# Gamma-convergence results for nematic elastomer bilayers: relaxation and actuation

Pierluigi Cesana, Andrés León Baldelli

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**Abstract**

We compute effective energies of thin bilayer structures composed by soft nematic elastic-liquid crystals in various geometrical regimes and functional configurations. Our focus is on order-strain interaction in elastic foundations composed of an isotropic layer attached to a nematic substrate. We compute Gamma-limits as the layers thickness vanishes in two main scaling regimes exhibiting spontaneous stress relaxation and shape-morphing, allowing in both cases out-of-plane displacements. This extends the plane strain modelling of [10], showing the asymptotic emergence of fully coupled macroscopic active-nematic foundations. Subsequently, we focus on actuation and compute asymptotic configurations of an active plate on nematic foundation interacting with an applied electric field. From the analytical standpoint, the presence of an electric field and its associated electrostatic work turns the total energy into a non-convex and non-coercive functional. We show that equilibrium solutions are min-max points of the system, that min-maximising sequences pass to the limit and, that the limit system can exert mechanical work under applied electric fields.

## Contents

# 1 Introduction

Nematic Liquid Crystal Elastomers (NLCEs) are classes of soft shape-memory alloys where order states and optical instabilities can be triggered, tuned, or suppressed by means of mechanical deformations and electrostatic fields. NLCEs, typically synthesised as thin strips, appear in the form of gels or rubbery solids. Structurally, they are constituted by polymeric chains which act as the structural backbone of the material, to which attach molecules of a nematic liquid crystal, the optically active units. Liquid crystal molecules have a two-fold response to stimuli. Indeed, they re-orient as a consequence of elastic deformations and stretches dictated by *internal energy* minimisation and they rotate collectively parallel to electric or magnetic forces, activated by *external fields*.

From the structural perspective, NLCE membranes enjoy mechanical instabilities which are typical of thin structures and interact with the rich multi-physical phenomenology of the material.

In the present contribution we test and analyse the interaction between elastic, optic, and electrostatic forces (characterising the *material* behaviour) versus geometric constraints (which cause *structural* instabilities) in two distinct physical regimes, and configurations relevant for technological applications.

We consider continuum models in the framework of linearised elasticity. Despite intrinsic limitations of an approach with infinitesimal displacements, a linearised model is particularly suitable for treating the coupling of multiphysyical phenomena, by superposition. In our setting, these are represented by the interplay between nematic ordering and elasticity, pertaining to the coupling of polymeric chains with optical states of the liquid crystal, and the opto-electric interaction, associated to the connection between the liquid crystal and the dielectric vector field. We consider an energy model introduced in [9] for the description of equilibrium states of NLCEs under an electric field. In the present work, we are interested in the asymptotic behaviour of sequences of functionals for bi-layer structures where a NLCE membrane sustains a stiff and thin isotropic film.

In this setting, two non-dimensional parameters (length scales) collapse several material and geometric parameters, allowing us to separately analyse two opposite phenomenological regimes, namely, that of thin films of large planar extent (the "large body" regime) and that of thin films of small area (referred to as "thin particles"). The main contribution in this paper is the derivation, the analysis, and the computation of effective two-dimensional plate models for multilayer structures comprising active soft nematic plates, in the regimes of spontaneous relaxation and shape-morphing actuation. In both limits the underlying elastic behaviour is represented by an effective linear plate energy of Kirchhoff-Love type, enriched by an additional nonlinear active foundation term which we explicitly characterise.

The first regime (that of large bodies) entails the relaxation of elastic stresses by formation of optimal optic microstructure, with full or partial coupling between in-plane and out-of-plane deformations depending on the thickness scaling. This process ultimately characterises the mechanical behaviour of large thin elastic sheets. This setting is explored in the first part of this paper, inspired by observations of pattern formation and opto-elastic relaxation in bilayer systems of nematic elastomers, see [18]. There, a complex material-structure interaction is observed in thin membranes of NLCE in contact with an overlying isotropic film, resulting in formation of micro-wrinkles competing with visible shear-band microstructures of the optical axis. The former manifestation is a typical structural instability, also observed in thin stretchable membranes [24, 5] whereas the latter is a material instability observed in various classes of media, encompassing, e.g. solid elastic crystals [4] besides liquid crystal elastomers. This paper complements and completes the analysis developed in [10] on planar-constrained nematic bilayers by exploring the full three-dimensional elastic model allowing for out-of plane elastic deformations.

The second regime (small particles), corresponds to the physical setting of small multilayer domains which can be actuated into nontrivial out-of-plane deformation modes by uniform fields, as we show in the second part of the paper. Elaborating on the relaxation result, we describe this asymptotic behaviour, where the optic director is free to rotate and re-align under the influence of applied external fields, albeit homogeneously. These small domains may be regarded as elementary "building blocks" for more complex morphing shapes that can be assembled via actuation of the frozen director, in presence of suitable boundary conditions. In the second part of this article we turn our attention on actuation, i.e. the capability of controlling the shape of a membrane thanks to the activation of liquid crystal molecules by external fields. Our investigation is inspired by a number of recent experimental realisations. In [23], [22] design of complex shapes is performed via thermal actuation of heterogeneously patterned nematic elastomer films, in [1], [19] opto-elasticity in nematic elastomers is investigated to show that "soft" NLCE robots can execute work cycles thanks to the cooperative interaction between light absorption and mechanical deformations. We refer to [25] for a review focussed on liquid crystalline materials from the view point of thermal-photo-elastic coupling.

From the mathematical perspective, we perform the exact computation of the Γ-limit of a family of energy functionals parametrised by the two scale parameters. For the actuation problem, because of the presence of an energetic contribution due to external fields possibly unbounded below, we face the issue of non-convexity and non-coercivity of the total energy functional which lacks sequential lower semicontinuity. The asymptotic analysis is nonetheless successful in showing that limit regime enjoys a saddle structure and, by computing the exact expression of asymptotic Lagrangians, we demonstrate that the limit system can indeed transform and convert work produced by electrostatic forces into shape deformation, which is, to date, a challenge in soft robotics. To illustrate our purpose, we numerically solve a simple actuation problem for membrane bending and show, as a mathematically relevant example, the equilibrium configuration of a nematic bilayer activated by an applied electrostatic potential.

The outline of the paper is as follows. After we formally present the functional setting in the Introduction, discussing the kinematics and the mechanics of the problem (Sec. 2) Section 3 is devoted to the analysis of relaxation results for thin and large NLCE bilayers. This part of the paper builds upon material produced in [10] which we systematically refer to. Finally, in Section 4 we analyse limit functionals for thin and small NLCE bi-layers. We characterise the asymptotic behaviour of the saddle points of the energy functionals and, as an example of our analytical work, we describe numerical calculations showing shape-actuation.

### 1.1 Notation

Throughout the paper, Greek indices run from 1 to 2 whereas Latin indices run from 1 to 3. The summation convention on repeated indices is assumed, unless explicitly stated. To highlight the dependency with respect to in-plane vs. out-of-plane coordinates, a prime sign indicates planar components of a vector, of second order tensor, and of differential operators, e.g. $v' = v_\alpha e_\alpha$, $B' = b_{\alpha\beta} e_\alpha e_\beta$, and $\nabla'(\cdot) = \partial_\alpha(\cdot) e_\alpha$, where $e_\alpha$ indicate the unit vectors in the $x_1 - x_2$ plane. In order to distinguish

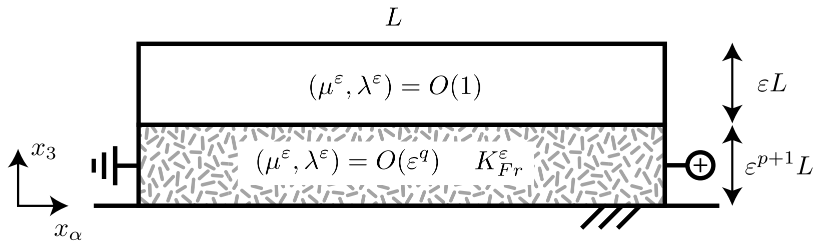


Figure 1: Physical three-dimensional domain of a thin bilayer system consisting in one nematic elastomer layer supporting a stiff thin film. The system undergoes in-plane (membrane) and out-of-plane (bending) displacements, subject to mechanical volume and surface loads as well as electrical work inducing nematic reorientation. We distinguish two physically relevant regimes, depending on the scaling laws of physical parameters: the relaxation regime (with formation of microstructure) and the actuation regime (with frozen optic tensor).

