_f(t)-1)\left[1 + (\alpha^{n_t+1}((n_t+1)^{\alpha_0} - n_t^{\alpha_0}) - \alpha) - \sum_{n=1}^{n_t}((n+1)^{\alpha_0} - n^{\alpha_0})(\alpha^{n+1}-\alpha^n)\right] \\
&\sim (\varphi_f(t)-1)(1-\alpha)\left[\sum_{n=0}^{n_t}((n+1)^{\alpha_0} - n^{\alpha_0})\alpha^n\right],
\end{aligned}$$

due to the dominated convergence theorem since, for all $n \geq 1$, $\varphi_f(nt) - 1 \sim n^{\alpha_0}(\varphi_f(t)-1)$ as $t \to 0$ and, for all $n = 1, ..., n_t$,

$$\frac{\varphi_f(nt)-1}{\varphi_f(t)-1} = n^{\alpha_0}\frac{L_0(|nt|^{-1})}{L_0(|t|^{-1})} = n^{\alpha_0}\frac{c(|nt|^{-1})}{c(|t|^{-1})}e^{-\int_{|nt|^{-1}}^{|t|^{-1}} \frac{\varepsilon_0(s)}{s}\, ds} = \mathcal{O}\left(n^{\alpha_0 + \sup_{|s|>|t(\log(t))^2|^{-1}} |\varepsilon_0(s)|}\right).$$

and finally by using Abel's summation by parts formula. Finally, using Abel's summation by parts formula a second time, we obtain that

$$\begin{aligned}
&\sum_{n\geq 0}\mathbb{E}_\mu\left[(P_t - P)\widetilde{N}_t^n(\mathbf{1})\right] \\
&\quad\sim (\varphi_f(t)-1)\left[\alpha^{n_t+1}(n_t+1)^{\alpha_0} - \sum_{n=0}^{n_t}(n+1)^{\alpha_0}(\alpha^{n+1}-\alpha^n)\right] \\
&\quad\sim (\varphi_f(t)-1)(1-\alpha)\left[\sum_{n\geq 0}(n+1)^{\alpha_0}\alpha^n\right],
\end{aligned}$$

which combined with Proposition 5.11 ends the proof of Example 5.12. □

Proof of Proposition 5.11. Note that the assumptions on $P_t - P$ imply also that $\widetilde{N}_t - \widetilde{N}_0$ is in $\mathcal{O}(t)$ in $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ and in $o(1)$ in $\mathcal{L}(\mathcal{B}_2, \mathbb{L}^1(\mu))$. Observe that $\widetilde{N}_0^n = P^n - \mathbb{E}_\mu$.

It follows from Theorem 3.3 applied to $(P_t)_t$ and to $(\widetilde{N}_t)_t$ that, for $\vartheta_1 \in]\vartheta, 1[$, up to reduce δ, Formulas (12) with (13) hold true on $\mathcal{B}_1$ and that for all $n \geq 1$ and all $|t| < \delta$, $\|\widetilde{N}_t^n\|_{\mathcal{L}(\mathcal{B}_1)} = \mathcal{O}(\vartheta_1^n)$. This will ensure that $(z\,\mathrm{Id} - \widetilde{N}_t)^{-1} = \sum_{n\geq 0} z^{-n-1}\widetilde{N}_t^n \in \mathcal{L}(\mathcal{B}_1)$ is uniformly bounded in $z \in \Gamma_1$ and that

$$\Pi_t = \frac{1}{2i\pi}\int_{\Gamma_1}(z\,\mathrm{Id} - P_t)^{-1}\, dz$$

is continuous from $]-\delta, \delta[$ to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$. Moreover $(z\,\mathrm{Id} - \widetilde{N}_0)^{-1} = \sum_{n\geq 0} z^{-n-1}\widetilde{N}_0^n \in \mathcal{L}(\mathcal{B}_2)$ for $z \in \Gamma_1$. Due to Fact 3.2,

$$\lambda_t - 1 = \frac{\mathbb{E}_\mu\left[(P_t - P)(\Pi_t(\mathbf{1}))\right]}{\mathbb{E}_\mu\left[\Pi_t(\mathbf{1})\right]} = \mathbb{E}_\mu\left[(P_t - P)\Pi_t(\mathbf{1})\right](1 + \mathcal{O}(t)), \quad \text{as } t \to 0.$$

A direct computation shows that

$$
\begin{aligned}
(z\,\mathrm{Id}-P_t)^{-1}(\mathbf{1}) &= \left(z\,\mathrm{Id}-\widetilde{N}_t\right)^{-1}\left(\mathbf{1}-\frac{\mathbb{E}_\mu[P_t(\widetilde{N}_t-z\,\mathrm{Id})^{-1}(\mathbf{1})]}{1+\mathbb{E}_\mu[(\widetilde{N}_t-z\,\mathrm{Id})^{-1}(\mathbf{1})]}\right)\\
&= (z\,\mathrm{Id}-\widetilde{N}_t)^{-1}(\mathbf{1})b_t(z), \quad \text{with } \ b_t(z):=\left(\frac{1+\mathbb{E}_\mu\left[(P_t-P)(z\,\mathrm{Id}-\widetilde{N}_t)^{-1}(\mathbf{1})\right]}{1-\mathbb{E}_\mu\left[(z\,\mathrm{Id}-\widetilde{N}_t)^{-1}(\mathbf{1})\right]}\right).
\end{aligned}
$$

Note that we can recover the fact that $(z\,\mathrm{Id}-P)^{-1}(\mathbf{1})=(z-1)^{-1}$. Let us prove that

$$
\sup_{z\in\Gamma_1}|b_t(z)-b_0(z)|=\mathcal{O}(|t|)\,. \tag{26}
$$

Since $\sup_{z\in\Gamma_1}\|(z\,\mathrm{Id}-\widetilde{N}_t)^{-1}\|_{\mathcal{L}(\mathcal{B}_1)}<\infty$ and $\|P_t-P\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}=\mathcal{O}(t)$, the numerator of $b_t(z)$ is in $1+\mathcal{O}(t)$ uniformly in $z\in\Gamma_1$. Moreover

$$
\left((z\,\mathrm{Id}-\widetilde{N}_t)^{-1}-(z\,\mathrm{Id}-\widetilde{N}_0)^{-1}\right)=(z\,\mathrm{Id}-\widetilde{N}_0)^{-1}(\widetilde{N}_t-\widetilde{N}_0)\left(z\,\mathrm{Id}-\widetilde{N}_t\right)^{-1}=\mathcal{O}(t) \tag{27}
$$

in $\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)$ uniformly in $z\in\Gamma_1$, since $\left\|\widetilde{N}_t-\widetilde{N}_0\right\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}=\mathcal{O}(t)$ and

$$
\sup_{|t|<\delta}\sup_{z\in\Gamma_1}\left(\|(z\,\mathrm{Id}-\widetilde{N}_0)^{-1}\|_{\mathcal{L}(\mathcal{B}_2)}+\|(z\,\mathrm{Id}-\widetilde{N}_t)^{-1}\|_{\mathcal{L}(\mathcal{B}_1)}\right)<\infty\,.
$$

This ends the proof of (26).

It follows also from (27), that $(z\,\mathrm{Id}-\widetilde{N}_t)^{-1}(\mathbf{1})=z^{-1}\mathbf{1}+\mathcal{O}(t)$ in $\mathcal{B}_2$. Since moreover $P_t-P=o(1)$ in $\mathcal{L}(\mathcal{B}_2,\mathbb{L}^1(\mu))$ and $\mathbb{E}_\mu[(P_t-P)(\mathbf{1})]=\varphi_f(t)-1=o(t)$ (since Y_1 is centered),

$$
\sup_{z\in\Gamma_1}\left|\mathbb{E}_\mu\left[(P_t-P)(z\,\mathrm{Id}-\widetilde{N}_t)^{-1}(\mathbf{1})\right]\right|=\sup_{z\in\Gamma_1}\left|\mathbb{E}_\mu\left[(P_t-P)(z^{-1}\mathbf{1})\right]\right|+o(t)=o(t)\,.
$$

Whence

$$
\begin{aligned}
\mathbb{E}_\mu\left[(P_t-P)\Pi_t(\mathbf{1})\right] &= \frac{1}{2i\pi}\int_{\Gamma_1}\mathbb{E}_\mu\left[(P_t-P)(z\,\mathrm{Id}-\widetilde{N}_t)^{-1}(\mathbf{1})\right]b_0(z)\,dz\\
&\quad+\mathcal{O}\left(o(t)\sup_{z\in\Gamma_1}|b_t(z)-b_0(z)|\right)\\
&=\mathbb{E}_\mu\left[(P_t-P)(\mathrm{Id}-\widetilde{N}_t)^{-1}(\mathbf{1})\right]+o(t^2)\,, \quad \text{as } t\to 0\,,
\end{aligned}
$$

where we used the fact that $b_0(z)=\frac{z}{z-1}$. □

6. Probabilistic limit theorems

Let $\delta_0>0$. Let $(S_n)_{n\ge 1}$ be a sequence of $\mathbb{X}$-valued random variables with $\mathbb{X}=\mathbb{R}^d$ or $\mathbb{Z}^d$ defined on a probability space $(\mathcal{M},\mathbb{P})$ such that

$$
\forall n\in\mathbb{N},\quad \forall t\in\mathbb{R}^d, |t|<\delta_0,\quad \mathbb{E}[e^{it.S_n}]=\lambda_t^n\Phi_t+M_{t,n}\,. \tag{28}
$$

We set $\mathbb{X}^*:=\mathbb{R}^d$ if $\mathbb{X}=\mathbb{R}^d$ and $\mathbb{X}^*:=[-\pi,\pi]^d$ if $\mathbb{X}=\mathbb{Z}^d$.

Remark 6.1. *In Remark 5.3, we have seen general situations in which $\mathbb{E}[e^{it.S_n}]=\mathbb{E}_\nu[P_t^n(h_0)]$ (see 10 and 11) for some h_0 and some family of operators $(P_t)_t$ such that 28 holds true with λ and $\Phi=\mathbb{E}_\nu[\Pi.(h_0)]$ continuous and with $\sup_{|t|<\delta_0}|M_{t,n}|=\mathcal{O}\left(\sup_{|t|<\delta_0}|||N_t^n(h)|||_{\mathcal{B}_2}\right)$ decaying exponentially fast in n. Recall moreover that further Taylor expansions have been studied in Theorem 5.5.*

The goal of this section is to establish probabilistic limit theorems for $(S_n)_{n\geq 1}$. More precisely, we will study situations in which $(S_n)_{n\geq 1}$ satisfies the same kind of limit theorems as sums of independent identically distributed random variables with characteristic function behaving at 0 as $t\mapsto \lambda_t$.

6.1. **Central and local limit theorems.** Let $\mathcal{W}$ be a $\mathbb{R}^d$-valued random variable.

Theorem 6.2 (Central Limit Theorem (CLT))**.** *Let $(A_n)_{n\geq 1}$ be a sequence of (normalizing) $d\times d$ matrices converging to 0. Assume that $\lim_{t\to 0}\Phi_t=1$, that $\lim_{n\to+\infty}\sup_{|t|<\delta_0}|M_{t,n}|=0$ and that $\lim_{n\to+\infty}\lambda^n_{A_n^*t}=\mathbb{E}[e^{it.\mathcal{W}}]$ for all $t\in\mathbb{R}^d$ (writing A_n^* for the transpose matrix of A_n). Then $(A_nS_n)_{n\geq 1}$ converges in distribution to $\mathcal{W}$.*

The condition $\lim_{n\to+\infty}\lambda^n_{A_n^*t}=\mathbb{E}[e^{it.\mathcal{W}}]$ means that λ_t behaves at 0 as tha characteristic function of a distribution belonging to the domain of attraction of the stable distribution of $\mathcal{W}$. In particular, if $\lambda_t-1\sim_0 -a|\Sigma t|_\alpha^\alpha$ with $|s|_\alpha^\alpha=\sum_{i=1}^d|s_i|^\alpha$ and with Σ an invertible matrix, then, setting $A_n=n^{-\frac{1}{\alpha}}\,\mathrm{Id}$ and considering $\mathcal{W}$ such that $\mathbb{E}[e^{it.\mathcal{W}}]=e^{-a|\Sigma t|_\alpha^\alpha}$, the following estimate holds true for any $t\in\mathbb{R}^d$

$$\begin{aligned}\left|\lambda^n_{A_n^*t}-\mathbb{E}[e^{it.\mathcal{W}}]\right| &= \left|\lambda^n_{t/n^{\frac{1}{\alpha}}}-e^{-an|\Sigma t/n^{\frac{1}{\alpha}}|_\alpha^\alpha}\right|\\ &\leq n\left|\lambda_{t/n^{\frac{1}{\alpha}}}-e^{-a|\Sigma t/n^{\frac{1}{\alpha}}|^\alpha}\right|\leq n\,o\left(\left|\Sigma t/n^{\frac{1}{\alpha}}|_\alpha^\alpha\right|\right)=o(1)\,,\end{aligned}$$

as n goes to infinity.

Remark 6.3. *Again, in the context of Remark 5.3,*

(a) *Under assumptions of Theorem 5.5 for $r\geq 1$ (and so of Proposition 5.8), Theorem 6.2 holds true with $A_n=\mathrm{Id}/\sqrt{n}$ and $\mathcal{W}$ a centered Gaussian random variable with variance*
$$a:=\sum_{n\in\mathbb{Z}}\mathcal{C}ov_{\mathcal{P}_\mu\otimes\mathbf{P}^{\otimes\mathbb{N}}}(Y_1,Y_{|n|+1})\,.$$

(b) *If Proposition 5.11 allows to prove that $\lambda_t-1\sim\mathbb{E}[e^{it.W_1}]-1$, where $(W_n)_{n\geq 1}$ is a sequence of i.i.d. random variables such that $(A_n\sum_{k=1}^n W_k)_n$ converges in distribution to $\mathcal{W}$, then Theorem 6.2 applies.*

Example 6.4 (Knudsen gas, convergence to gaussian or stable distributions)**.** *Consider the simple Knudsen gas model introduced in Example 5.2. The continuity of $t\mapsto\Phi_t=\mathbb{E}_\mu[\Pi_t(\mathbf{1})]$ as well as the fact that $\sup_{|t|<\delta}\mathbb{E}_\mu[N_t^n(\mathbf{1})]$ decays exponentially fast as $n\to+\infty$ have been proved in Example 5.4 thanks to Theorem 3.3 and Remark 6.1.*

If $r\geq 1$, then $(\sum_{k=0}^{n-1}f(X_k)/\sqrt{n})_{n\geq 1}$ converges in distribution to a centered Gaussian random variable with variance matrix $\frac{1+\alpha}{1-\alpha}\mathbb{E}[f^{\otimes 2}]$. Indeed the second order Taylor expansion of $t\mapsto\lambda_t$ has been proved in Example 5.9.

Consider now the situation of Example 5.12, that is $r=0$, $d=1$ and there exists $\alpha_0\in]1,2]$ and a function L_0 slowly varying at infinity such that the characteristic function φ_f of f with respect to μ satisfies

$$\varphi_f(t)-1\sim|t|^{\alpha_0}(1-i\beta\,\mathrm{sgn}(t))L_0(|t|^{-1})$$

as $t\to 0$, for some $|\beta|<\tan(\alpha_0\pi/2)$. Then $(A_n\sum_{k=0}^{n-1}f(X_k))_{n\geq 1}$ converges in distribution to the stable law with characteristic function $t\mapsto e^{-c|t|^{\alpha_0}(1-i\beta\,\mathrm{sgn}(t))}$, for $c=(1-\alpha)^2\sum_{n\geq 0}\alpha^n(n+1)^{\alpha_0}$ and where $A_n\to 0$ is so that $\lim_{n\to+\infty}n|A_n|L_0(A_n^{-1})=1$.

Indeed, due to Example 5.12, $\lambda_t - 1 \sim c|t|^{\alpha_0} L_0(|t|^{-1})(1 - i\beta\,\mathrm{sgn}(t))$. *As in the usual proof of convergence of sum of independent identically distributed random variables to stable law (see* [27, Theorem 2.6.5]*), it follows that*

$$\forall t \in \mathbb{R}, \quad \lambda^n_{A_n t} = e^{-nc|A_n t|^{\alpha_0} L_0(|A_n t|^{-1})(1+o(1)))} \sim e^{-c|t|^{\alpha_0}(1-i\beta\,\mathrm{sgn}(t))},$$

as $n \to +\infty$.

Proof of Theorem 6.2. We prove the convergence of characteristic functions. We fix $t \in \mathbb{R}^d$ and write

$$\mathbb{E}\left[e^{it.(A_n S_n)}\right] - \mathbb{E}[e^{it.\mathcal{W}}] = \lambda^n_{A_n^* t} \Phi_{A_n^* t} + M_{A_n^* t, n} - \mathbb{E}[e^{it.\mathcal{W}}] = o(1)\,,$$

since $\lim_{n\to+\infty} \Phi_{A_n^* t} = 1$, $\lim_{n\to+\infty} \lambda^n_{A_n^* t} = \mathbb{E}[e^{it.\mathcal{W}}]$ and since

$$\lim_{n\to+\infty} M_{A_n^* t,n} \le \lim_{n\to+\infty} \sup_{|u|<\delta_0} |M_{u,n}| = 0\,.$$

Thus $(A_n S_n)_{n\ge1}$ converges in distribution to $\mathcal{W}$. □

Theorem 6.5 (Local Limit Theorem)**.** *Assume Assumptions of Theorem 6.2 with* A_n *invertible,* $\sup_{|t|<\delta_0} |\Phi(t)| < \infty$, $\sup_{|t|<\delta_0} |M_{t,n}| = o(\det A_n)$ *and* $|\lambda^n_{A_n^* t}| \le g(t)$ *if* $|A_n^* t| < \delta_0$, *with* g *integrable on* $\mathbb{R}^d$. *Assume moreover that* $\mathcal{W}$ *has density* $h_{\mathcal{W}}$ *and integrable characteristic function and that* f *takes its values in* $\mathbb{Z}^d$ *and that* $\sup_{t\in[-\pi,\pi]^d\setminus B(0,\delta_0)} \left|\mathbb{E}[e^{it.S_n}]\right| = o(\det A_n)$. *Then*

$$\sup_{k\in\mathbb{Z}} |\mathbb{P}(S_n = k) - \det(A_n) h_{\mathcal{W}}(A_n k)| = o(\det(A_n))\,.$$

Remark 6.6. *Consider the context of Remark 6.3. We have already seen that* $\sup_{|t|<\delta_0} |\Phi(t)| < \infty$ *and* $\sup_{|t|<\delta_0} |M_{t,n}| = o(\det A_n)$ *for some* $\delta_0 > 0$.
The integrability of $t \mapsto \sup_n |\lambda^n_{A_n^* t} \mathbf{1}_{|A_n^* t|<\delta_0}|$ *follows in practice from the control of* $|\lambda_t - 1|$ *(e.g. in case (a) of Remark 6.3, if* $\mathfrak{a}$ *is invertible,* $|\lambda_t| \le e^{-\frac{\mathfrak{a}.t^{\otimes 2}}{4}}$ *as soon as* $|t|$ *is small enough).*
Finally, the condition $\sup_{t\in[-\pi,\pi]^d\setminus B(0,\delta_0)} \left|\mathbb{E}[e^{it.S_n}]\right| = o(\det A_n)$ *is usually a consequence of the fact that, for* $t \notin 2\pi\mathbb{Z}$, $\rho_{ess}(P_t) < 1$, $\rho(P_t) \le 1$ *and that* P_t *admits no eigenvalue of modulus 1, which imply that* $\|P_t^{n_t}\| < 1$ *(for some* n_t*) which, combined with a continuity argument of* $t \mapsto \|P_t\|$ *on the compact* $[-\pi,\pi]^d$, *leads to the existence of a positive integer* n' *such that* $\sup_{t\in[-\pi,\pi]^d\setminus B(0,\delta_0)} \|P_t^{n'}\| < 1$ *and implies the exponential decay of* $\sup_{t\in[-\pi,\pi]^d\setminus B(0,\delta_0)} \left|\mathbb{E}[e^{it.S_n}]\right|$ *as* $n \to +\infty$.

To complete this remark, let us indicate that, under the assumptions of Theorem 3.3 (Keller and Liverani theorem), it has been proved in [26, Propositions 5.3 and 5.4] that the non nonlattice property together with an additional reasonnable condition imply the exponential decay of $\sup_{t\in[-\pi,\pi]^d\setminus B(0,\delta_0)} \left|\mathbb{E}[e^{it.S_n}]\right|$ as n goes to infinity.

Example 6.7 (Knudsen gas, LLT)**.** *Theorem 6.5 applies in the situation considered in Example 6.4, provided* f *takes its values in* $\mathbb{Z}^d$ *but is not supported by a sublattice of* $\mathbb{Z}^d$.

Proof of Example 6.7. The domination of $|\lambda^n_{A_n^* t}|$ follows from the estimates already established in Example 6.4. It remains to prove that $\sup_{t\in[-\pi,\pi]^d\setminus B(0,\delta_0)} \left|\mathbb{E}[e^{it.S_n}]\right| | = o(\det A_n)$. To this end, we follow the strategy explained in Remark 6.6. Let $t \in \mathbb{R}\setminus 2\pi\mathbb{Z}$ and $p \in]1, +\infty[$. Since $\|P_t\|_{\mathbb{L}^p(\mu)} \le \alpha\|h\| + (1-\alpha)\|h\|_{\mathbb{L}^1(\mu)}$, it follows by standard arguments (see e.g. [28, 18]) that the essential spectral radius of P_t is strictly smaller than 1. Consider now $h \in \mathbb{L}^p(\mu)$ and $\lambda = e^{id} \in \mathbb{C}$ with $d \in \mathbb{R}$ and $\lambda h = P_t(h) = \alpha e^{it.f} h + (1-\alpha)\mathbb{E}_\mu[e^{it.f} h]$. Then $\mathbb{E}_\mu[|h|] \le \mathbb{E}_\mu[|\alpha e^{it.f} h|] + |(1-\alpha)\mathbb{E}_\mu[e^{it.f} h]| \le \mathbb{E}_\mu[|h|]$ and we conclude that $\lambda h = e^{it.f} h = \mathbb{E}_\mu[e^{it.f} h]$ μ-a.e.,

thus $h = e^{-id}e^{it.f}h$ is constant. So either $h = 0$ or $e^{id} = e^{it.f}$. But $e^{id} = e^{it.f}$ would mean that $t.f \in d + 2\pi\mathbb{Z}$, which contradicts the fact that f is not contained in a sublattice of $\mathbb{Z}^d$.
Since P_t has essential spectral radius strictly smaller than 1 and does not admit any eigenvalue of modulus 1, we conclude that its spectral radius is strictly smaller than 1. So there exists $n_t \geq 1$ such that $|\mathbb{E}_{\mathcal{P}_\mu}[e^{itS_{n_t}}]| \leq \|P_t^n\mathbf{1}\|_{\mathbb{L}^p(\mu)} < 1$. But $u \mapsto \mathbb{E}_{\mathcal{P}_\mu}[e^{iuS_{n_t}}] = \mathbb{E}_\mu[P_u^{n_t}(\mathbf{1})]$ is continuous at t (since $u \mapsto P_u \in \mathcal{L}(\mathbb{L}^{a'}(\mu), \mathbb{L}^{b'}(\mu))$ is continuous, for all $a' > b' \geq 1$, using also the fact that $\sup_u \|P_u\|_{\mathcal{L}(\mathbb{L}^a(\mu))} \leq 1$). □

Proof of Theorem 6.5. Observe that

$$\begin{aligned}\mathbb{P}(S_n = k) &= \mathbb{E}\left[\mathbf{1}_{\{S_n-k=0\}}\right] = \mathbb{E}\left[\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{it.(S_n-k)}\,dt\right]\\ &= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{-itk}\mathbb{E}[e^{it.S_n}]\,dt\\ &= \frac{1}{(2\pi)^d}\int_{B(0,\delta_0)} e^{-itk}\lambda_t^n\Phi_t\,dt + o\left(\det A_n\right)\,,\end{aligned}$$

where we used the Fubini theorem for integrable functions and the fact that $\sup_{|t|<\delta_0}|M_{t,n}| = o\left(\det A_n\right)$ and that $\sup_{t\in[-\pi,\pi]^d\setminus B(0,\delta_0)}\left|\mathbb{E}[e^{it.S_n}]\right| = o\left(\det A_n\right)$. Now, making the change of variable $t = A_n^* s$, we obtain

$$\mathbb{P}(S_n = k) = \frac{\det(A_n)}{(2\pi)^d}\int_{(A_n^*)^{-1}B(0,\delta_0)} e^{-iA_n^*s.k}\lambda^n_{A_n^*s}\Phi_{A_n^*s}\,ds + o\left(\det A_n\right)\,,$$

and

$$\begin{aligned}&\sup_{k\in\mathbb{Z}}\left|\int_{(A_n^*)^{-1}B(0,\delta_0)} e^{-iA_n^*s.k}\lambda^n_{A_n^*s}\Phi_{A_n^*s}\,ds - \int_{\mathbb{R}^d} e^{-is.A_nk}\mathbb{E}[e^{is.\mathcal{W}}]\,ds\right|\\ &\leq \int_{\mathbb{R}^d}\left|\mathbf{1}_{(A_n^*)^{-1}B(0,\delta_0)}\lambda^n_{A_n^*s}\Phi_{A_n^*s} - \mathbb{E}[e^{is.\mathcal{W}}]\right|\,ds = o(1)\,,\end{aligned}$$

due to the dominated convergence theorem since g and $s \mapsto \mathbb{E}[e^{-is\mathcal{W}}]$ are integrable. We end the proof by using $h_{\mathcal{W}}(u) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-is.u}\mathbb{E}[e^{is.\mathcal{W}}]\,ds$. □

6.2. **Edgeworth expansions ($d = 1$).** We assume $d = 1$ throughout this section. We recall now some general Edgeworth expansions results coming from [10]. We first introduce some assumptions.

Assumption $(\alpha')[\widehat{r}]$ **(Smoothness):** *Assume λ, Φ and M enjoys the following Taylor expansions*

$$\lambda_t = 1 - \frac{\sigma^2t^2}{2} + \sum_{k=3}^{\widehat{r}+2}\alpha_k t^k + o(t^{\widehat{r}+2}),\quad \Phi_t = \sum_{k=0}^{\widehat{r}} B_k t^k + \mathcal{O}\left(|t|^{\widehat{r}+1}\right)\,,$$

$$\left|M_{t,n} - \sum_{k=0}^{\widehat{r}} C_{k,n}t^k\right| \leq K_n|t|^{\widehat{r}+1}$$

with $\sigma^2 > 0$ and with $\sup_{k,n}\left(|C_{k,n}| + K_n\right) = \mathcal{O}(n^{-p})$, for all $p > 0$.