homologue quantities defined in the two layers, we superpose a hat to those which refer to the nematic layer, as in $k, \hat{k}$ to indicate limit rescaled strains in the film and nematic layer, respectively. The inner product is denoted by a dot. In general (but with some exceptions, like $\nu$), material parameters or effective coefficients are indicated by sans serif letters, cf. Table 1 for a collection of relevant parameters and physical constants. With $u \otimes_s v$ we signify the symmetrised outer product $\frac{1}{2}(u_i v_j + u_j v_i)$ between vectors $u, v$ and by $I$ the identity matrix in $\mathbb{R}^{n\times n}$. Throughout the paper, $C$ stands for a generic constant which may change from line to line. Thickness averages are indicated by an overbar, as in $\bar{v}(x') := 1/H \int v(x', x_3) dx_3$ where $H$ denotes the size of the (transverse) integration domain. We adopt standard notation for functional spaces, such as $L^2(\Omega, \mathbb{R}^n)$, $L^2(\Omega, \mathbb{R}^{n\times n})$, and $H^1(\Omega, \mathbb{R}^n)$, $H^1(\Omega, \mathbb{R}^{n\times n})$, for the Lebesgue spaces of square integrable maps from $\Omega$ onto $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, and the Sobolev space of square integrable maps with square integrable weak derivatives on $\Omega$. Concisely, we write $L^2(\Omega)$ and $H^1(\Omega)$ whenever $n = 1$.

All $\varepsilon$-dependent quantities refer to the physical three-dimensional system, a thin bilayer structure whose thickness depends on $\varepsilon$. After introducing appropriate scalings for all material quantities and rescaling the physical domain we drop the $\varepsilon$-dependence.

# 2 Setting of the problem

**Domain.** Let $\Omega^\varepsilon$ be a sufficiently smooth three-dimensional domain, constituted by the union of two thin layers: a linearly elastic film occupying $\Omega^\varepsilon_f = \omega \times (0, \varepsilon L)$ and a soft nematic elastomer occupying $\Omega^\varepsilon_b = \omega \times (-\varepsilon^{p+1} L, 0]$. The two layers are attached to a rigid substrate which imposes a hard condition of place, see Figure 1. The basis of the cylindrical three-dimensional domain is $\omega \subseteq \mathbb{R}^2$ with characteristic size $L$. We focus on *thin* limit systems as $\varepsilon \to 0$, by requiring that $p + 1 > 0$.

The elastic film can deform both in-plane through membrane deformations and out-of-plane, by bending. The nematic elastomer is subject to a hard boundary condition at the interface with the underlying (rigid) substrate.

**Order tensors.** We define the set of biaxial (De Gennes) tensors [14] as

$$\mathcal{Q}_B := \Big\{Q \in \mathbb{R}^{3\times 3}, \operatorname{tr} Q = 0, Q = Q^T : \ -\frac{1}{3} \leq \lambda_{\min}(Q) \leq \lambda_{\max}(Q) \leq \frac{2}{3}\Big\}, \tag{1}$$

where $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ denote the smallest and largest eigenvalue of the matrix $Q$. We remind that $\mathcal{Q}_B$ is convex, closed and bounded.

We introduce $\mathcal{Q}_U$, that is, the subset of $\mathcal{Q}_B$ populated by all uniaxial tensors [16]. These are matrices that can be written in the form

$$Q = s(n \otimes n - \frac{1}{3}I) \tag{2}$$

for some $n \in \mathbb{R}^3$ with $|n| = 1$ and some $-\frac{1}{2} \leq s \leq 1$. Without loss of generality, the set of uniaxial tensors is defined as follows

$$\mathcal{Q}_U := \Big\{Q \in \mathcal{Q}_B : \ |Q|^6 = 54(\det Q)^2\Big\}. \tag{3}$$

Indeed, the two algebraic characterizations in (2) and (3) can be proven to be equivalent. We remark (3) is non-convex, pointwise closed and therefore closed in all the strong topologies. Also, we introduce the set of (uniaxial) Frank tensors [17] which uses only the eigenframe of $Q$ as the nematic state variable, constrained to have eigenvalues $2/3, -1/3, -1/3$. Uniaxial tensors range in the set

$$\mathcal{Q}_{Fr} := \Big\{Q \in \mathcal{Q}_U : \lambda_{\max}(Q) = \frac{2}{3}, \ \lambda_{\min}(Q) = -\frac{1}{3}\Big\}. \tag{4}$$

Observe that any tensor in 4 can be represented in the following manner:

$$Q = n \otimes n - \frac{1}{3}I \tag{5}$$

for some $|n| = 1$ (which is equivalent to requiring $s = 1$ in (2)). This clarifies an order tensor taken in the illustrative example of Fig. 2 is indeed a Frank tensor (see Caption of Fig. 2.) It is important to remark that, whenever a liquid crystal system is described by a tensor in the form (5), then $n$ (which is called the *director*) represents the common direction of the perfectly aligned nematic molecules. Instead, $\mathcal{Q}_U$ and $\mathcal{Q}_B$ describe, additionally, order states, that is, configurations where the liquid crystal fails to be perfectly ordered and aligned to a director. Instead, the description of such systems should be performed in probabilistic terms, and $\mathcal{Q}_U$ and $\mathcal{Q}_B$ model probabilistic information derived from the theories of Ericksen and de Gennes, respectively. Notice that, since $\operatorname{tr} Q = 0$, this suffices to describe the spectrum of $Q$. It follows by the definition that $\mathcal{Q}_{Fr}$ is a closed and non-convex set and the inclusion $\mathcal{Q}_{Fr} \subset \mathcal{Q}_U \subset \mathcal{Q}_B$ holds. Importantly, $\mathcal{Q}_B$ coincides with the convex envelope of $\mathcal{Q}_{Fr}$ and of $\mathcal{Q}_U$.

**Mechanical model.** The total energy of the system is modelled on the classical theory of linearised elasticity. Thus, we may assume physical forces are additive and their effects are algebraically superposed. The total energy combines a film contribution (measured on $\Omega_f^\varepsilon$) to the contribution of the nematic bonding layer (defined on $\Omega_b^\varepsilon$). The latter, in turn, is the sum of three terms: a bulk energy density which measures the strain-order interaction of nematic elastomers according to the well-known model defined in [9] and analysed in [7, 6, 8]; a curvature term (or Frank energy) proportional to the square of the gradient of the $Q$-tensor which induces molecules to be parallel to each other; and finally, a loading term representing the external work, the only possibly non-positive contribution to the energy.

Considering here only electrostatic work and summing all contributions, the total energy reads

$$
\begin{aligned}
E_\varepsilon(v,Q) := \frac{1}{2}\int_{\Omega_f^\varepsilon} \frac{\mathsf{E}^\varepsilon}{1+\nu}\left(|e(v)|^2 + \frac{1}{1-2\nu}\operatorname{tr}^2 e(v)\right) dy \\
+ \frac{1}{2}\int_{\Omega_b^\varepsilon} \frac{\mathsf{E}^\varepsilon}{1+\nu}\left(|e(v)-Q|^2 + \frac{1}{1-2\nu}\operatorname{tr}^2 e(v)\right) dy \\
+ \frac{1}{2}\int_{\Omega_b^\varepsilon} K^\varepsilon_{Fr}|\nabla_\varepsilon Q|^2 dy - \frac{1}{2}\int_{\Omega_b^\varepsilon} \nabla\tilde{\varphi}^T\tilde{\mathsf{D}}(Q)\nabla\tilde{\varphi} dy, \quad (6)
\end{aligned}
$$

where admissible spaces for displacements $v$, the optic tensor $Q$, and the electrostatic potential $\tilde{\varphi}$ read

$$
v \in \mathcal{V}_\varepsilon := \{H^1(\Omega^\varepsilon, \mathbb{R}^3), v(x', -\varepsilon^{p+1}) = 0 \text{ a.e. } x' \in \omega\}, \quad Q \in H^1(\Omega_b^\varepsilon, \mathcal{Q}_{Fr}), \quad \tilde{\varphi} \in H^1(\Omega_b^\varepsilon).
$$

**Material regime (assumptions on the scaling of material parameters)** We explicitly, for definiteness, the assumptions on material parameters by fixing a parametric scaling law defining the relative stiffness and the relative curvature energy. Considering that the nematic bonding layer is much softer than the overlying film, while being elastically compressible, we assume the following

$$
(\mathsf{E}^\varepsilon, \nu^\varepsilon)(x) = \begin{cases} (\mathsf{E}, \nu), & x \text{ in } \Omega_f \\ (\varepsilon^q \mathsf{E}, \nu), & x \text{ in } \Omega_b \end{cases}, \quad \text{with } q > 0, \qquad K^\varepsilon_{Fr} = \frac{\varepsilon^q \mathsf{E}}{1+\nu}\tilde{\delta}^2_\varepsilon. \quad (7)
$$

Here, $\mathsf{E}$ is the Young modulus of the elastic film and $-1 < \nu < 1/2$ its Poisson ratio. From now on, to simplify the notation without any loss of generality we assume $\mathsf{E} = 1$, leaving explicit reference to the only meaningful elastic nondimensional parameter, the Poisson ratio $\nu$. Note that this is always licit and amounts to a rescaling of displacements. In the expression above, $\tilde{\delta}_\varepsilon$ represents the characteristic length scale which emerges from the competition of the shear modulus of nematic rubber vs. the Frank constant of the liquid crystal. For the purpose of our analysis, $\tilde{\delta}_\varepsilon$ identifies a critical material parameter which, as $\varepsilon$ goes to zero, may vanish or blow up, leading to the two separate regimes of relaxation or of director actuation, respectively. In order to bootstrap the asymptotic procedure and focus on the interplay between membrane and bending modes, we further scale dependent and independent variables as follows.

$$
v(y', y_3) = \begin{cases} L(\varepsilon u_\alpha(Lx', L\varepsilon x_3), u_3(Lx', L\varepsilon x_3)) & \text{in } \Omega_f \\ L(\varepsilon u_\alpha(Lx', L\varepsilon^{p+1} x_3), \varepsilon^r u_3(Lx', L\varepsilon^{p+1} x_3)) & \text{in } \Omega_b, \end{cases} \quad (8)
$$

where $r$ is the magnitude of vertical displacements, a parameter that ultimately depends on the loads. The scaling above has a twofold goal, that of mapping the physical, $\varepsilon$-dependent domain onto a fixed, unit, domain, and that of exposing the interplay between in-plane vs. out-of-plane displacements which, in turn, depends on the type and intensity of the loads.

Similarly, we introduce the nondimensional (rescaled) electrostatic potential $\varphi$

$$
\tilde{\varphi}(y', y_3) = \varphi_0^\varepsilon \varphi(Lx', L\varepsilon^{p+1} x_3) \quad \text{in } \Omega_b, \quad (9)
$$

where $\varphi_0^\varepsilon$ is the electrostatic scale gauge. Note that, because the electric field is solved independently of the opto-elastic problem, its scale is imparted by its boundary conditions which, in turn, can be freely chosen in such a way that the electric energy is of the same order of magnitude as the elastic terms, which ultimately depend upon $\varepsilon$.

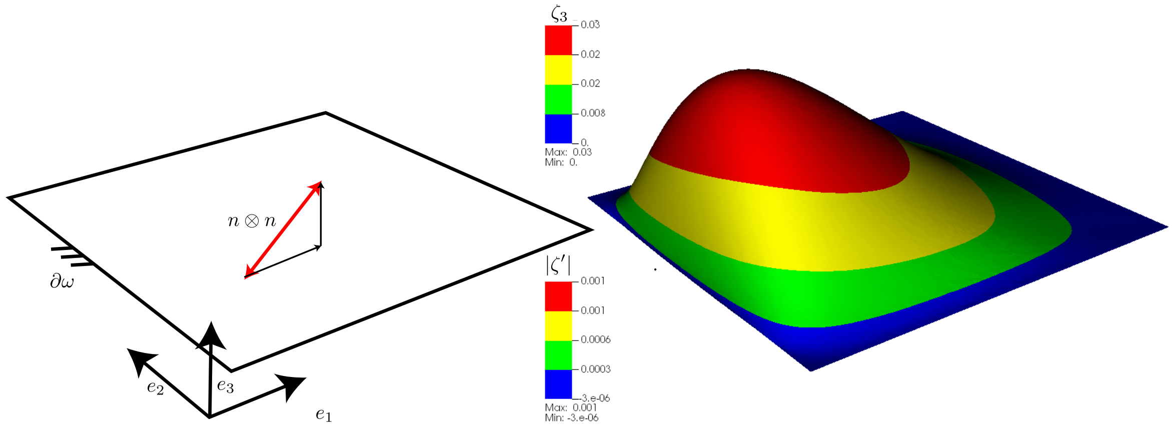


Figure 2: Illustrative numerical calculation of a thin nematic bilayer plate in the actuation regime, cf. Section 4, Theorems 3. The effective model given by the asymptotic theory is a fully coupled macroscopic opto-elastic plate. By exploiting strain-order coupling, spontaneous deformations of a multilayer composite achieve out-of-plane bending deformations under external electric stimulation. Here, an initially flat and thin active bilayer, clamped at the boundaries, is actuated by a uniform optic tensor $Q = n_0 \otimes n_0 - \frac{1}{3}I$ described by the homogeneous director $n_0 = (e_1 + e_3)/\sqrt{2}$ (cf. image left). Colour coding in figure refers to the euclidean norm of in-plane deformations $|\zeta'|$, the legend for $\zeta_3$ is displayed to indicate the relative scaling of transverse displacements.

**Film energy.** Writing the energy (6) in terms of the scaled quantities identified in (8), the film contribution reads

$$\varepsilon^3 L^3 J_f^\varepsilon(u) = \varepsilon^3 L^3 \frac{1}{2} \int_{\Omega_f} \left( \left( |e_{\alpha\beta}(u)|^2 + (\varepsilon^{-2} e_{33}(u))^2 + 2|\varepsilon^{-1} e_{\alpha 3}(u)|^2 \right) \right) dx$$
$$+ \varepsilon^3 L^3 \frac{1}{2} \int_{\Omega_f} \frac{1}{1-2\nu} \left( (e_{\alpha\alpha}(u)) + \varepsilon^{-2} e_{33}(u) \right)^2 dx \quad (10)$$

**Nematic energy.** On the other hand, using (8), the nematic contribution to the total energy (6) reads

$$\varepsilon^3 L^3 \varepsilon^{q+p-2} J_b^\varepsilon(u, Q) :=$$
$$L^3 \frac{1}{2} \varepsilon^q \varepsilon^{p+1} \int_{\Omega_b} \left( |\varepsilon e_{\alpha\beta}(u) - Q_{\alpha\beta}|^2 + \left( \frac{\varepsilon^r}{\varepsilon^{p+1}} e_{33}(u) - Q_{33} \right)^2 + 2 \left| \frac{1}{2} \left( \varepsilon^r \partial_\alpha u_3 + \frac{\varepsilon}{\varepsilon^{p+1}} \partial_3 u_\alpha \right) - Q_{\alpha 3} \right|^2 \right) dx$$
$$+ L^3 \frac{1}{2} \varepsilon^q \varepsilon^{p+1} \int_{\Omega_b} \frac{1}{1-2\nu} \left( \varepsilon e_{\alpha\alpha}(u) + \frac{\varepsilon^r}{\varepsilon^{p+1}} e_{33}(u) \right)^2 dx, \quad (11)$$

while the curvature energy writes

$$\frac{1}{2} L \int_{\Omega_b} K_{Fr}^\varepsilon \left( |\nabla' Q|^2 + \left| \frac{\partial_3 Q}{\varepsilon^{p+1}} \right|^2 \right) \varepsilon^{p+1} dx = \frac{1}{2} L \int_{\Omega_b} \frac{\tilde{\delta}_\varepsilon^2}{\varepsilon^{p+1}} \left( \varepsilon^{2p+2} |\nabla' Q|^2 + |\partial_3 Q|^2 \right) dx.$$