Remark 6.8. *In the Markovian context of Section 5, Assumption $(\alpha')[\widehat{r}]$ will follow from Theorem 5.5 and Corollary 5.6 with $r := \widehat{r} + 1$ (up to assume the positivity of σ^2, the expression of which is given in Proposition 5.8).*

Assumption (β') **(Non-arithmeticity):** *For any compact K of $\mathbb{X}^*\setminus\{0\}$,*

$$\forall p>0,\quad \sup_{s\in K}\left|\mathbb{E}\left[e^{isS_n}\right]\right| = \mathcal{O}(n^{-p})\,,$$

As already mentioned in Remark 6.6, in the Markovian context of Section 5, Assumption (β') will follow from the fact that $\rho_{ess}(P_t)<1$ (which can be established by using (4) if $\mathcal{B}_1\hookrightarrow\mathcal{B}_2$ is compact, due to a Theorem by Hennion [18]), combined with the fact that $\rho(P_t)\le 1$ and that P_t admits no eigenvalue of modulus 1 (for t in the compact K).

Assumption (γ')**:** *Either $\mathbb{X}=\mathbb{Z}$, or there exist $\alpha_1,\alpha_1',\hat\delta$ such that, for all p,*

$$\forall |s|>K,\quad \left|\mathbb{E}\left[e^{isS_n}\right]\right| = \mathcal{O}\left(n^{-p}+|s|^{1+\alpha_1'}e^{-n^{\alpha_1}\hat\delta|s|^{-\alpha_1'}}\right)\,.$$

Assumption $(\delta')[r']$**:** *$\mathbb{X}=\mathbb{R}$ and for any $B>0$, there exists $K>0$ such that*

$$\int_{K<|s|<Bn^{\frac{r'-1}{2}}}\frac{\left|\mathbb{E}_\mu\left(e^{isS_n}\right)\right|}{|s|}\,ds = o(n^{-r'/2})\,.$$

Remark 6.9. *Note that, both Conditions (γ') and $(\delta')[r']$ hold true provided there exist positive s_0, n_0, $\widehat{C}>0$, $\alpha_1'>0$ such that $r'<{\alpha_1'}^{-1}+\frac12$ and such that*

$$\forall |s|>s_0,\ \forall n\ge n_0,\quad \left|\mathbb{E}[e^{isS_n}]\right|<e^{-n\widehat{C}|s|^{-\alpha_1'}}\,. \tag{29}$$

In the context of Section 5 with $\mathbb{E}\left[e^{itS_n}\xi_n\right]=\mathbb{E}_\nu\left[gP_t^n(h)\right]$, (29) holds true if there exist Banach spaces $\mathcal{B}_1\hookrightarrow\mathcal{B}_2$ containing h and the dual of which contains $\mathbb{E}_\nu[g\cdot]$ such that

$$\forall n\ge n_0,\ \forall|s|>s_1,\quad \|P_s^n\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}\le Ce^{-Cn|s|^{-\alpha_1'}}\,.$$

(see [10, Lemma 4.7]). Note that this holds true if $\|P_s^{n_0}\|_{\mathcal{B}_1}<1-\frac{C}{|s|^{\alpha_1'}}$. This condition generalizes the $\alpha-$Diophantine property of supp X: $\mathbb{E}[e^{isY_1}]<1-\frac{\widehat{C}}{|s|^{\alpha_1'}}$ of the i.i.d. case (see [8]).

Example 6.10 (Knudsen gas)**.** *Consider again Example 5.2. Recall that $P\left(h-\mathbb{E}_\mu[h]\right)=\alpha(h-\mathbb{E}_\mu[h])$ with $\alpha\in]0,1[$ and that $Y_n=f(X_n)$, with $f:\Omega\to\mathbb{R}$ centered. Assume f admits moments of order $(\widehat{r}+2)$ with respect to $\nu=\mu$.*
The fact that Assumption $(\alpha')[\widehat{r}]$ holds true follows from Theorem 5.5 and Example 5.7.
Assumption (β') will hold true if, for all $t\in\mathbb{X}^*\setminus\{0\}$, $\left|\mathbb{E}_\mu[e^{itf}]\right|<1$. Indeed, as seen in Example 6.7, it is enough to prove that P_t (for $t\in\mathbb{X}^*$) admits no eigenvalue λ of modulus 1. If it was the case, there would exist $h\in\mathcal{B}_1\setminus\{0\}$ such that $\lambda h=e^{itf}h=\mathbb{E}_\mu[e^{itf}h]$ μ-almost surely, contradicting $\left|\mathbb{E}_\mu[e^{itf}]\right|<1$.
Finally, when $\mathbb{X}=\mathbb{R}$, Assumptions (γ') (with $\alpha_1=1$) and $(\delta')[\lceil\frac12+{\alpha_1'}^{-1}\rceil-1]$ hold true as soon as $\mathbb{E}[e^{itf}]<1-\frac{\widehat{C}_0}{|t|^{\alpha_1'}}$. Indeed

$$P_t^2(h)(x)=\alpha^2e^{i2tf(x)}h(x)+\alpha(1-\alpha)e^{itf}\mathbb{E}_\mu[e^{itf}h]+\alpha(1-\alpha)\mathbb{E}_\mu[e^{i2tf}h]+(1-\alpha)^2\mathbb{E}_\mu[e^{itf}]\mathbb{E}_\mu[e^{itf}h]\,,$$

and so, for all $p\in[1,+\infty]$, it follows that

$$\begin{aligned}\|P_t^2(h)\|_{\mathbb{L}^p(\mu)}&\le\left(1-(1-\alpha)^2\right)\|h\|_{\mathbb{L}^p(\mu)}+(1-\alpha)^2\left(1-\frac{\widehat{C}_0}{|t|^{\alpha_1'}}\right)\left|\mathbb{E}_\mu[e^{itf}h]\right|\\&\le\left(1-(1-\alpha)^2\frac{\widehat{C}_0}{|t|^{\alpha_1'}}\right)\|h\|_{\mathbb{L}^p(\mu)}\,.\end{aligned}$$

Set $\widehat{g}(s) := \int_{\mathbb{X}} e^{-isx} g(x)\, d\lambda(x)$ for $s \in \mathbb{X}^*$, where λ is the Lebesgue measure if $\mathbb{X} = \mathbb{R}$ and where λ is the counting measure if $\mathbb{X} = \mathbb{Z}$. If $\mathbb{X} = \mathbb{R}$, we say $g \in \mathfrak{F}_k^m$ if $g : \mathbb{R} \to \mathbb{R}$ is continuous, λ-integrable and if $\hat{g} : \mathbb{X}^* \to \mathbb{C}$ is k times continuously differentiable with

$$C_k^m(g) := C^m(g) + C_k(g). < \infty\,,$$

$$\text{with} \quad C^m(g) := \sup_{s\in\mathbb{X}^*} \frac{|\widehat{g}(s)|}{\min(1, |s|^{-m})} \quad \text{and} \quad C_k(g) := \|\widehat{g}^{(k)}\|_\infty\,.$$

If $\mathbb{X} = \mathbb{Z}$, $\mathfrak{F}_k^m = \mathfrak{F}_k^0$ is the set of functions $g : \mathbb{Z} \to \mathbb{C}$ satisfying the following summability condition

$$\sum_{n\in\mathbb{Z}} |n|^k |g(n)| < \infty\,.$$

Note that $C_k(g) \le \max_{0\le j\le k} \int_{\mathbb{X}} |x|^j |g(x)|\, d\lambda(x)$. When $\mathbb{X} = \mathbb{R}$, $C^m(g) \le \max_{0\le j\le m} \|g^{(j)}\|_{\mathrm{L}^1(\mathbb{R})}$. Under our assumptions, we set $\mathfrak{N}$ for the distribution function of a centered Gaussian random variable with variance σ^2 and $\mathfrak{n}$ for the corresponding probability density function (that is $\mathfrak{n}$ is the derivative of $\mathfrak{N}$). Let us recall now the general results of [10].

Theorem 6.11. [10, Theorem 1.1] *Let $\widehat{r}$ be a nonnegative integer and $q > \alpha_1'(1 + \frac{\widehat{r}}{2\alpha_1})$. Assume $(\alpha')[\widehat{r}]$, (β') and (γ') hold. Then there exist polynomials R_j such that, for all $g \in \mathfrak{F}_0^{q+2}$,*

$$\mathbb{E}\left[g(S_n)\right] = \sum_{j=0}^{\widehat{r}} \frac{1}{n^{(j-1)/2}} \int_{\mathbb{X}} (R_j \cdot \mathfrak{n})(x/\sqrt{n}) g(x)\, d\lambda(x) + C^{q+2}(g) \cdot o(n^{-\widehat{r}/2})\,.$$

Theorem 6.12. [10, Theorem 1.2] *Let $\widehat{r}$ be a nonnegative integer. Let $q > \alpha_1'(1 + \frac{\widehat{r}+1}{2\alpha_1})$. Assume $(\alpha')[\widehat{r}]$, (β') and (γ') hold. Then there exist polynomials Q_j such that, for all $g \in \mathfrak{F}_{\widehat{r}+1}^{q+2}$,*

$$\sqrt{n}\mathbb{E}\left[g(S_n)\right] = \sum_{j=0}^{\lfloor \widehat{r}/2 \rfloor} \frac{1}{n^j} \int_{\mathbb{X}} g(x) Q_j(x)\, d\lambda(x) + C_{\widehat{r}+1}^{q+2}(g) \cdot o(n^{-\widehat{r}/2})\,.$$

Theorem 6.13. [10, Theorem 1.6] *Let $\widehat{r}$ be a positive integer and $r' \ge 1$ be a real number. Assume $(\alpha')[\widehat{r}]$, (β') and $(\delta')[r']$ hold. Then there exist polynomials P_k such that*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \le x \right) - \mathfrak{N}(x) - \mathfrak{n}(x) \sum_{k=1}^{\min(\widehat{r}, \lfloor r' \rfloor)} \frac{P_k(x)}{n^{k/2}} \right| = o(n^{-\min(\widehat{r}, r')/2})\,.$$

Corollary 6.14. [10, Corollary 1.7] *Assume $(\alpha')[1]$ and (β') hold with $\mathbb{X} = \mathbb{R}$. Then*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \le x \right) - \mathfrak{N}(x) - \frac{P_1(x)}{n^{1/2}} \mathfrak{n}(x) \right| = o(n^{-1/2})\,.$$

Corollary 6.15. [10, Corollary 1.8] *Assume $(\alpha')[2]$, (β') and $(\delta')[r_0]$ hold for some real number $r_0 \in (1, 2)$. Then*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \le x \right) - \mathfrak{N}(x) - \frac{P_1(x)}{\sqrt{n}} \mathfrak{n}(x) \right| = o\left(n^{-r_0/2}\right)\,.$$

Example 6.16 (Knudsen gas, Edgeworth expansions in CLT, LLT)**.** *Consider Example 5.2. Recall that Assumptions (α'), (β'), (γ') and (δ') have been checked in Example 6.10. Assume f admits moments of order $(\widehat{r}+2)$ with respect to $\nu = \mu$ and that $\left|\mathbb{E}_\mu[e^{itf}]\right| < 1$ for all $t \in \mathbb{X}^* \setminus \{0\}$.*

- *(Expansions of order $\widehat{r} - 1$ and $\widehat{r}$ in the LLT) Assume either $\mathbb{X} = \mathbb{Z}$ or $|\mathbb{E}_\mu[e^{itf}]| < 1 - \frac{\widehat{C}_0}{|t|^{\alpha_1'}}$ for some $\alpha_1' > 0$, then the conclusions of Theorems 6.11 and 6.12 hold true with respectively $q > \alpha_1'(1 + \frac{\widehat{r}}{2})$ and $q > \alpha_1'(1 + \frac{\widehat{r}+1}{2})$.*

- *(Edgeworth expansion of order $\widehat{r}$) Assume $\mathbb{X}=\mathbb{R}$ and $|\mathbb{E}_\mu[e^{itf}]|<1-\frac{\widehat{C}_0}{|t|^{\alpha'_1}}$ for some $\alpha'_1<(\widehat{r}-\frac{1}{2})^{-1}$. Then the conclusion of Theorem 6.13 holds true.*
- *(First order Edgeworth expansion) Assume $\mathbb{X}=\mathbb{R}$ and $r=1$. Then the conclusion of Theorem 6.14 holds true.*
- *(Edgeworth expansion of order $r_0\in]1,2[$) Assume $\mathbb{X}=\mathbb{R}$, $\widehat{r}=0$ and that $|\mathbb{E}_\mu[e^{itf}]|<1-\frac{\widehat{C}_0}{|t|^{\alpha'_1}}$ for some $\alpha'_1<(r_0-\frac{1}{2})^{-1}$. Then the conclusion of Corollary 6.15 holds true.*

7. Limit theorems for Markov random walks

We focus again in this section on the context of Markov random walks, that is the general Markovian setting of 5. Recall that $(X_n)_{n\geq 0}$ is a Markov chain with states space Ω and with invariant distribution μ and initial distribution ν and that $(Z_k)_{k\geq 1}$ is a sequence of independent identically distributed random variables with common distribution $\mathbf{P}$ and independent of the Markov chain $(X_n)_{n\geq 0}$. Recall that we are interested in the behaviour of $S_n:=\sum_{k=1}^n Y_k$, with $Y_k=f(X_{k-1},X_k,Z_k)$.

In a first subsection, we establish probabilistic limit theorems in the general context as a direct consequence of the results of Section 6. In the three following subsections, we apply our approach for classical families of Markov chains: the ρ-mixing Markov chains, the V-geometrically ergodic Markov chains and Lipschitz iterative model. More precisely, we prove Theorem 1.1 in these three last subsections.

7.1. **General results.** We set $\mathcal{P}_\nu$ for the Markov distribution with transition operator P and initial probability measure ν. We assume that $((x_k)_{k\geq 0},\omega)\mapsto f(x_0,x_1,\omega)$ is $\mathcal{P}_\mu\otimes\mathbf{P}$-centered. We set

$$P_t(h)(x)=\mathbb{E}\left[e^{it.Y_1}h(X_1)|X_0\right]\,.$$

We establish probabilistic limit theorems under the assumptions of Theorem 5.5 that we recall in the statement for reader convenience.

Theorem 7.1. *Let $\delta_0>0$. Let r be a positive integer and $Y_1\in\mathbb{L}^{r+1}(\mathcal{P}_\mu\otimes\mathbf{P})$. Let $(\mathcal{B}_j,\|\cdot\|_{(j)})$, $j=0,...,r+1$ be a chain of $(r+2)$ Banach spaces such that $\mathbf{1}\in\mathcal{B}_0$ and that for all $j=1,...,r+1$, $\mathcal{B}_{j-1}\hookrightarrow\mathcal{B}_j$, $\mathcal{B}_j\hookrightarrow\mathbb{L}^{\frac{r+1}{j}}(\mu)$. Assume that P_t (for $|t|<\delta_0$) acts continuously on $\mathcal{B}_1,...,\mathcal{B}_{r+1}$ and that P_0 acts continuously on $\mathcal{B}_0$ and that, for all $m=0,...,r$, $P_t-\sum_{k=0}^m\frac{P_0^{(k)}}{k!}.t^{\otimes k}$ is both in $\mathcal{O}(t^m)$ in $\bigcap_{j=0}^{r-m}\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+m})$ and in $o(t^m)$ in $\mathcal{L}(\mathcal{B}_{r-m},\mathcal{B}_{r+1})$, with*

$$P_0^{(k)}(h)(x)=i^k\mathcal{Q}_k(h):=\int_E P\left((if(x,\cdot,\omega))^{\otimes k}h(\cdot)\right)(x)\,d\mathbf{P}(\omega)\in\bigcap_{j=0}^{r-k}\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+k}^{d^{\otimes k}})\,.$$

Finally we assume that, in $\mathcal{L}(\mathcal{B}_1)$,

$$\exists\vartheta_1\in]0,1[,\ \forall n\in\mathbb{N}^*,\ \forall|t|<\delta_0,\quad P_t^n=\lambda_t^n\Pi_t+N_t^n\,,\quad \text{with }\sup_{|t|<\delta_0}\|N_t^n\|_{\mathcal{B}_1}=\mathcal{O}(\vartheta_1^n)\,,\tag{30}$$

with $\Pi_0=\mathbb{E}_\mu[\cdot]\mathbf{1}$, $\lambda_0=1$ and

$$\Pi_t:=\frac{1}{2i\pi}\int_{\Gamma_1}(z\,\mathrm{Id}-P_t)^{-1}\,dz,\quad N_t^n:=\frac{1}{2i\pi}\int_{\Gamma_0}z^n(z\,\mathrm{Id}-P_t)^{-1}\,dz\,,\tag{31}$$

with Γ_1 the oriented circle $\mathcal{C}(1,\delta)$ and Γ_0 the oriented circle $\mathcal{C}(0,a)$, with $\vartheta_1 < a < a+\delta < 1$ and that[8]

$$\sup_{j=1,\dots,r+1}\sup_{|t|<\delta_0}\sup_{z\in\Gamma_0\cup\Gamma_1}\Big(\|(z\,\mathrm{Id}-P_t)^{-1}\|_{\mathcal{B}_j}+\|(z\,\mathrm{Id}-P)^{-1}\|_{\mathcal{B}_0}\Big)<\infty\,. \tag{32}$$

Assume either $\mathbb{E}_\nu \in \mathcal{B}_1^*$ or, more generally, that there exists some Banach space $\widetilde{\mathcal{B}}_0$ (that can be intermediate between $\mathcal{B}_0$ and $\mathcal{B}_1$) such that and $\mathbb{E}_\nu \in \widetilde{\mathcal{B}}_0^*$ and such that $t\mapsto \Pi_t \in \mathcal{L}(\mathcal{B}_0,\widetilde{\mathcal{B}}_0)$ is continuous at 0 (using e.g. Theorem 3.3) , then

(i) *Theorem 6.2 (CLT) applies and we conclude that $(S_n/\sqrt{n})_{n\ge 1}$ converges in distribution to a centered Gaussian random variable with variance matrix $\sum_{n\in\mathbb{Z}} \mathcal{C}ov_{\mu\otimes\mathbf{P}^{\otimes\mathbb{N}}}\Big(Y_1,Y_{|n|+1}\Big)$.*
(ii) *Theorem 6.5 (LLT) applies if Y_1 is $\mathbb{Z}^d$ valued and if the non-arithmetic condition (β') is satisfied.*

If moreover $\mathbb{E}_\nu \in \mathcal{B}_r^*$, then Assumption $(\alpha')[r-1]$ is satisfied. In particular:

(iii) *($d=1$) Theorems 6.11 and 6.12 (Expansions of order $\widehat{r}-1=r-2$ and $\widehat{r}=r-1$ in the LLT) apply if Y_1 is $\mathbb{Z}$-valued and if the non-arithmetic condition (β') is satisfied.*
(iii') *($d=1$) Theorems 6.11 and 6.12 (Expansions of order $\widehat{r}-1=r-2$ and $\widehat{r}=r-1$ in the LLT) apply if conditions (β') (non-arithmeticity) and (γ') (Diophantine-type condition) are satisfied.*
(iv) *($d=1$) Corollary 6.14 (First order Edgeworth expansion) holds true if $r\ge 2$ and if conditions (β') (non-arithmeticity) is satisfied on $\mathbb{X}=\mathbb{R}$.*
(v) *($d=1$) Theorem 6.13 (Edgeworth expansion of order $\widehat{r}=r-1$) applies if $r\ge 2$ and if the non-arithmetic condition (β') and Condition $(\delta')[r-1]$ hold true.*

7.2. **ρ-mixing Markov chains.** We consider here the case of Markov chains that are ρ-mixing in the discrete time, i.e. when the transfer operator P is geometrically ergodic on $\mathbb{L}^2(\mu)$, that is satisfies

$$\exists C>0,\ \exists\vartheta\in]0,1[,\quad \forall g\in\mathbb{L}^2(\mu),\quad \|P^n(g)-\mathbb{E}_\mu[\cdot]\mathbf{1}\|_{\mathbb{L}^2(\mu)}\le C\vartheta^n\,.$$

Recall that this also implies the geometric ergodicity on each $\mathbb{L}^p(\mu)$ for $p\in]1,+\infty[$ (see [38]). Let us observe that the study of Markov random walks driven by a ρ-mixing Markov chain $(X_k)_k$ can be simplified in a additive function of a ρ-mixing Markov chain.[9]

Proposition 7.2. *If $(X_k)_{k\ge 0}$ is ρ-mixing, then the Markov chain $(\widetilde{X}_k := (X_{k-1},X_k,Z_k))_{k\ge 0}$ with invariant probability measure $\widetilde{\mu}$ the distribution of (X_0,X_1,Z_0) with respect to $\mathcal{P}_\mu$, which is given by*

$$\widetilde{\mu}(A\times B\times C)=\mathbb{E}_\mu\left[P(\mathbf{1}_B)\mathbf{1}_A\right]\mathbf{P}(C)$$

is also ρ-mixing, with same rate.

Proof. Let $\widetilde{P}$ be the transfer operator of $\widetilde{X}$. Then, for all $n\ge 1$,

$$\begin{aligned}\widetilde{P}^n(G)(x,y,z)&=\mathbb{E}\left[G(X_{n-1},X_n,Z_n)|\,X_0=y\right]\\&=\int_E (P^{n-1}(H(\cdot,z)))(y)\,d\mathbf{P}(z)\,,\end{aligned}$$

[8] recall that, in practice, (30), (31) and the first part of (32) follow from the Keller-Liverani theorem (Theorem 3.3) applied with the Banach spaces $\mathcal{B}_j\hookrightarrow\mathcal{B}_{j+1}$ up to consider an additional Banach space $\mathcal{B}_{r+2}$.

[9] Note that this change induces a change in the definition of non-lattice.

where $H(x,z)=\mathbb{E}\left[G(X_0,X_1,z)|\,X_0=x\right]$, and so

$$\begin{aligned}\|\widetilde{P}^n(G)-\mathbb{E}_{\widetilde{\mu}}[G]\|^2_{L^2(\widetilde{\mu})} &= \int_\Omega \left|\int_E \left(P^{n-1}(H(\cdot,z))(y)-\mathbb{E}_\mu[H(\cdot,z)]\right)\,d\mathbf{P}(z)\right|^2\,d\mu(y)\\ &\le \int_\Omega\int_E \left|P^{n-1}(H(\cdot,z))(y)-\mathbb{E}_\mu[H(\cdot,z)]\right|^2\,d\mathbf{P}(z)\,d\mu(y)\\ &\le \int_E \left\|P^{n-1}(H(\cdot,z))(y)-\mathbb{E}_\mu[H(\cdot,z)]\right\|^2_{\mathbb{L}^2(\mu)}\,d\mathbf{P}(z)\\ &\le \int_E C^2\vartheta^{2(n-1)}\|H(\cdot,z)\|^2_{\mathbb{L}^2(\mu)}\,d\mathbf{P}(z)\\ &= \int_E C^2\vartheta^{2(n-1)}\|\mathbb{E}\left[G(X_0,X_1,z)|\,X_0\right]\|^2_{\mathbb{L}^2(\mu)}\,d\mathbf{P}(z)\le C^2\vartheta^{2(n-1)}\|G\|^2_{\mathbb{L}^2(\widetilde{\mu})}\,.\end{aligned}$$

□

Thus, without any loss of restriction, from now on, in this subsection, we replace $f(x,y,z)$ by $f(y)$ (up to replace the Markov chain X by the Markov chain $\widetilde{X}$). Note that this replacement changes the notion of non-lattice, which can be corrected by using [24].

Theorem 7.3. *Assume P is geometrically ergodic on $\mathbb{L}^2$. Assume the initial measure ν is the stationary measure μ and $Y_k=f(X_k)$, with $f:\Omega\to\mathbb{R}^d$. Let r be a positive integer. Assume f is μ-centered and in $\mathbb{L}^{r+1}(\mu)$. Then the assumptions of Theorem 7.1 hold true with $\mathcal{B}_0=\mathbb{C}.\mathbf{1}$ endowed with the infinite norm and with $\mathcal{B}_j=\mathbb{L}^{\frac{r+1}{j}}(\mu)$ for all $j\in\{1,...,r\}$ and $\mathcal{B}_{r+1}=\mathbb{L}^{\frac{2r+1}{2r}}(\mu)$ (note that $1<\frac{2r+1}{2r}<\frac{r+1}{r}$). In particular*

- (i) *Theorem 6.2 (CLT) applies and we conclude that $(S_n/\sqrt{n})_{n\ge1}$ converges in distribution to a centered Gaussian random variable with variance matrix $\sum_{n\in\mathbb{Z}}\mathcal{C}ov_{\mathcal{P}_\mu}\left(Y_1,Y_{|n|+1}\right)$.*
- (ii) *Theorem 6.5 (LLT) applies if Y_1 is $\mathbb{Z}^d$-valued and non-lattice in $\mathbb{Z}^d$.*[10]
- (iii) *($d=1$) Theorems 6.11 and 6.12 (Expansions of order $\widehat{r}-1=r-2$ and $\widehat{r}=r-1$ in the LLT) apply if Y_1 is $\mathbb{Z}$-valued and non-lattice in $\mathbb{Z}$.*
- (iii') *($d=1$) Theorems 6.11 and 6.12 (Expansions of order $\widehat{r}-1=r-2$ and $\widehat{r}=r-1$ in the LLT) apply if Assumption (γ') holds true and if Y_1 is non-lattice in $\mathbb{R}$.*[11]
- (iv) *($d=1$) Corollary 6.14 (First order Edgeworth expansion) holds true if $r\ge2$ and if Y_1 is non-lattice in $\mathbb{R}$.*
- (v) *($d=1$) Theorem 6.13 (Edgeworth expansion of order $\widehat{r}=r-1$) applies if $r\ge2$, if Y_1 is non-lattice in $\mathbb{R}$ and if Assumption (δ')$[r-1]$ holds true.*

Proof. Since $P=\mathrm{Id}$ on $\mathcal{B}_0$, 1 is the single spectral value of $P_{|\mathcal{B}_0}$ and $\|(z\,\mathrm{Id}-P)^{-1}\|_{\mathcal{L}(\mathcal{B}_0)}=|z-1|^{-1}$. We know that for all $p\in]1,+\infty[$, $\|P^n-\mathbb{E}_\mu[\cdot]\mathbf{1}\|_{\mathcal{L}(\mathbb{L}^p)}$ decreases exponentially fast. This implies in particular that P is quasi-compact on $\mathcal{B}_1$ and on $\mathcal{B}_2$ with a single dominating eigenvalue 1, which is simple and also that

$$\|P_t^n(g)\|_{\mathbb{L}^p(\mu)}\le\|P^n(|g|)\|_{\mathbb{L}^p(\mu)}\le\|P^n-\mathbb{E}_\mu[\cdot]\mathbf{1}\|_{\mathcal{L}(\mathbb{L}^p)}\,\|g\|_{\mathbb{L}^p(\mu)}+\|g\|_{\mathbb{L}^1(\mu)}\,,$$

[10] non-latticity for $\mathbb{Z}^d$-valued observables means that there exist no triple (a,H,θ) with $a\in\mathbb{Z}^d$, $H\ne\mathbb{Z}^d$ a closed subgroup in $\mathbb{Z}^d$ and $\theta:\Omega\to\mathbb{Z}^d$ such that $Y_1+\theta(X_1)-\theta(X_0)\in a+H$ $\mathcal{P}_\mu\times\mathbf{P}$-a.s.