| Symbol | Quantity |
|---|---|
| $K^{\varepsilon}_{Fr}, K_{Fr}$ | Frank constant |
| $\tilde{\mathsf{D}}(Q), \mathsf{D}(Q), \overline{\mathsf{D}}(Q)$ | dimensional matrix of dielectric coefficients, nondimensional, averaged |
| $\overline{\mathsf{B}}(\overline{Q})$ | relaxed matrix of dielectric coefficients |
| $\mathsf{d}_{ij}(Q)$ | Coefficients of $\mathsf{D}(Q)$ |
| $\epsilon_o$ | Dielectric constant (in vacuum) |
| $\epsilon_\perp, \epsilon_{\|\|}$ | Relative perpendicular and parallel, dielectric constant |
| $\Omega^{\varepsilon}_f, \Omega_f$ | Film layer |
| $\Omega^{\varepsilon}_b, \Omega_b$ | Bonding layer |
| $\Omega^{\varepsilon}, \Omega$ | Bilayer (union of film and bonding layer) |
| $\omega$ | Membrane planar section |
| $L$ | Diameter |
| $\nu, (\lambda, \mu)$ | Elastic constants: Poisson ratio, Lamé coefficients |
| $J^{\varepsilon}_b$ | Rescaled energy, nematic layer |
| $J^p_{\varepsilon}, \widetilde{J}^p_{\varepsilon}$, | Energy functional, mechanical model |
| $J^0, J^-$ | Relaxation limit energies |
| $I_\varepsilon, I_0$ | Electrostatic work, limit |
| $\mathsf{K}(\zeta', \zeta_3)$ | Effective stiffness of nematic foundation |

Table 1: Material and geometric parameters.

We choose to keep an explicit dependence upon $p$ because, depending on its value (the relative thickness of the film layers), we identify phenomenologically different limit regimes.

**Electrostatic work.** The electrostatic work density is the (scalar) product of the electrostatic vector $e$ by the dielectric displacement vector $d$. Based on the linear model of nematic liquid crystals, the relation between $d$ and $e$ is obtained introducing the tensor of dielectric coefficients $\tilde{\mathsf{D}}(Q)$ so that $d := \tilde{\mathsf{D}}(Q)e$, see [14], [21].

Upon introduction of the electrostatic potential, related to the electric field by $e = -\nabla\tilde{\varphi}$, the electrostatic work density is given by $d \cdot e = e^T\tilde{\mathsf{D}}(Q)e$ where $\tilde{\mathsf{D}}(Q) = \epsilon_0\mathsf{D}(Q)$. The tensor of dielectric coefficients depends linearly on the order tensor $Q$. In turn, the electric field is obtained by optimisation and depends, in an intricate way, upon $Q$. We shall elaborate on their connection in the Actuation, Section 4. The scaled electrostatic work density reads

$$J^{\varepsilon}_{ele}(Q,\tilde{\varphi}) = \frac{1}{2}\int_{\Omega^{\varepsilon}_b} \nabla\tilde{\varphi}^T\tilde{\mathsf{D}}(Q)\nabla\tilde{\varphi}\,dy = \varepsilon^3(\varphi^{\varepsilon}_0)^2\varepsilon^{p-2}\epsilon_0 L\frac{1}{2}\int_{\Omega_b} \nabla^T_{\varepsilon}\varphi\mathsf{D}(Q)\nabla_{\varepsilon}\varphi\,dx. \tag{12}$$

where we have concisely denoted by $\nabla_\varepsilon\varphi$ the scaled gradient of a scalar function, namely $\nabla_\varepsilon\varphi := \left(\nabla'\varphi, \frac{1}{\varepsilon^{p+1}}\partial_3\varphi\right)$. Note that the work integral above, for a fixed $Q \in \mathcal{Q}_X$ ($X$ being a place-holder for $Fr, U$ or $B$) and for a fixed $\varepsilon > 0$, is a standard elliptic functional modelled upon the symmetric positive definite matrix of nondimensional dielectric coefficients $\mathsf{D}(Q)$.

**Additional mechanical loads.** Finally, we consider applied body and surface loads by prescribing two force densities, $f^\varepsilon$ in the interior and $g^\varepsilon$ on the upper face of the film domain. Without loss of generality, we scale imposed loads in such a way that the corresponding work is of the order of magnitude of the elastic film energy. Accordingly, we set $f^\varepsilon := \varepsilon^2 f, g^\varepsilon := \varepsilon^3 g$ with $f : L^2(\Omega_f^\varepsilon, \mathbb{R}^3), g : L^2(\Omega_f^{\varepsilon +}, \mathbb{R}^3)$ so that the scaled linear form corresponding to mechanical loads reads

$$\mathcal{L}^\varepsilon(v) = \varepsilon^3 \mathcal{L}(v) = \varepsilon^3 L^3 \int_{\Omega_f} f v dy + \varepsilon^3 L^2 \int_{\omega^+} g v dy', \tag{13}$$

where $\omega^+ = \omega \times \{1\}$, see [12]. Our assumption on the scaling of loads is not restrictive owing to the fact that the mechanical work is a continuous perturbation to the total energy.

## 2.1 Scaling regimes

We specialise the scaling laws introduced in (7), (8) in order to focus on the material regime in which there is possible coupling between membrane and bending deformation modes, as well as with the optoelastic behaviour of the nematic layer. Heuristically, the bending energy of the film scales like $\varepsilon^3$, thus we fix the scaling parameters of the system in such a way that both i) the energy of the nematic bonding layer is of the same order of magnitude of the bending energy of the film, ii) we focus on vertical displacements which are of the same order of the thickness of the overlying film, and iii) the electrostatic work is of the same order of magnitude of the membrane energy of the film. Respectively, we set

$$\text{i) } q + p - 2 = 0, \qquad \text{ii) } r = p + 1, \qquad \text{iii) } \varphi_0^\varepsilon = \varepsilon^{-1-p/2} {\epsilon_0}^{-1/2}. \tag{14}$$

Under these assumptions, the total energy, i.e., the sum of film and nematic layer energies minus the external work, as defined in (10), (11) and (12), reads

$$\begin{aligned}
\tilde{J}^p_\varepsilon(v, Q, \varphi) &:= J^\varepsilon_f(v) + J^\varepsilon_b(v, Q) - J^\varepsilon_{ele}(Q, \varphi) \\
&= \frac{1}{2}\int_{\Omega_f} \left(|e_{\alpha\beta}(v)|^2 + (\varepsilon^{-2} e_{33}(v))^2 + 2|\varepsilon^{-1} e_{\alpha 3}(v)|^2\right) dx + \frac{1}{1-2\nu}\left((e_{\alpha\alpha}(v)) + \varepsilon^{-2} e_{33}(v)\right)^2 dx \\
&\quad + \frac{1}{2}\int_{\Omega_b} \left(|\varepsilon e_{\alpha\beta}(v) - Q_{\alpha\beta}|^2 + (e_{33}(v) - Q_{33})^2\right) dx \\
&\quad + \frac{1}{2}\int_{\Omega_b} 2\left|\frac{1}{2}\left(\varepsilon^{p+1}\partial_\alpha v_3 + \varepsilon^{-p}\partial_3 v_\alpha\right) - Q_{\alpha 3}\right|^2 dx + \frac{1}{1-2\nu}\left(\varepsilon e_{\alpha\alpha}(v) + e_{33}(v)\right)^2 dx \\
&\quad + \frac{1}{2}\int_{\Omega_b} \frac{\tilde{\delta}^2_\varepsilon}{L^2\varepsilon^{2p+2}}\left(L^2\varepsilon^{2p+2}|\nabla' Q|^2 + |\partial_3 Q|^2\right) dx - \frac{1}{L^2}\frac{1}{2}\int_{\Omega_b}\left(\nabla'\varphi, \frac{1}{\varepsilon^{p+1}}\partial_3\varphi\right)^T \mathsf{D}(Q)\left(\nabla'\varphi, \frac{1}{\varepsilon^{p+1}}\partial_3\varphi\right) dx.
\end{aligned} \tag{15}$$

In the expression above, the quantity

$$\delta^2_\varepsilon := \frac{\tilde{\delta}^2_\varepsilon}{L^2\varepsilon^{2p+2}}, \tag{16}$$

identifies a material length scale stemming from the ratio between Frank's curvature constant and the bonding layer's stiffness, relatively to the size of the domain $L$ and the thickness of the nematic layer. Notice that this quantity is scaled with respect to the thickness, hence, depending on the material regime and geometric dimensions may either vanish or blow up, as $\varepsilon \to 0$. These two scenarios indeed correspond to the distinct material regimes of actuation (with fixed orientation of the director) and that of spontaneous relaxation (with emergence micro-textured patterns).