[11] non-latticity in $\mathbb{R}$ means that there exist no triple $a,b\in\mathbb{R}$ and $\theta:\Omega\to\mathbb{R}$ such that $Y_1+\theta(X_1)-\theta(X_0)\in a+b\mathbb{Z}$ $\mathcal{P}_\mu\times\mathbf{P}$-a.s..

implying the uniform Doeblin-Fortet inequality (4) for $\mathcal{B}_j \hookrightarrow \mathcal{B}_{j+1}$. Moreover, for all $j = 0, ..., r$ and $m \in \{1, ..., r-j\}$,

$$\left\|P_t(g) - \sum_{k=0}^{m} \frac{P((if)^k g)}{k!} t^k\right\|_{\mathcal{B}_{j+m}} \leq \left\|P\left(\left(e^{itf} - \sum_{k=0}^{m} \frac{(if)^k}{k!} t^k\right) g\right)\right\|_{\mathcal{B}_{j+m}}$$
$$\leq \left\|e^{itf} - \sum_{k=0}^{m} \frac{(if)^k}{k!} t^k\right\|_{\mathcal{B}_m} \|g\|_{\mathcal{B}_j} \leq o(t^m) \|g\|_{\mathcal{B}_j} ,$$

due to the dominated convergence theorem. Note that this last inequality implies in particular the continuity assumption of $t \mapsto P_t \in \mathcal{L}(\mathcal{B}_j, \mathcal{B}_{j+1})$ required in Theorem 3.3. Moreover $P_0^{(k)} = P((if)^k \cdot) \in \bigcap_{j=0}^{r-k} \mathcal{L}(\mathcal{B}_j, \mathcal{B}_{j+k})$. We also apply Theorem 3.3 with $\mathcal{B}_r \hookrightarrow \mathcal{B}_{r+1}$ and with $\mathcal{B}_{r+1} \hookrightarrow \mathbb{L}^1(\mu)$. The above inequalities combined with the Keller and Liverani perturbation theorem (Theorem 3.3) ensure that the assumptions of Items (C) and (D) of Proposition 4.1. Thus Corollary 5.6 with $\nu = \mu$ and Proposition 5.8 apply. For (i), we apply Theorem 6.2 thanks to the previous facts and we identify the variance matrix of the limit using Proposition 5.8.
Condition (P) of [26] is satisfied since the $\mathbb{L}^p(\mu)$ are contained and dense in $\mathbb{L}^1(\mu)$. Condition $(\widehat{K})$ of [26] follows from the ρ-mixing. Thus, due to [26, Proposition 5.4], the non-lattice property implies the non-arithmeticity (exponential decay of $\sup_{t\in[-\pi,\pi]^d\setminus B(0,\delta_0)} \left|\mathbb{E}[e^{it.S_n}]\right|$ | as $n \to +\infty$). We conclude by applying Theorem 7.1. □

7.3. ***V*-geometrically ergodic Markov chains.** $V : \Omega \to [1, +\infty[$ be an unbounded measurable function. The random walk $(X_n)_n$, or equivalently its transition operator P, is said to be V-geometrically ergodic if there exists $C > 0$ and $\vartheta \in]0, 1[$ such that

$$\forall n \geq 1, \quad \left\|\frac{P^n(\cdot) - \mathbb{E}_\pi[\cdot]}{V}\right\|_\infty \leq C\vartheta^n \|\cdot/V\|_\infty .$$

Again the study of Markov random walks driven by a V-geometrically ergodic random walk $(X_k)_{k\geq 0}$ can be reduced to an additive function of a V-geometrically ergodic Markov chain.

Proposition 7.4. *If the Markov chain $(X_k)_{k\geq 1}$ is V-geometrically ergodic, then the Markov chain $(\widetilde{X}_k = (X_{k-1}, X_k, Z_k))_{k\geq 0}$ with invariant measure $\widetilde{\mu}$ defined in Proposition 7.2 is $\widetilde{V}$-geometrically ergodic with same rate, with $\widetilde{V}(x, y, z) = V(x) + V(y)$.*

Proof. Let $G : \Omega^2 \times E \to \mathbb{C}$ be a bounded measurable function. We have seen in the proof of Proposition 7.2 that the transfer operator $\widetilde{P}$ of $\widetilde{X}$ satisfies

$$\forall n \geq 2, \quad \widetilde{P}^n((G\widetilde{V})(x, y, z)) = \int_E (P^{n-1}(H(\cdot, z)))(y)\, d\mathbf{P}(z) ,$$

where $H(x, z) = \mathbb{E}\left[G(X_0, X_1, z)(V(x) + V(X_1))\right| X_0 = x]\, V(x) =: K(x, z)V(x)$. Moreover

$$\|K\|_\infty \leq \frac{\mathbb{E}\left[\|G\|_\infty(V(X_0) + V(X_1))|X_0 = x\right]}{V(x)}$$
$$\leq \|G\|_\infty \left(1 + \|P(V)/V\|_\infty\right) .$$

Thus

$$\left\|\frac{\widetilde{P}^n(G\widetilde{V}) - \mathbb{E}_{\widetilde{\mu}}[G\widetilde{V}]}{\widetilde{V}}\right\|_\infty = \left\|\frac{1}{\widetilde{V}} \int_E \left(P^{n-1}(K(\cdot, z)V) - \mathbb{E}_\mu[K(\cdot, z)V]\right)\, d\mathbf{P}(z)\right\|_\infty$$
$$\leq C\vartheta^{n-1} \int_E \|K(\cdot, z)\|_\infty\, d\mathbf{P}(z) \leq C\vartheta^{n-1} \|G\|_\infty \left(1 + \|P(V)/V\|_\infty\right) .$$

This ends the proof of the lemma since P acts continuously on the space $V.L^\infty$ endowed with the norm $\|\cdot/V\|_\infty$. $\square$

Thus, without any loss of restriction, from now on, in this subsection, we replace $f(x,y,z)$ by $f(y)$ (again this complicates the notion of non-lattice, see [25] for a simple one).

Theorem 7.5. *Let r be a positive integer and a real number $r'\in]r,r+1]$ and $f:\Omega\to\mathbb{R}^d$ is μ-centered and in $\mathbb{L}^{r+1}(\mu)$. Let $V:\Omega\to[1,+\infty[$ be an unbounded measurable function such that $\mathbb{E}_\mu[V]<\infty$. Assume P is V-geometrically ergodic and*

$$\max_{u\in\{0,r'-r\}}\sup_{j=0,...,r}\sup_{m=1,...,r-j}\left\|V^{-\frac{j+m+u}{r+1}}P\left(|f|^{m+u}V^{\frac{j}{r+1}}\right)\right\|_\infty<\infty\,, \tag{33}$$

then the conclusions of Items (C) and (D) of Propositions 4.1, Corollary 5.6 (for $\nu=\mu$) and Proposition 5.8 hold true with $\mathcal{B}_0=\mathbb{C}.\mathbf{1}$ endowed with the infinite norm and with, for $j=1,...,r$, $\mathcal{B}_j:=V^{\frac{j}{r+1}}$ and $\mathcal{B}_{r+1}:=V^{\frac{r'}{r+1}}.L^\infty$ endowed with the norm $\|\cdot\|_{(j)}:=\|\cdot/V^{\frac{j}{r+1}}\|_\infty$ and $\|\cdot\|_{(r+1)}:=\|\cdot/V^{\frac{r'}{r+1}}\|_\infty$.
If $\mathbb{E}_\nu[V^\varepsilon]<\infty$ for some $\varepsilon>0$, then

- (i) *Theorem 6.2 (CLT) applies and we conclude that $(S_n/\sqrt{n})_{n\ge1}$ converges in distribution to a centered Gaussian random variable with variance matrix $\sum_{n\in\mathbb{Z}}\mathcal{C}ov_{\mu\otimes\mathbf{P}^{\otimes\mathbb{N}}}\left(Y_1,Y_{|n|+1}\right)$.*
- (ii) *Theorem 6.5 (LLT) applies if Y_1 is $\mathbb{Z}^d$ valued and non-lattice in $\mathbb{Z}^d$.*

If moreover $\mathbb{E}_\nu[V^{\frac{r}{r+1}}]<\infty$, then

- (iii) *($d=1$) Theorems 6.11 and 6.12 apply with $\widehat{r}=r-1$ if f is $\mathbb{Z}$-valued and non-lattice in $\mathbb{Z}$.*
- (iii') *($d=1$) Theorems 6.11 and 6.12 (Expansions of order r in the LLT) apply with $\widehat{r}=r-1$ if Assumption (γ') holds true for $\mathbb{E}_{\mathcal{P}_\nu}[e^{isS_n}]$ and if Y_1 is non-lattice in $\mathbb{R}$.*
- (iv) *($d=1$) Corollary 6.14 (First order Edgeworth expansion) holds true if $r=2$ and f is non-lattice in $\mathbb{R}$.*
- (v) *($d=1$) Theorem 6.13 (Edgeworth expansion of order $\widehat{r}=r-1$) applies if Y_1 is non-lattice in $\mathbb{R}$ and if Assumption $(\delta')[r-1]$ holds true for $\mathbb{E}_{\mathcal{P}_\nu}[e^{isS_n}]$.*

Proof. Recall that P is also V^γ-geometrically ergodic for any $\gamma\in]0,1]$ and that Theorem 3.3 applies with the Banach spaces $V^\gamma L^\infty\hookrightarrow\mathbb{L}^1(\mu)$ (see e.g. [26, Lemma 10.1]). Observe that $\mathcal{B}_j\subset\mathbb{L}^{\frac{r+1}{j}}(\mu)\cap\mathbb{L}^{\frac{r+1}{j}}(\nu)$ since $V\in\mathbb{L}^1(\mu)\cap\mathbb{L}^1(\nu)$. In particular, ν defines a linear continuous form on $\mathcal{B}_{r+1}$. For $j\in\{1,...,r+1\}$,

$$\|P^n(g)-\mathbb{E}_\pi[g]\|_{(j)}\le C\rho^n\|g\|_{(j)}\,,$$

$$\|P_t^n(g)\|_{(j)}\le\|P^n(|g|)\|_{(j)}\le C\rho^n\|g\|_{(j)}+\|g\|_1\,,$$

ensuring the Doeblin-Fortet estimate (4) for $\mathcal{B}_j\hookrightarrow\mathcal{B}_{j+1}$ for $j=1,...,r$. For $k=0,...,r$, $P_0^{(k)}:=P((if)^k)\in\cap_{j=0}^{r-k}\mathcal{L}(\mathcal{B}_j,\mathcal{B}_{j+k})$ since $f^k\in\mathcal{B}_k$ and since P acts continuously on $\mathcal{B}_{j+k}$ and

$$\|f^kg\|_{(j+k)}\le\|f^k\|_{(k)}\|g\|_{(j)}\,.$$

It remains to check the regularity assumptions. For all $j = 0, ..., r$ and $m \in \{1, ..., r-j\}$,

$$\begin{aligned}\left\|P_t(g) - \sum_{k=0}^{m} \frac{P((if)^k g)}{k!} t^k\right\|_{(j+m)} &\le \left\|P\left(\left(e^{itf} - \sum_{k=0}^{m} \frac{(if)^k}{k!} t^k\right) V\right)\right\|_{(j+m)} \\ &\le \left\|V^{-\frac{j+m}{r+1}} P\left(|tf|^m \|g\|_{(j)} V^{\frac{j}{r+1}}\right)\right\|_{\infty} \\ &\le \mathcal{O}(t^m)\, \|g\|_{(j)} ,\end{aligned}$$

and

$$\begin{aligned}\left\|P_t(g) - \sum_{k=0}^{m} \frac{P((if)^k g)}{k!} t^k\right\|_{(r+1)} &\le \left\|P\left(|tf|^{m+r'-r} V^{\frac{j}{r+1}}\right) V^{-\frac{j+m+r'-r}{r+1}}\right\|_{\infty} \|g\|_{(j)} \\ &\le \mathcal{O}(t^{m+r'-r})\, \|g\|_{(j)} = o(t^m)\, \|g\|_{(j)} .\end{aligned}$$