More precisely, the relaxation scenario is dominated by the *rescaled* length scale $\delta_\varepsilon$, in the regime $\delta_\varepsilon \to 0$. In this setting, $\delta_\varepsilon$ is the smallest scale of the system well below the layers' thickness and allows for vanishing transition layers with negligible energetic cost. Contrarily, the actuation regime is characterised by the *macroscopic* length scale $\tilde{\delta}_\varepsilon$, in the limit $\tilde{\delta}_\varepsilon \to \infty$. In this context the optic tensor is much more rigid, its homogeneity is forced under the influence of applied external fields.

Because an electric field generated by an external device acts on the nematic elastomer by orienting the LC molecules and thus performing work, the sign of the functional is undefined. A careful analysis is required to study critical points of the total energy which are of saddle-type. We devote Section 4 to the analysis of the nematic elastic foundations and electric fields, whereas we focus our attention in the next section to the analysis of the regime of nematic relaxation where optoelastic patterns spontaneously emerge, without external stimuli, in such a way to relax mechanical stresses. Accordingly, we set $\varphi \equiv 0$ in (15) and compute the asymptotics as $\varepsilon \to 0$ of the energy $\tilde{J}^p_\varepsilon(v, Q, 0)$.

# 3 Relaxation

The relaxation regime for nematic multilayers is characterised by the spontaneous emergence of textured microstructures and a strong two-way coupling between optic axis and elastic displacements. This scenario, in turn, occurs as Frank's curvature energy is small and transitions between differently oriented microscale domains can be accommodated with little energetic cost. Indeed, in this case, Frank's stiffness provides the smallest length scale of the system. Therefore, the relaxation regime is characterised by a vanishing $\delta_\varepsilon$ which corresponds to the regime of a large plates with low bending constant. Thus, we assume $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, the precise rate of decay will be specified in the following.

The program is to explicitly compute the effective stress relaxation induced by a local accommodation of the optical texture under mechanical deformation, a mechanism which is responsible of the emergence of fine scale, possibly periodic, optical patterns. In energetic terms, this amounts to first computing locally-optimal nematic textures at microscale and then performing the dimension reduction to derive a macroscopic two-dimensional model, as described by the following one-variable model [10, 11, 7, 15]

$$J^p_\varepsilon(u) := \begin{cases} \inf\limits_{Q \in H^1(\Omega_b, \mathcal{Q}_X)} \tilde{J}^p_\varepsilon(u, Q, 0) & \text{if } u \in \mathcal{V} \\ +\infty & \text{if } u \in L^2(\Omega, \mathbb{R}^3) \setminus \mathcal{V}, \end{cases} \tag{17}$$

where

$$\mathcal{V} := \{H^1(\Omega, \mathbb{R}^3), u(x', -1) = 0\} \tag{18}$$

is the set of kinematically admissible three-dimensional displacements and $X$ stands for either $Fr, U$ or $B$, according to the available order tensor models. We will show that, remarkably, the $\Gamma$-limit of (17) is common to all available models.

**Remark 1.** *By introducing scaled strain tensors in the film $\kappa_\varepsilon$ and in the nematic layer $\hat{\kappa}_\varepsilon$, reading respectively*

$$\kappa_\varepsilon(u) = \begin{pmatrix} e_{\alpha\beta}(u) & \frac{1}{2\varepsilon}\left(\partial_\alpha u_3 + \partial_3 u_\alpha\right) \\ sym & \varepsilon^{-2} e_{33}(u) \end{pmatrix} \quad \textit{and} \quad \hat{\kappa}_\varepsilon(u) = \begin{pmatrix} \varepsilon e_{\alpha\beta}(u) & \frac{1}{2\varepsilon}\left(\varepsilon^{p+1}\partial_\alpha u_3 + \varepsilon^{-p}\partial_3 u_\alpha\right) \\ sym & e_{33}(u) \end{pmatrix}, \tag{19}$$

*we may rewrite* (17) *in compact notation, namely for* $u \in \mathcal{V}$

$$J^p_\varepsilon(u) = \frac{1}{2}\int_{\Omega_f} \left(|\kappa_\varepsilon|^2 + \frac{1}{1-2\nu}\operatorname{tr}^2 \kappa_\varepsilon\right) dx \\ + \frac{1}{2}\inf_{Q\in H^1(\Omega_b,\mathcal{Q}_X)} \int_{\Omega_b} \left(|\hat\kappa_\varepsilon - Q|^2 + \frac{1}{1-2\nu}\operatorname{tr}^2 \hat\kappa_\varepsilon + \frac{1}{2}\delta_\varepsilon^2\left(\varepsilon^{p+2}|\nabla' Q|^2 + |\partial_3 Q|^2\right)\right) dx. \quad (20)$$

The convergence properties of minimising sequences of displacements associated to the functional above characterise the limit space of displacements, which is independent of the thickness ratio $p$. However, it is *the rate* of convergence of minimising sequences (depending on values of $p$) which identifies the contributions entering into the limit asymptotic models. For this reason, we choose to carry explicit dependence on $p$ in the total energy functional. Also note that, because it is the boundedness of scaled terms that implies the sharp convergence properties of displacements, the formulation via rescaled strains (20) proves to be effective in clarifying and rendering explicit the compactness of minimising sequences in (17).

## 3.1 Estimates and compactness

We start with two preliminary results frequently invoked in the reminder of the article.

**Lemma 1.** *Let* $u \in L^2(\Omega, \mathbb{R}^3)$*, with* $\Omega = \omega \times (-1, 1)$ *and* $u \in H^1(-1, 1)$ *with* $u(x', -1) = 0$ *a.e.* $x' \in \omega$*. Then there exists a constant* $C > 0$ *depending only on* $\omega$*, such that*

$$||u||^2_{L^2(\Omega)} \le C||\partial_3 u||^2_{L^2(\Omega)}$$

The next result, proved in [10, Section 4.1], allows to characterise the weak limit of the (suitably rescaled) gradient of a bounded displacement field within the nematic layer.

**Lemma 2** (Convergence of gradients)**.** *Let* $f_\varepsilon \in H^1(\Omega_b, \mathbb{R}^3)$ *for every* $\varepsilon$*. Let* $K > 0$*. Suppose* $f_\varepsilon \in L^2(\Omega_b, \mathbb{R}^3)$ *uniformly bounded in* $\varepsilon$ *and* $\varepsilon^K \|\partial_\alpha f_\varepsilon\|_{L^2(\Omega_b,\mathbb{R}^3)} \le C$*, with* $C$ *independent of* $\varepsilon$*. Then* $\varepsilon^K \partial_i f_\varepsilon \rightharpoonup 0$ *weakly in* $L^2(\Omega_b, \mathbb{R}^3)$*, for* $i = 1, 2, 3$*.*

*Proof.* See Paragraph Compactness of Section 4.1 in [10]. □

Remark, importantly, that considering admissible minimising sequences $(u^\varepsilon) \subset L^2(\Omega, \mathbb{R}^3)$ that leave the energy uniformly finite alongside Lemma 1 implies the uniform boundedness of three-dimensional displacements in $L^2(\Omega, \mathbb{R}^3)$, hence there exists a compact set of $L^2(\Omega, \mathbb{R}^3)$ such that minimising sequences are compact therein.

The two lemmas above allow to establish the following characterisation of limit strains. In what follows, we denote thickness averages by an overline (cf. notation in Section 1.1). Also, observe thanks to Jensen inequality we have, for $f \in L^2(\Omega_f)$, that $\|f\|_{L^2(\Omega_f)} \ge \|\overline{f}\|_{L^2(\omega)}$.