Condition (P) of [26] is satisfied since the Banach spaces are stables under complex modulus and Condition $(\widehat{K})$ of [26] follows from the V-geometric ergodicity. Thus, due to [26, Proposition 5.4], the non-lattice property implies the exponential decay of $\sup_{t\in[-\pi,\pi]^d\setminus B(0,\delta_0)} \left|\mathbb{E}[e^{it.S_n}]\right|$ | as $n \to +\infty$. We end the proof by applying Theorem 7.1 with $\widetilde{\mathcal{B}}_0 := V^{\varepsilon}.L^{\infty}$ for (applying Theorem 3.3 for $V^{\frac{\varepsilon}{2}}.L^{\infty} \hookrightarrow \widetilde{\mathcal{B}}_0 = V^{\varepsilon}.L^{\infty}$). □

Remark 7.6. *Assume $\nu = \mu$. Observe that* (33) *is satisfied as soon as $\|f^{r+1}/V\|_\infty < \infty$ (which also ensures that $f \in \mathbb{L}^{r+1}(\mu)$) with $r \ge 1$, since for $j + m \le r$ and $u \in \{0, r'-r\}$:*

$$\left\|V^{-\frac{j+m+u}{r+1}} P\left(|f|^{m+u} V^{\frac{j}{r+1}}\right)\right\|_\infty \le \left\|V^{-\frac{j+m+u}{r+1}} P\left(V^{\frac{j+m+u}{r+1}}\right)\right\|_\infty \left\|f^{r+1}/V\right\|_\infty^{\frac{m+u}{r+1}} < \infty .$$

Moreover, for the Markov chain (X_{k-1}, X_k, Z_k) considered in Proposition 7.4 with the reference function $\widetilde{V}$, (33) *holds true if $\left\|\int_E |f(\cdot,\cdot,\omega)|^{r+1}\, d\mathbf{P}(\omega)/\widetilde{V}\right\|_\infty < \infty$. Indeed, setting $g(x,y) = \frac{\int_E |f(x,y,\omega)|^{r+1}\, d\mathbf{P}(\omega)}{V(x)+V(y)}$,*

$$\begin{aligned}\left|\widetilde{P}\left(|f|^{m+u} \widetilde{V}^{\frac{j}{r+1}}\right)(x)\right| &= \left|P\left(\int_E |f(x,\cdot,\omega)|^{m+u}\, d\mathbf{P}(\omega) (V(x)+V(\cdot))^{\frac{j}{r+1}}\right)(x)\right| \\ &\le \left|P\left(\left(\int_E |f(x,\cdot,\omega)|^{r+1}\, d\mathbf{P}(\omega)\right)^{\frac{m+u}{r+1}} (V(x)+V(\cdot))^{\frac{j}{r+1}}\right)(x)\right| \\ &\le \left|P\left((g(x,\cdot))^{\frac{m+u}{r+1}} (V(x)+V(\cdot))^{\frac{j+m+u}{r+1}}\right)(x)\right| \\ &\le \|g\|_\infty^{\frac{m+u}{r+1}} \left(1 + \left\|V^{-\frac{j+m+u}{r+1}} P(V^{\frac{j+m+u}{r+1}})\right\|_\infty\right) V(x)^{\frac{j+m+u}{r+1}} ,\end{aligned}$$

since P is $V^{\frac{j+m+u}{r+1}}$-geometrically ergodic.

7.4. **Lipschitz iterative models.** We consider a non-compact metric space (E, d) in which every closed ball is compact, and endow it with its Borel σ-field $\mathcal{E}$. Let $(G, \mathcal{G})$ be a measurable space, let $(\theta_n)_{n\ge 1}$ be a sequence of independent identically distributed G-valued random variables. Let $F : E \times G \to E$ be a measurable function such that, for all $g \in G$, $F_g := F(\cdot, g)$ is Lipschitz continuous with Lipschitz constant $\mathcal{C}_g$. We consider the random walk defined by

$$\forall n \ge 1, \quad X_n = F(X_{n-1}, \theta_n) ,$$

the sequence $(\theta_n)_{n\ge 1}$ being independent of the initial value X_0 of the random walk. This random walk is called an iterative Lipschitz model [5] [7] and has transition operator

$$P(g)(x) = \mathbb{E}[\, g(F(x, \theta_1))] .$$

Note that this context includes the autoregressive chains on $\mathbb{R}^d$ of the form

$$\forall n \geq 1, \quad X_n = A_n\, X_{n-1} + \theta_n\,,$$

where $(A_n, \theta_n)_{n\geq 1}$ is a i.i.d. sequence of r.v. taking values in $\mathcal{M}_d(\mathbb{R}) \times \mathbb{R}^d$, independent of X_0. ($\mathcal{M}_d(\mathbb{R})$ denotes the set of real $d \times d$-matrices.) with e.g. $|A_1| < 1$ ($|\cdot|$ being a matrix norm. For one-dimensional autoregressive chains, convergence to stable laws has been investigated in [17] for $f(x) = x$. Let x_0 be a fixed point in E. For $x \in E$, we set $p(x) = 1 + d(x, x_0)$. We set, for all $g \in G$, $\mathcal{M}_g := \mathcal{C}_g + p(F(x_0, g))$. We are interested in the asymptotic behaviour of $S_n = \sum_{k=1}^n f(X_k)$, with $f : \Omega \to \mathbb{R}^d$ satisfying the following condition for some $C_1, s \geq 0$:

$$\forall (x,y) \in E \times E, \quad |f(x) - f(y)| \leq C_1\, d(x,y)\, \left(p(x) + p(y)\right)^s\,. \tag{34}$$

Recall that the case f is Lipschitz continuous (i.e. $s = 0$ in (34)) has been studied e.g. in [7] and in [2]. Fix $\alpha \in]0,1]$. We set $\widetilde{\mathcal{B}}^{(0)}_{\alpha,b,a}$ for the set of functions $g : E \to \mathbb{C}$ such that $|f|^{(0)}_a + \|f\|^{(0)}_{\alpha,b} < \infty$, with

$$|f|^{(0)}_a := \|f/p^a\|_\infty \quad \text{and} \quad m^{(0)}_{\alpha,b}(f) := \sup_{x,y\in E,\, x\neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha\, \max(p(x), p(y))^b}\,.$$

Remark 7.7. *Observe that* (34) *means that the coordinates of f belong to* $\widetilde{\mathcal{B}}^{(0)}_{1,s,s+1} \subset \widetilde{\mathcal{B}}^{(0)}_{\alpha,s+1-\alpha,s+1}$.

In the sequel, we set $\mathcal{M} := \mathcal{M}_{\theta_1}$ and $\mathcal{C} := \mathcal{C}_{\theta_1}$. Recall that it has been proved in [20, Th. I] that if

$$\mathbb{E}\left[\mathcal{M}^{\alpha(s+1)}\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\mathcal{C}^\alpha\, \max\{\mathcal{C}, 1\}^{\alpha s}\right] < 1\,,$$

then the Markov chain $(X_n)_n$ admits a unique stationary distribution μ and $p \in \mathbb{L}^{\alpha(s+1)}(\mu)$ (see also e.g [5] [7]), which implies in particular that $f \in \mathbb{L}^\alpha(\mu)$.

Theorem 7.8. *Let r be a positive integer and a real number $r' \in]r, r+1]$. Let ν be a probability measure on Ω. Assume that $p \in \mathbb{L}^{(s+1)(r+1)}(\mu)$ and $\alpha \in]0, \frac{s+1}{s+2}]$. Assume that f is μ-centered, satisfies* (34) *and that*[12]

$$\mathbb{E}\left[\,\mathcal{M}^{(s+1)(r+1)} + \mathcal{C}^\alpha\, \mathcal{M}^{(s+1)r'+s\alpha}\right] < +\infty \tag{35}$$

and

$$\mathbb{E}\left[\mathcal{C}^\alpha\, \max\{\mathcal{C}, 1\}^{(s+1)r'+s\alpha}\right] < 1\,. \tag{36}$$

If $p \in \mathbb{L}^\varepsilon(\nu)$ for some $\varepsilon > 0$, then

(i) *Theorem 6.2 (CLT) applies and we conclude that $(S_n/\sqrt{n})_{n\geq 1}$ converges in distribution to a centered Gaussian random variable with variance matrix $\sum_{n\in\mathbb{Z}} \mathcal{C}ov_{\mu\otimes\mathbf{P}^{\otimes\mathbb{N}}}\left(Y_1, Y_{|n|+1}\right)$.*

(ii) *Theorem 6.5 (LLT) applies if Y_1 is $\mathbb{Z}^d$ valued and non-lattice in $\mathbb{Z}^d$.*

If moreover $p \in \mathbb{L}^{(s+1)r}(\nu)$, then

(iii) *($d = 1$) Theorems 6.11 and 6.12 apply with $\widehat{r} = r - 1$ if f is $\mathbb{Z}$-valued and non-lattice in $\mathbb{Z}$.*

(iii') *($d = 1$) Theorems 6.11 and 6.12 (Expansions of order r in the LLT) apply with $\widehat{r} = r - 1$ if Assumption (γ') holds true for $\mathbb{E}_\nu[e^{isS_n}]$ and if Y_1 is non-lattice in $\mathbb{R}$.*

(iv) *($d = 1$) Corollary 6.14 (First order Edgeworth expansion) holds true if $r = 2$ and f is non-lattice in $\mathbb{R}$.*

(v) *($d = 1$) Theorem 6.13 (Edgeworth expansion of order $\widehat{r} = r-1$) applies if Y_1 is non-lattice in $\mathbb{R}$ and if Assumption (δ')[$r - 1$] holds true for $\mathbb{E}_\nu[e^{isS_n}]$.*

[12] Observe that these assumptions ensure that $p \in \mathbb{L}^{(s+1)(r+1)}(\mu)$, thus that $f \in \mathbb{L}^{r+1}(\mu)$.

Let us prove this result. We consider the following notion of weighted Hölder-type spaces due to D. Guibourg [14] and used in [26] generalizing those introduced [32] (used also in [34, 37]). For positive real numbers β,γ such that $0<\beta\le\gamma$ and for $(x,y)\in E^2$, we set

$$\Delta_{\alpha,\beta,\gamma}(x,y):=p(x)^{\alpha\gamma}\,p(y)^{\alpha\beta}+p(x)^{\alpha\beta}\,p(y)^{\alpha\gamma}.$$

Then $\mathcal{B}_{\alpha,\beta,\gamma}$ denotes the space of $\mathbb{C}$-valued functions g on E satisfying the following condition

$$m_{\alpha,\beta,\gamma}(g):=\sup\left\{\frac{|g(x)-g(y)|}{d(x,y)^\alpha\,\Delta_{\alpha,\beta,\gamma}(x,y)},\ x,y\in E,\ x\neq y\right\}\ <\ +\infty.$$

Set $|g|_{\alpha,\gamma}:=\sup_{x\in E}\dfrac{|g(x)|}{p(x)^{\alpha(\gamma+1)}}$ and $\|g\|_{\alpha,\beta,\gamma}:=m_{\alpha,\beta,\gamma}(g)+|g|_{\alpha,\gamma}$. Then $(\mathcal{B}_{\alpha,\beta,\gamma},\|\cdot\|_{\alpha,\beta,\gamma})$ is a Banach space. Moreover

$$\gamma\le\gamma'\quad\Rightarrow\quad\mathcal{B}_{\alpha,\beta,\gamma}\hookrightarrow\mathcal{B}_{\alpha,\beta,\gamma'}\,,\tag{37}$$

and $\mathcal{B}_{\alpha,\beta,\gamma}\hookrightarrow\mathbb{L}^{\frac{K}{\gamma+1}}(\mu)$ if $p\in\mathbb{L}^{\alpha K}(\mu)$. Theorem 7.8 relies on the following propositions combined with Theorem 7.1

Proposition 7.9. *Assume the Assumptions of Theorem 7.8. Then the assumptions of Corollary 5.6 are satisfied with* $\mathcal{B}_0=\mathbb{C}.\mathbf{1}$, $\mathcal{B}_j=\mathcal{B}_{\alpha,s+1,\frac{j(s+1)}{\alpha}-1}$ *for* $j=1,...,r$, $\mathcal{B}_{r+1}=\mathcal{B}_{\alpha,s+1,\frac{r'(s+1)}{\alpha}-1}$, $\mathcal{B}_{r+2}=\mathbb{L}^1(\mu)$.

Proof. Since $\alpha\in]0,\frac{s+1}{s+2}]$, $s+1\le\frac{s+1}{\alpha}-1$. Due to Lemmas 7.11, we know that, for all for $j=1,...,r+1$, P is geometrically ergodic on $\mathcal{B}_j$ and that $(P_t)_t$ satisfies the Doeblin Fortet assumption of Theorem 3.3 for the Banach spaces $\mathcal{B}_j\hookrightarrow\mathbb{L}^1(\mu)$. Moreover $\mathbf{1}\in\mathcal{B}_0$, and P coincide with Id on $\mathcal{B}_0$ and so $(z\,\mathrm{Id}-P)^{-1}=(z-1)^{-1}\,\mathrm{Id}$ on $\mathcal{B}_0$. Due to Lemma 7.12, $P_t-\sum_{j=0}^k\frac{P_0^{(j)}.t^{\otimes j}}{j!}$ is in $\mathcal{O}(|t|^k)$ in $\mathcal{L}(\mathcal{B}_i,\mathcal{B}_{i+k})$ for all $i=0,...,r-k$ and in $\mathcal{O}(|t|^{k+r'-r})=o(t^k)$ in $\mathcal{L}(\mathcal{B}_{r-k},\mathcal{B}_{r+1})$, with $P_0^{(j)}:=P\left((if)^{\otimes j}\cdot\right)$.