**Proposition 1** (Characterisation of limit strains)**.** *Consider a sequence* $u^\varepsilon \subset L^2(\Omega, \mathbb{R}^3)$ *for every* $\varepsilon$ *such that* $u^\varepsilon(\cdot, -1) = 0$ *and* $u^\varepsilon \to u$ *strongly in* $L^2(\Omega_f, \mathbb{R}^3)$ *as* $\varepsilon \to 0$ *and plug* $u^\varepsilon$ *into* $J^p_\varepsilon(u^\varepsilon)$*. Uniform boundedness* $J^p_\varepsilon(u^\varepsilon) \le C$ *implies that*

*a) there exists a limit* $\hat k \in L^2(\Omega_b, \mathbb{R}^{3\times3})$ *such that* $\hat\kappa_\varepsilon \rightharpoonup \hat k$ *in* $L^2(\Omega_b, \mathbb{R}^{3\times3})$*.*

*b) there exists* $k \in L^2(\Omega_f, \mathbb{R}^{3\times3})$ *such that* $\kappa_\varepsilon \rightharpoonup k$ *in* $L^2(\Omega_f, \mathbb{R}^{3\times3})$*, and*

$$k_{33} = -\frac{1}{2-2\nu} e_{\alpha\alpha}(u), \qquad k_{\alpha\beta} = e_{\alpha\beta}(u)$$

*c)* $e_{i3}(u^\varepsilon) \to 0$ *strongly in* $L^2(\Omega_f, \mathbb{R}^{3\times3})$*,*

*d) there exists* $e \in L^2(\Omega_f, \mathbb{R}^{3\times3})$ *such that* $e_{\alpha\beta}(u^\varepsilon) \rightharpoonup e_{\alpha\beta}$ *weakly in* $L^2(\Omega_f, \mathbb{R}^{3\times3})$*,*

*e)* $||e_{33}(u^\varepsilon)||_{L^2(\Omega_b, \mathbb{R}^{3\times3})} \leq C$*,*

*f)* $\varepsilon e_{\alpha\beta}(u^\varepsilon) \rightharpoonup 0$ *weakly in* $L^2(\Omega_b, \mathbb{R}^{2\times2})$*,*

*g)* $\varepsilon^{p+1}\partial_\alpha \bar{u}_3^\varepsilon \rightharpoonup 0$*, weakly in* $L^2(\omega)$*.*

*Proof.* To carry the proof of the items above we systematically use Jensen's inequality to obtain lower bounds upon integration of a convex function along the thickness, as in $||\bar{f}||_{L^2(\omega)} \leq ||f||_{L^2(\Omega_b)}, \forall f \in L^2(\Omega_b)$, where the overbar stands for the thickness average. Item $a$) simply follows from the uniform boundedness of $\|\hat{\kappa}^\varepsilon\|_{L^2(\Omega_b, \mathbb{R}^{3\times3})}$, and the boundedness of $Q$. Furthermore, $b$) derives from the uniform boundedness $\|\kappa^\varepsilon\|_{L^2(\Omega_f, \mathbb{R}^{3\times3})}$ by optimising with respect to the $k_{33}$ component and noticing that the convergence of minimising sequences $u^\varepsilon$ in $\Omega_f$ is actually weak $H^1(\Omega_f, \mathbb{R}^3)$, hence limit rescaled strains can be identified with scaled components of the strain in the limit. To prove $c$) observe that $b$) implies the existence of constants $C, C'$

$$\|\varepsilon^{-1} e_{\alpha3}\|_{L^2(\Omega_f, \mathbb{R}^{3\times3})} \leq C, \text{ and } \|\varepsilon^{-2} e_{33}\|_{L^2(\Omega_f, \mathbb{R}^{3\times3})} \leq C'.$$

To prove $d$) observe that $a$) implies $\|e_{\alpha\beta}(u^\varepsilon)\|_{L^2(\Omega_f, \mathbb{R}^{3\times3})} \leq C$. Then, $e$) is implied by $b$). To prove $f$) we need to use $\|\varepsilon e_{\alpha\beta}(u^\varepsilon)\|_{L^2(\Omega_b, \mathbb{R}^{3\times3})} \leq C$, uniformly in $\varepsilon$ (implied by $a$)), and then invoke Lemma 2.

To prove $g$) we first claim the following: there exist constants $C, C'$ such that

$$\varepsilon^{p+1}||\partial_\alpha \bar{u}_3^\varepsilon||_{L^2(\Omega_b)} \leq C \text{ and } \varepsilon^{-p}||\partial_3 \bar{u}_\alpha^\varepsilon||_{L^2(\Omega_b)} \leq C'. \tag{21}$$

These terms vanish in the limit energy owing to the boundedness of the gradient terms and the fact that they are multiplied by a vanishing sequence. To establish the estimate (21) it suffices to integrate the energy estimate for the shear term exploiting convexity and use $a$). Indeed,

$$||\varepsilon^{p+1}\partial_\alpha \bar{u}_3^\varepsilon + \varepsilon^{-p}\partial_3 \bar{u}_\alpha^\varepsilon||_{L^2(\omega)} \leq ||\varepsilon^{p+1}\partial_\alpha u_3^\varepsilon + \varepsilon^{-p}\partial_3 u_\alpha^\varepsilon||_{L^2(\Omega_b)} \leq C, \tag{22}$$

then use triangle inequality. By explicit integration we obtain a boundary norm whose boundedness in $H^1(\Omega(\omega))$ is ensured by the compactness of trace operator, the continuity of displacements, and their weak convergence, through the use of the trace theorem [12, Theorem 6.1-7]. We can thus write

$$\varepsilon^{p+1}||\partial_\alpha \bar{u}_3^\varepsilon||_{L^2(\omega)} \leq ||\varepsilon^{p+1}\partial_\alpha \bar{u}_3^\varepsilon + \varepsilon^{-p}\partial_3 \bar{u}_\alpha^\varepsilon||_{L^2(\omega)} + ||\varepsilon^{-p} u_\alpha^\varepsilon(x', 0)||_{L^2(\omega)} \leq C, \tag{23}$$

hence $\partial_\alpha \bar{u}_3^\varepsilon$ goes to zero weakly in $L^2(\omega)$ thanks to Lemma 2.

In (23), notice that $u_\alpha^\varepsilon(x', 0) \to u_\alpha(x', 0)$ strongly in $L^2(\omega)$ by the trace theorem and therefore $||u_\alpha^\varepsilon(x', 0)||_{L^2(\omega)}$ is uniformly bounded in $\varepsilon$. □

### 3.1.1 Kirchhoff-Love sets of displacements $KL$ and $KL^\sharp$

The structure of limit displacements is determined upon integration with respect to $z_3$ of the film relations (see $c$) in Prop. 1)

$$e_{33} = 0 \Longrightarrow \partial_3 u_3 = 0, \qquad e_{\alpha 3} = 0 \Longrightarrow \frac{\partial u_3}{\partial x_\alpha} = -\frac{\partial u_\alpha}{\partial x_3}. \tag{24}$$

The first implies that $u_3$ is a function of $x'$ only, that is $u_3(x) = \zeta_3(x')$. For such functions, the latter relations yield, upon integration in $x_3$,

$$(u'(x', x_3), u_3(x')) = (\zeta'(x') - x_3 \nabla' \zeta_3(x'), \zeta_3(x')). \tag{25}$$