Moreover $\mathcal{B}_j\hookrightarrow p^{(s+1)j}L^\infty\hookrightarrow\mathbb{L}^{\frac{r+1}{j}}(\mu)$ since $p^{(s+1)(r+1)}\in\mathbb{L}^1(\mu)$; and $t\mapsto P_t\in\mathcal{L}(\mathcal{B}_{r+1},\mathcal{B}_{r+2})$ is continuous (we can use e.g. Lemma 7.12). □

Lemme 7.10. ([26, Lemma B.1]) *Let $a\ge\alpha$ and $b\in[0,a+\alpha\gamma]$ and $0\le\beta\le\gamma$. Assume*

$$\mathbb{E}\left[\mathcal{M}^{a+\alpha(\gamma+1)}+\mathcal{C}^\alpha\mathcal{M}^{a+\alpha(\gamma+\beta)}+\mathcal{C}^\alpha\mathcal{M}^{b+\alpha(\gamma+1)}\right]<\infty\,.$$

There exists $C'>0$ such that, for all $g\in\widetilde{\mathcal{B}}^{(0)}_{\alpha,a,b}$,

$$\|P(g\cdot)\|_{\mathcal{L}(\mathcal{B}_{\alpha,\beta,\gamma},\mathcal{B}_{\alpha,\max(\beta,\frac{b-a+\alpha}{\alpha}),\gamma+\frac{a}{\alpha}})}\le C'\|g\|_{\widetilde{\mathcal{B}}^{(0)}_{\alpha,b,a}}\,,$$

and

$$\|P(g\cdot)\|_{\mathcal{L}(\mathbb{C}.\mathbf{1},\widetilde{\mathcal{B}}_{\alpha,b,a})}\le C'\|g\|_{\widetilde{\mathcal{B}}^{(0)}_{\alpha,b,a}}\,.$$

More precisely

$$|P(gh)|_{\alpha,\gamma+\frac{a}{\alpha}}\le\mathbb{E}[\mathcal{M}^{a+\alpha(\gamma+1)}]|g|_a^{(0)}|h|_{\alpha,\gamma}$$

and

$$m_{\alpha,\max(\beta,\frac{b-a+\alpha}{\alpha}),\gamma+\frac{a}{\alpha}}(gh)\le\mathbb{E}[\mathcal{C}^\alpha\mathcal{C}_1^{a+\alpha(\gamma+\beta)}]|g|_a^{(0)}m_{\alpha,\beta,\gamma}(h)+\mathbb{E}[\mathcal{C}^\alpha\mathcal{M}^{b+\alpha(\gamma+1)}]m_b^{(0)}(g)|h|_{\alpha,\gamma}\,,$$

with $\mathcal{C}_1:=max(\mathcal{C},1)+d(F(x_0,\theta_1),x_0)\le\mathcal{M}$.

Proof. The fact that $\mathcal{C}_1 \le \mathcal{M}$ follows from [20, p.1945] We start from [26, Lemma B.1] which states that

$$|P(gh)| \le \mathbb{E}[\mathcal{M}^{a+\alpha(\gamma+1)}]|g|_a^{(0)}|h|_{\alpha,\gamma}p^{a+\alpha(\gamma+1)}$$

and that

$$\frac{|P(gh)(x)-P(gh)(y)|}{d(x,y)^\alpha} \le \mathbb{E}[\mathcal{C}^\alpha\mathcal{C}_1^{a+\alpha(\gamma+\beta)}]|g|_a^{(0)}m_{\alpha,\beta,\gamma}(h)p(x)^a\Delta_{\alpha,\beta,\gamma}(x,y) + \mathbb{E}[\mathcal{C}^\alpha\mathcal{M}^{b+\alpha(\gamma+1)}]m_b^{(0)}(g)|h|_{\alpha,\gamma}p(x)^bp(y)^{\alpha(\gamma+1)}\,,$$

if $p(y) \le p(x)$. We conclude by noticing that

$$\begin{aligned} p(x)^a\Delta_{\alpha,\beta,\gamma}(x,y) &= p(x)^a(p(x)^{\alpha\beta}p(y)^{\alpha\gamma}+p(y)^{\alpha\beta}p(x)^{\alpha\gamma}) \\ &\le p(x)^{a+\alpha\gamma}p(y)^{\alpha\beta}+p(y)^{\alpha\beta}p(x)^{a+\alpha\gamma} \end{aligned}$$

since $\beta \le \gamma$ and

$$p(x)^bp(y)^{\alpha(\gamma+1)} \le p(x)^{a+\alpha\gamma}p(y)^{\alpha+b-a}$$

since $\alpha(\gamma+1) \ge \alpha+b-a$. □

Lemme 7.11. **(**[20, Theorem 11.5]**,** [26, Propositions 11.2 and 11.4]**)** *Assume $s+1 \le \beta \le \gamma$ and*

$$\mathbb{E}\left[\mathcal{M}^{\alpha(\gamma+1)}+\mathcal{C}^\alpha\,\mathcal{M}^{\alpha(\gamma+\beta)}\right] < +\infty \quad \textit{and} \quad \mathbb{E}\left[\mathcal{C}^\alpha\,\max\{\mathcal{C},1\}^{\alpha(\gamma+\beta)}\right] < 1\,. \tag{38}$$

Then P is geometrically ergodic on $\mathcal{B}_{\alpha,\beta,\gamma}$: $\|P^n-\mathbb{E}_\mu\|_{\mathcal{L}(\mathcal{B}_{\alpha,\beta,\gamma})} = \mathcal{O}(\vartheta^n)$ and $(P_t)_t$ satisfies the assumptions of Theorem 3.3 with $\mathcal{B}_1 = \mathcal{B}_{\alpha,\beta,\gamma}$ and $\mathcal{B}_2 = \mathbb{L}^1(\mu)$.

We prove the following result (generalizing [26, Proposition 11.6], see also ([26, Propositions 11.5 and 11.7] for $\mathcal{C}^m$-smoothness).

Lemme 7.12. *If $s+1 \le \beta \le \gamma < \gamma'$ satisfies $\alpha(\gamma'-\gamma) \le s+1$ and*

$$\mathbb{E}[\mathcal{M}^{\alpha(\gamma'+1)}+\mathcal{C}^\alpha\mathcal{M}^{\alpha(\gamma'+\beta)}] < \infty\,,$$

then $t \mapsto P_t \in \mathcal{L}(\mathcal{B}_{\alpha,\beta,\gamma},\mathcal{B}_{\alpha,\beta,\gamma'})$ is continuous at 0.

If m is a positive integer and $m' \in [m,m+1]$ and β,γ are such that $0 \le \beta \le \gamma$ and

$$\mathbb{E}\left[\mathcal{M}^{m'(s+1)+\alpha(\gamma+1)}+\mathcal{C}^\alpha\mathcal{M}^{m'(s+1)+\alpha(\gamma+\beta)}+\mathcal{C}^\alpha\mathcal{M}^{m'(s+1)+\alpha\gamma}\right] < \infty\,.$$

Then $P_t-\sum_{k=0}^m \frac{P_0^{(k)}.t^{\otimes k}}{k!}$ is in $\mathcal{O}\left(t^{m'}\right)$ in $\mathcal{L}\left(\mathcal{B}_{\alpha,\beta,\gamma},\mathcal{B}_{\alpha,\beta,\gamma+\frac{m'(s+1)}{\alpha}}\right)$ and in $\mathcal{L}\left(\mathbb{C}.\mathbf{1},\widetilde{\mathcal{B}}^{(0)}_{\alpha,m'(s+1)-\alpha,m'(s+1)} = \mathcal{B}_{\alpha,0,\frac{m}{}}\right.$ with $P_0^{(k)} := P\left((if)^{\otimes k}\cdot\right)$.

Proof. Due to Lemma 7.10, the continuity at 0 of $t \mapsto P_t \in \mathcal{L}(\mathcal{B}_{\alpha,\beta,\gamma},\mathcal{B}_{\alpha,\beta,\gamma'})$ will follow from the control of $\|e^{it.f}-1\|_{\widetilde{\mathcal{B}}^{(0)}_{\alpha,s\alpha,a=\alpha(\gamma'-\gamma)}}$ (since $\frac{s\alpha-a+\alpha}{\alpha} \le s+1 \le \beta$ and $0 \le s\alpha \le a+\alpha\gamma = \alpha\gamma'$). We conclude by noticing, on the first hand (using $\frac{a}{s+1} \le 1$), that

$$\begin{aligned} |e^{it.f}-1| &\le \min(2,|t|.|f|) \le \min\left(2,|t|.\left(|f(x_0)|+C_1p^{s+1}\right)\right) \\ &\le 2^{1-\frac{a}{s+1}}|t|^{\frac{a}{s+1}}\left(|f(x_0)|+C_1\right)p^{\frac{a}{s+1}(s+1)}\,, \end{aligned}$$

which implies that $|e^{it.f}-1|_a^{(0)} = \mathcal{O}\left(|t|^{\min(\frac{a}{s+1},\alpha)}\right)$ and, on the second hand, that

$$|e^{it.f(x)}-e^{it.f(y)}| \le \min\left(2,|t|.|f(x)-f(y)|\right) \le 2^{1-\alpha}|t|^\alpha C_1^\alpha d(x,y)^\alpha(p(x)+p(y))^{s\alpha}\,,$$

ensuring that $m_b^{(0)}(f)$ This ends the proof of the first point. Let us prove the second point. To this end, we first observe that

$$P_t - \sum_{k=0}^{m} \frac{P_0^{(k)}.t^{\otimes k}}{k!} = P\left(\left(e^{it.f} - \sum_{k=0}^{m} \frac{(it.f)^k}{k!}\right)\cdot\right).$$

Due to Proposition 7.10, it is enough to prove that $\left\|e^{it.f} - \sum_{k=0}^{m} \frac{(it.f)^k}{k!}\right\|_{\widetilde{\mathcal{B}}^{(0)}_{\alpha,m'(s+1)-\alpha,m'(s+1)}} = \mathcal{O}(|t|^{m'})$. We set $h(u) := e^{iu} - \sum_{k=0}^{m} \frac{(iu)^k}{k!}$ and notice that

$$|h(t.f)| \le 2\frac{|t.f|^{m'}}{m!} \le \frac{2|t|^{m'}\left(|f|^{(0)}_{s+1}\right)^{m'}}{m!} p^{(s+1)m'}.$$

Observe that there exists $C_0' > 0$ such that $|h'(iu)| = \left|i\left(e^{iu} - \sum_{k=0}^{m-1} \frac{(iu)^k}{k!}\right)\right| \le C_0'|u|^{m'-1}$ for all $u \in \mathbb{R}$. Thus

$$\begin{aligned}|h(t.f(x)) - h(t.f(y))| &\le C_0' \max\left(|t.f(x)|, |t.f(y)|\right)^{m'-1} |t.(f(x) - f(y))| \\ &\le C_1''|t|^{m'-1} \max\left(p(x)^{s+1}, p(y)^{s+1}\right)^{m'-1} |t| d(x,y)^\alpha \max(p(x),p(y))^{s+1-\alpha} \\ &\le |t|^{m'} d(x,y)^\alpha \max(p(x),p(y))^{m'(s+1)-\alpha}.\end{aligned}$$

□

Proof of Theorem 7.8. Proposition 7.9 ensures that assumptions of Theorem 7.1 (except maybe those on ν) are satisfied. Observe that the condition $p \in \mathbb{L}^{(s+1)r}(\nu)$ ensures that $\mathbb{E}_\nu \in \mathcal{B}_r^*$ since $\mathcal{B}_r \hookrightarrow p^{r(s+1)}.L^\infty$.
For Items (i)-(ii) (TCL and LLT), assuming $\varepsilon < 1/3$ and $\varepsilon < \alpha(2s+2)$, we take $\widetilde{\mathcal{B}}_0 := \mathcal{B}_{\frac{\varepsilon}{2s+2},s+1,2s+1}$ and apply Theorem 3.3 for $\mathcal{B}_{\frac{\varepsilon}{2s+2},s+1,s+1} \hookrightarrow \widetilde{\mathcal{B}}_0 = \mathcal{B}_{\frac{\varepsilon}{2s+2},s+1,2s+1}$. This will ensure the continuity of $t \mapsto \Pi_t \in \mathcal{L}(\mathcal{B}_0, \widetilde{\mathcal{B}}_0)$. The fact that $\widetilde{\mathcal{B}}_0 := \mathcal{B}_{\frac{\varepsilon}{2s+2},s+1,2s+1} \hookrightarrow \mathbb{L}^1(\nu)$ will follow from $\widetilde{\mathcal{B}}_0 \hookrightarrow p^\varepsilon.L^\infty$ and $p^\varepsilon \in \mathbb{L}^1(\nu)$. Let us check that the assumptions of Theorem 3.3 are satisfied for $\mathcal{B}_{\frac{\varepsilon}{2s+2},s+1,s+1} \hookrightarrow \widetilde{\mathcal{B}}_0$. The continuity assumption of Theorem 3.3 is ensured by Lemma 7.12 (the integrability conditions follows from $\varepsilon < 1/3$). The other assumptions follow from Lemma 7.11. Indeed Conditions (35) and (36) imply (38) for $\beta = \gamma = s+1$ since $\alpha \in]0, \frac{s+1}{s+2}]$ implies $\alpha(2s+2) \le (s+1)r' + s\alpha$ and $\alpha(s+2) \le (s+1)(r+1)$; and moreover (38) implies that (38) holds also true with α being replaced by $\varepsilon/(2s+2) < \alpha$ (due to the Hölder inequality).
Finally it has been proved in [26, Proposition 11.8] that the non-lattice property implies the exponential decay of $\max_{j=1,\dots,r+1} \sup_{t\in[-\pi,\pi]^d\setminus B(0,\delta_0)} \|P_t^n\|_{\mathcal{B}_j}$ as $n \to +\infty$. □