These relations identify the limit space as the set of (Kirchhoff-Love) displacements

$$KL := \{v \in H^1(\Omega_f, \mathbb{R}^3) : v' = \zeta' - x_3 \nabla' \zeta_3, v_3 = \zeta_3, \\ \text{with } \zeta' \in H^1(\omega, \mathbb{R}^2), \zeta_3 \in H^2(\omega), x_3 \in (0,1)\} \tag{26}$$

which is equivalent (cf. [12]) to set of functions $u \in \mathcal{V}$ for which (24) holds. Observe then that items $c$), $d$) of Proposition 1), and Korn's inequality [12, Theorem 6.3-3] imply the weak convergence of $u_\varepsilon$ to $\overline{u} \in KL$. In the definition above, $\zeta'$ coincides with the trace of the three-dimensional displacement $u$ at interface between the two layers $\omega \times \{0\}$. By analogy, we introduce the set of *shifted* limit displacements

$$KL^\sharp := \{v \in H^1(\Omega_f, \mathbb{R}^3) : v' = \zeta'_\sharp - (x_3 - \frac{1}{2}) \nabla' \zeta_3, v_3 = \zeta_3, \\ \text{with } \zeta'_\sharp \in H^1(\omega, \mathbb{R}^2), \zeta_3 \in H^2(\omega), x_3 \in (0,1)\}, \tag{27}$$

where the functions $\zeta'_\sharp$ represent the trace of the three-dimensional displacement $u$ in correspondence to the mid-section of the film $\omega \times \{1/2\}$. Note that, from the topological and functional standpoint $KL$ coincides with $KL^\sharp$ and the functions representing in-plane displacements are related by

$$\zeta'(x') = \zeta'_\sharp(x') + \frac{1}{2} \nabla' \zeta_3(x'), \qquad \text{a.e. } x' \in \omega \tag{28}$$

## 3.2 Gamma-limits of nematic plate foundations

We now turn to the mathematical analysis and mechanical discussion of the two physically relevant material regimes, as a function of the aspect ratio represented by $p$, referred to as the 'thin nematic', for $p = 0$, and the 'thick nematic' case, for $-1 < p < 0$. This first setting leads to a full opto-elastic coupling between the nematic layer and the overlying elastic plate, whilst the second scenario involves only a partial (transverse) opto-elastic coupling. The following is the main result of this section.

**Theorem 1** (Fully coupled, thin nematic)**.** *Let* $J^0_\varepsilon(u)$ *be the energy defined in* (17)*, with* $p = 0$*. Then,*

$$J^0(u) = \Gamma\text{-}\lim_{\varepsilon \to 0} J^0_\varepsilon(u)$$

*in the strong* $L^2(\Omega_f, \mathbb{R}^3)$*-topology, where* $u \in KL$,

$$J^0(u) = \begin{cases} \frac{1}{2}\int_\omega \left( |e'(\zeta')|^2 - e'(\zeta)\nabla'\nabla'\zeta_3 + \frac{1}{3}|\nabla'\nabla'\zeta_3|^2 \right) dx' \\ \qquad +\frac{1}{2}\int_\omega \frac{2}{3-2\nu} \left( (\mathrm{tr}(\zeta'))^2 + \mathrm{tr}(\zeta')\Delta'\zeta_3 + \frac{1}{3}(\Delta'\zeta_3)^2 \right) dx' \\ +\frac{1}{2}\int_\omega \left( \mathrm{dist}^2(\mathsf{K}(\zeta',\zeta_3), \mathcal{Q}_B) + \frac{1}{1-2\nu}\zeta_3^2 \right) dx' & \text{if } (\zeta',\zeta_3) \in H^1(\omega,\mathbb{R}^2) \times H^2(\omega) \\ +\infty & \text{otherwise in} \in L^2(\omega,\mathbb{R}^3), \end{cases} \tag{29}$$

$(\zeta', \zeta_3)$ *are the in-plane and transverse displacements at the interface* $\omega \times \{0\}$ *defined in* (27),

$$\mathsf{K}(\zeta',\zeta_3) = \begin{pmatrix} 0 & 0 & \frac{1}{2}\zeta_1 \\ 0 & 0 & \frac{1}{2}\zeta_2 \\ \frac{1}{2}\zeta_1 & \frac{1}{2}\zeta_2 & \zeta_3 \end{pmatrix}, \tag{30}$$

*and*

$$\mathrm{dist}^2(\mathsf{K}(\zeta',\zeta_3), \mathcal{Q}_B) = \inf_{\bar{Q}\in\mathcal{Q}_B} |\bar{Q} - \mathsf{K}(\zeta',\zeta_3)|^2. \tag{31}$$

*Proof.* This is based on computing and matching a structural lower bound (the Γ-liminf inequality) with a suitably constructed upper bound which shows the existence of an optimal recovery sequence (Γ-limsup inequality) to $\Gamma\text{-}\lim_{\varepsilon\to 0} J^0_\varepsilon(u)$.

Imposing $p = 0$ in Proposition 2 we have the Γ-liminf inequality. Then, from Proposition 3 fixing $p = 0$, we obtain the Γ-limsup inequality and the result follows. □

**Theorem 2** (Weakly coupled, thick nematic)**.** *Let* $-1 < p < 0$ *and* $J^p_\varepsilon(u)$ *as in* (15) *and* (17) *respectively. Then,*

$$J^-(u) = \Gamma\text{-}\lim_{\varepsilon\to 0} J^p_\varepsilon(u)$$

*in the strong* $L^2(\Omega_f, \mathbb{R}^3)$*-topology, where* $u \in KL^\sharp$ *and*

$$J^-(u) = \begin{cases} \frac{1}{2}\int_\omega \left( |e'(\zeta'_\sharp)|^2 + \frac{1}{3}|\nabla'\nabla'\zeta_3|^2 + \frac{2}{3-2\nu}\left( \mathrm{tr}^2\, e'(\zeta'_\sharp) + \frac{1}{3}(\Delta'\zeta_3)^2 \right) \right) dx' \\ \qquad +\frac{1}{2}\int_\omega \left( \mathrm{dist}^2(\mathsf{K}(0,\zeta_3), \mathcal{Q}_B) + \frac{1}{1-2\nu}\zeta_3^2 \right) dx' \quad \text{if } (\zeta'_\sharp, \zeta_3), \in H^1(\omega,\mathbb{R}^2) \times H^2(\omega) \\ +\infty, \qquad \text{otherwise in } L^2(\omega,\mathbb{R}^3) \end{cases} \tag{32}$$

*where* $\mathsf{K}$ *is defined in* (30) *and* $(\zeta'_\sharp, \zeta_3)$ *are the in-plane and transverse displacements at the mid-section of the film* $\omega \times \{1/2\}$ *defined in* (27).

Theorem 2 is a consequence of Proposition 2 (Γ-liminf inequality) and of Proposition 3 (Γ-limsup inequality) for a functional defined on displacements at the mid-section of the film. The proof of Theorem 2 is postponed to Section 3.3.1.

### 3.3 Proof of Gamma-convergence theorems for $-1 < p \leq 0$

We analyse thin and thick models of nematic foundations condensing two relaxation results. Propositions 2 (lower bound) and 3 (upper bound) suffice to characterize Γ-limits for nematic foundations for $-1 < p \leq 0$ comprehensively, by characterizing the asymptotic plate regime in terms of $KL$-displacements measured at the interface between nematic and film layer. While this is precisely the requested result for coupled nematic foundations ($p = 0$), we are left with performing a final shift mapping from $KL$ to $KL^\sharp$ to represent the Γ-limit in terms of the film mid-section for plates without shear coupling ($-1 < p < 0$). This is done in Section 3.3.1.