Appendix A. Markov Additive processes

In [24], the authors considered Markov processes $(X_n, \tilde{S}_n)_n$ on $\Omega \times \mathbb{R}^d$ such that

$$\mathbb{E}\left[h(X_n, \tilde{S}_n)\Big| X_{n-1} = x, \tilde{S}_{n-1} = y\right] = \mathbb{E}\left[h(X_n, y + \tilde{S}_n)\Big| X_{n-1} = x, \tilde{S}_{n-1} = 0\right]. \quad (39)$$

They also considered a continuous version of this form. We concentrate here on the discrete time process. The first coordinate $(X_n)_{n\ge0}$ of this process is also a Markov process (driving the process $(\widetilde{S}_n)_{n\ge1}$).
We explain here how, starting with the process $(X, \tilde{S})$ and using an independent sequence

$(Z_k)_{k\geq 1}$ of i.i.d. random variables, we can construct a process $(S_n)_{n\geq 0}$ with

$$S_n := \sum_{k=1}^{n} Y_k, \quad Y_k := f\left(X_{k-1}, X_k, Z_k\right)$$

such that $(X_n, \tilde{S}_0 + S_n)_{n\geq 0}$ is Markov with the same transition operator as $(X_n, \tilde{S}_n)_{n\geq 0}$. Let $\left(Z_n = \left(Z_n^{(1)}, ..., Z_n^{(d)}\right)\right)_{n\geq 1}$ be a sequence of i.i.d. random variables uniformly distributed on $]0,1[^d$ independent of $(\tilde{S}_0, (X_n)_{n\geq 0})$. Let $F_k(x, x', s_1, ..., s_{k-1}, \cdot)$ be the distribution function of $\tilde{S}_1^{(k)}|(X_0 = x, X_1 = x', \tilde{S}_0 = 0, \tilde{S}_1^{(1)} = s_1, ..., \tilde{S}_1^{(k-1)} = s_{k-1})$ (by Jirina's theorem). Consider $f_k(x, x', s_1, ..., s_{k-1}, \cdot)$ its inverse, i.e.

$$f_k(x, x', s_1, ..., s_{k-1}, u) = \inf\left\{z \in \mathbb{R} | F_k(x, x', s_1, ..., s_{k-1}, z) \geq u\right\}.$$

Then we define the coordinates $Y_n^{(k)}$ of $Y_n = \left(Y_n^{(1)}, ..., Y_n^{(d)}\right)$ inductively on k by setting, for all $k = 1, ..., d$,

$$Y_n^{(k)} := f_k\left(X_{n-1}, X_n, Y_n^{(1)}, ..., Y_n^{(k-1)}, Z_n^{(k)}\right).$$

A crucial well known fact is that $f_k(..., z) \leq u \Leftrightarrow z \leq F_k(..., u)$. In particular

$$Y_n^{(k)} \leq u_k \quad \Leftrightarrow \quad Z_n^{(k)} \leq F_k(X_{n-1}, X_n, Y_n^{(1)}, ..., Y_n^{(k-1)}, u_k).$$

This ensures that the distribution of $Y_n^{(k)}$ given $\left(X_{n-1}, X_n, S_{m-1}, (Y_n^{(j)})_{j=1,...,k-1}\right)$ is the same as the one of $\tilde{S}_n^{(k)}$ given $\left(X_{n-1}, X_n, \tilde{S}_{n-1} = 0, (\tilde{S}_n^{(j)} = Y_n^{(j)})_{j=1,...,k-1}\right)$. Thus, we prove by induction that, for all $k = 1, ..., d$, the distribution of $(Y_n^{(1)}, ..., Y_n^{(k)})$ given X_{n-1}, X_n, S_{n-1} coincide with the one of $(\tilde{S}_n^{(1)}, ..., \tilde{S}_n^{(k)})$ given $(X_{n-1}, X_n, \tilde{S}_{n-1} = 0)$. Indeed this holds true for $k = 1$ and, moreover assuming the result holds true at some rank $k \in \{1, ..., d-1\}$, it follows that

$$\begin{aligned}
&\mathbb{E}\left[\prod_{j=1}^{k+1} \mathbf{1}_{]-\infty,u_j]}(Y_n^{(j)}) \middle| X_{n-1}, X_n, S_{n-1}\right] \\
&= \mathbb{E}\left[\prod_{j=1}^{k} \mathbf{1}_{]-\infty,u_j]}(Y_n^{(j)}) \mathbf{1}_{\{Z_n^{(k+1)} \leq F_{k+1}(X_{n-1},X_n,Y_n^{(1)},...,Y_n^{(k)},u_{k+1})\}} \middle| X_{n-1}, X_n, S_{n-1}\right] \\
&= \mathbb{E}\left[\prod_{j=1}^{k} \mathbf{1}_{]-\infty,u_j]}(Y_n^{(j)}) F_{k+1}(X_{n-1}, X_n, Y_n^{(1)}, ..., Y_n^{(k)}, u_{k+1}) \middle| X_{n-1}, X_n, S_{n-1}\right] \\
&= \mathbb{E}\left[\prod_{j=1}^{k} \mathbf{1}_{]-\infty,u_j]}(\tilde{S}_n^{(j)}) F_{k+1}(X_{n-1}, X_n, \tilde{S}_n^{(1)}, ..., \tilde{S}_n^{(k)}, u_{k+1}) \middle| X_{n-1}, X_n, \tilde{S}_{n-1} = 0\right] \\
&= \mathbb{E}\left[\prod_{j=1}^{k+1} \mathbf{1}_{]-\infty,u_j]}(\tilde{S}_n^{(j)}) \middle| X_{n-1}, X_n, S_{n-1}\right],
\end{aligned}$$

and thus the result holds also at rank $k+1$. Therefore the distribution of Y_n given X_{n-1}, X_n, S_{n-1} coincide with the one of $\tilde{S}_n$ given $(X_{n-1}, X_n, \tilde{S}_{n-1} = 0)$. Moreover $(X_n, S_n)_n$ is a Markov process.

Acknowledgements. The author wishes to thank Loïc Hervé for his interest in this work and for stimulating discussions.

References

[1] P. Bálint, S. Gouëzel, S., Limit theorems in the stadium billiard. Commun. Math. Phys. 263, 461–512 (2006).
[2] M. Benda M, A central limit theorem for contractive stochastic dynamical systems, J. App. Prob. 35 (1998) 200–205.

[3] S. Boatto, F. Golse, Diffusion approximation of a Knudsen gaz model : dependence of the diffusion constant upon a boundary condition. Asymptotic. Anal., 31 (2002), no 2, 93–111.
[4] F. E. Browder, On the spectral theory of elliptic differential operators, Math. Annalen, vol. 142 (1961), 22–130.
[5] P. Diaconis P, D. Freedman, Iterated random functions, SIAM Rev. 41 (1999), no. 1, 45–76 (1999).
[6] W. Doeblin, Wolfgang; R. Fortet; Sur des chaînes à liaisons complètes. (French) Bull. Soc. Math. France 65 (1937), 132–148.
[7] M. Duflo, Random Iterative Models, Applications of Mathematics, Springer-Verlag Berlin Heidelberg (1997).
[8] D. Dolgopyat, K. Fernando, *An error term in the Central Limit Theorem for sums of discrete random variables.* preprint.
[9] K. Fernando, C. Liverani, Edgeworth expansions for weakly dependent random variables, to appear in Annales de l'Institut Henri Poincaré.
[10] K. Fernando, F. Pène, Expansions in the local and the central limit theorems for dynamical systems, ArXiv 2008.08726, 66 p.
[11] D. Ferré, Théorème de Keller-Liverani et forte ergodicité, preprint, hal-00538107.
[12] S. Gouëzel, C. Liverani, Banach spaces adapted to Anosov systems, Ergodic Theory Dynam. Systems 26 (2006) 189–217.
[13] M. I. Gordin, The central limit theorem for stationary processes (Russian), Dokl. Akad. Nauk SSSR 188 (1969) 739–741.
[14] D. Guibourg, L. Hervé, A renewal theorem for strongly ergodic Markov chains in dimension $d \geq 3$ and centered case, Potential Anal. 34 (2011), No. 4, 385–410.
[15] Y. Guivarc'h, Application d'un théorème limite local à la transience et à la récurrence de marches de Markov, in Théorie du potentiel (Orsay, 1983), Lecture Notes in Math., vol. 1096, Springer, 1984, p. 301–332.
[16] Y. Guivarc'h and J. Hardy, Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov, Ann. Inst. H. Poincaré Probab. Statist. 24 (1988), p. 73–98.
[17] Y. Guivarc'h, E. Le Page, On spectral properties of a family of transfer operators and convergence to stable laws for affine random walks. Ergodic Theory Dynam. Systems, 28 (2008) no. 2, pp. 423–446.
[18] H. Hennion, Sur un théorème spectral et son application aux noyaux lipschitziens (French), Proc. AMS, 118 (1993), No 2, 627–634.
[19] H. Hennion and L. Hervé, Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness, Lecture Notes in Math., vol. 1766, Springer, 2001
[20] H. Hennion, L. Hervé L, Central limit theorems for iterated random lipschitz mappings, Annals of Proba. 32 (2004), No. 3A, 1934–1984.
[21] H. Hennion and L. Hervé, Stable laws and products of positive random matrices, J. Theoret. Probab. 21 (2008), no. 4, 966–981.
[22] L. Hervé, Théorème local pour chaînes de Markov de probabilité de transition quasi-compacte. Applications aux chaînes V -géométriquement ergodiques et aux modèles itératifs, Annales de l'Institut Henri Poincaré - Probabilités et Statistiques, 41 (2005) 179-196.
[23] L. Hervé, Vitesse de convergence dans le théorème limite central pour des chaînes de Markov fortement ergodiques, Annales de l'Institut Henri Poincaré - Probabilités et Statistiques, 44 (2008), No. 2, 280–292.
[24] L. Hervé, D. Ferré, J. Ledoux, Limit theorems for stationary processes with L2-spectral gap, Annales de l'Institut Henri Poincaré - Probabilités et Statistiques, 48, No. 2, (2012) 396–423.
[25] L. Hervé, J. Ledoux, V. Patilea, A Berry-Esseen theorem of M-estimators for V-geometrical Markov chains, Bernoulli 18 (2), (2012) 703–734.
[26] Loïc Hervé, Françoise Pène : The Nagaev-Guivarc'h method via the Keller-Liverani theorem, Bulletin de la Société Mathématique de France 138, 415-489 (2010).
[27] I. A. Ibragimov, Y.V. Linnik, Independent and Stationary Sequences of Random Variables (J. F. C. Kingman, ed.). Wolters-Noordhoff, Groningen.
[28] C. T. Ionescu Tulcea, G. Marinescu, Théorie ergodique pour des classes d'opérations non complètement continues. (French) Ann. of Math. (2) 52 (1950), 140–147.
[29] J. Karamata, Sur un mode de croissance régulière. Théorèmes fondamentaux, Bulletin de la SMF 61 (1933), 55–62.
[30] G. Keller, Un théorème de la limite centrale pour une classe de transformations monotones par morceaux, C. R. Acad. Sci., Paris, Sér. A 291 (1980), 155–158 .
[31] G. Keller and C. Liverani, Stability of the Spectrum for Transfer Operators, Annali della Scuola Normale Superiore di Pisa Classe di Scienze Sér. 4, 28 (1999),141–152.
[32] E. Le Page E, Théorèmes de renouvellement pour les produits de matrices aléatoires. Equations aux différences aléatoires, Séminaires de Rennes (1983).

[33] S.P. Meyn, R.L. Tweedie, Markov chains and stochastic stability. Springer Verlag, New York, Heidelberg, Berlin (1993).
[34] X. Milhaud, A. Raugi, Etude de l'estimateur du maximum de vraisemblance dans le cas d'un processus autorégressif: convergence, normalité asymptotique, vitesse de convergence, Ann. Inst. H. Poincaré 25 (1989), No 4, 383–428.
[35] S. V. Nagaev, Some limit theorems for stationary Markov chains (Russian), Teor. Veroyatnost. i Primenen. 2 (1957), p. 389–416.
[36] S. V. Nagaev , More exact limit theorems for homogeneous Markov chains, Teor. Verojatnost. i Primenen. 6 (1961), p. 67–86.
[37] M. Peigné M, Iterated function schemes and spectral decomposition of the associated Markov operator, Séminaires de Probabilité de Rennes (1993).
[38] M. Rosenblatt, Markov processes. Structure and asymptotic behavior. Springer-Verlag. New York (1971).
[39] D. Szász, T. Varjù, Limit laws and recurrence for the planar Lorents process with infinite horizon., J. Stat. Phys. 129 (2007), 59–80.

Univ Brest, Université de Brest LMBA, UMR CNRS 6205, 6 avenue Le Gorgeu, 29238 Brest cedex, France