**Proposition 2** (Lower bound inequality)**.** *Consider $J^p$ as in* (17)*, for $-1 < p \leq 0$. Then for sequences $(u^\varepsilon) \subset L^2(\Omega, \mathbb{R}^3)$ converging to $u$ strongly in $L^2(\Omega_f, \mathbb{R}^3)$ we have*

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} J^p_\varepsilon(u) \geq J^p(u), \tag{33}$$

*where*

$$J^p(u) = \begin{cases} \frac{1}{2}\int_\omega \left( |e'(\zeta')|^2 - e'(\zeta')\nabla'\nabla'\zeta_3 + \frac{1}{3}|\nabla'\nabla'\zeta_3|^2 \right) dx' \\ \qquad +\frac{1}{2}\int_\omega \frac{2}{3-2\nu}\left( \operatorname{tr}^2 e'(\zeta') - \operatorname{tr} e'(\zeta')\Delta'\zeta_3 + \frac{1}{3}(\Delta'\zeta_3)^2 \right) dx' \\ +\frac{1}{2}\int_\omega \left( \operatorname{dist}^2(\mathsf{K}((\zeta^{[p]})', \zeta_3), \mathcal{Q}_B) + \frac{1}{1-2\nu}\zeta_3^2 \right) dx', \qquad \text{if } (\zeta', \zeta_3) \in H^1(\omega, \mathbb{R}^2)\times H^2(\omega) \\ +\infty, \qquad \text{otherwise in} \in L^2(\omega, \mathbb{R}^3) \end{cases} \tag{34}$$

*where we write $(\zeta^{[p]})' = \zeta'$ if $p = 0$ and $(\zeta^{[p]})' = 0$ if $p \in (-1, 0)$.*

*Proof.* We consider a general sequence $(u^\varepsilon) \subset L^2(\Omega, \mathbb{R}^3)$ converging to $u$ in $L^2(\Omega_f, \mathbb{R}^3)$ and such that $J^0_\varepsilon(u^\varepsilon)$ is uniformly bounded in $\varepsilon$. Thanks to Proposition 1, it necessarily follows $u^\varepsilon \to u \in KL$ strongly in $L^2(\Omega_f, \mathbb{R}^3)$ and we have

$$\liminf_{\varepsilon\to 0} J^p_\varepsilon(u^\varepsilon) \geq \liminf_{\varepsilon\to 0} \Bigg\{ \frac{1}{2}\int_{\Omega_f} \left[ |e_{\alpha\beta}(u_\varepsilon)|^2 + \frac{2}{3-2\nu}(e_{\alpha\alpha}(u_\varepsilon))^2 \right] dx + \inf_{Q\in H^1(\Omega_b, \mathcal{Q}_{Fr})} \frac{1}{2}\int_{\Omega_b} \Big[ |\varepsilon e_{\alpha\beta}(u^\varepsilon) - Q_{\alpha\beta}|^2 + 2|\tfrac{1}{2}(\varepsilon^{-p}\partial_3 u^\varepsilon_\alpha + \varepsilon^{p+1}\partial_\alpha u^\varepsilon_3) - Q_{\alpha 3}|^2 + \frac{1}{1-2\nu}(\varepsilon e_{\alpha\alpha}(u^\varepsilon) + e_{33}(u^\varepsilon))^2 + (e_{33}(u^\varepsilon) - Q_{33})^2 \Big] dx \Bigg\}. \tag{35}$$

The inequality in (35) is obtained by neglecting shear terms in film, and optimising with respect to transverse component $\varepsilon_{33}$ of the strain gradient in the film layer, which implies

$$\frac{1}{\varepsilon^2} e_{33}(u^\varepsilon) = -\frac{2}{3-2\nu} e_{\alpha\alpha}(u^\varepsilon) \tag{36}$$

Integrating with respect to $x_3$ applying Jensen's inequality, we expose all averaged quantities (indicated by an overhead bar). Observe that $\varepsilon^{p+1}\nabla'\overline{u^\varepsilon_3} \rightharpoonup 0$ weakly in $L^2(\omega)$, as proved in Proposition 1-$g$) and $\varepsilon^{-p}\int_{-1}^0 \partial_3 u^\varepsilon_\alpha dx_3 \rightharpoonup 0$ weakly in $L^2(\omega)$ for $-1 < p < 0$, and $\varepsilon^{-p}\int_{-1}^0 \partial_3 u^\varepsilon_\alpha dx_3 \rightharpoonup \zeta_\alpha(x')$ weakly in $L^2(\omega)$ for $p = 0$. We obtain a further lower bound by extending the optical minimisation from $H^1(\Omega_b, \mathcal{Q}_{Fr})$ to the larger $L^2(\Omega_b, \mathcal{Q}_B)$. This leads to

$$\liminf_{\varepsilon\to 0} J^p_\varepsilon(u^\varepsilon) \geq \liminf_{\varepsilon\to 0} \Bigg\{ \frac{1}{2}\int_\omega \left[ |e_{\alpha\beta}(\overline{u}_\varepsilon)|^2 + \frac{2}{3-2\nu}(e_{\alpha\alpha}(\overline{u}_\varepsilon))^2 \right] dx + \inf_{Q\in L^2(\Omega_b, \mathcal{Q}_B)} \frac{1}{2}\int_\omega \Big[ |\overline{Q_{\alpha\beta}}|^2 + 2|\tfrac{1}{2}\zeta^{[p]}_\alpha - \overline{Q_{\alpha 3}}|^2 + \frac{1}{1-2\nu}(e_{33}(\overline{u^\varepsilon}))^2 + (e_{33}(\overline{u^\varepsilon}) - \overline{Q_{33}})^2 \Big] dx \Bigg\}. \tag{37}$$

where we use the short-hand notation $(\zeta^{[p]})' = \zeta'$ if $p = 0$ and $(\zeta^{[p]})' = 0$ if $-1 < p < 0$. Using the characterization of the set of limit displacements introduced in Paragraph 3.1.1, we read the energy in terms of the traces of displacements $(\zeta', \zeta_3)$ at the interface $\omega \times 0$.

By taking the infimum over all sequences in (35), owing to the lower semicontinuity of all convex terms we have

$$
\begin{aligned}
\Gamma\text{-}\liminf_{\varepsilon\to 0} J^p_\varepsilon(u^\varepsilon) \geq \frac{1}{2}\int_\omega \left( |e'(\zeta')|^2 - e'(\zeta')\nabla'\nabla'\zeta_3 + \frac{1}{3}|\nabla'\nabla'\zeta_3|^2 \right) dx' \\
+ \frac{1}{2}\int_\omega \frac{2}{3-2\nu}\left( \operatorname{tr}^2 e'(\zeta') - \operatorname{tr} e'(\zeta')\Delta'\zeta_3 + \frac{1}{3}(\Delta'\zeta_3)^2 \right) dx' \\
+ \inf_{\overline{Q}\in L^2(\omega,\mathcal{Q}_B)} \frac{1}{2}\int_\omega \left[ |\overline{Q}'|^2 + 2\left|\tfrac{1}{2}\zeta^{[p]}_\alpha(x') - \overline{Q_{\alpha 3}}\right|^2 + \frac{1}{1-2\nu}\zeta_3^2 + \left(\zeta_3 - \overline{Q_{33}}\right)^2 \right] dx'. \quad (38)
\end{aligned}
$$

where we remind $\zeta^{[p]}_\alpha = \zeta_\alpha$ if $p = 0$ and $\zeta^{[p]}_\alpha = 0$ if $p \in (-1, 0)$. Notice that in (38) we pass to infimum over $L^2(\omega, \mathcal{Q}_B)$ because the integrand is independent of $x_3$. Finally the claim follows because

$$
\inf_{\overline{Q}\in L^2(\omega,\mathcal{Q}_B)} \frac{1}{2}\int_\omega \left[ |\overline{Q}'|^2 + 2\left|\tfrac{1}{2}\zeta^{[p]}_\alpha(x') - \overline{Q_{\alpha 3}}\right|^2 + \left(\zeta_3 - \overline{Q_{33}}\right)^2 \right] dx' \equiv \frac{1}{2}\int_\omega \operatorname{dist}^2\left( \mathsf{K}\left( (\zeta^{[p]})', \zeta_3 \right), \mathcal{Q}_B \right) dx', \quad (39)
$$

which holds by virtue of the convexity of the set $\mathcal{Q}_B$. □

**Remark 2.** *Observe that the energy* (39) *is written in terms of the trace of displacements at the common interface $\omega \times \{0\}$, which is necessarily well defined by the limits from above (in the film) and below (in the nematic), owing to the compactness of displacements. In-plane and out-of-plane terms are coupled through cross products between the first in-plane derivatives of in-plane displacements and the second in-planes derivatives of the transverse component.*

**Proposition 3.** *[Upper bound inequality, $-1 < p \leq 0$] Let $J^p_\varepsilon$ as in* (17). *For every $u \in KL$, there exists a sequence $(v^\varepsilon) \subset L^2(\Omega, \mathbb{R}^3)$ such that $v^\varepsilon \to u = (\zeta' - x_3\nabla'\zeta_3, \zeta_3)$ strongly in $L^2(\Omega_f, \mathbb{R}^3)$ such that*

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} J^p_\varepsilon(v^\varepsilon) \leq \frac{1}{2}\int_\omega \left( |e'(\zeta')|^2 - e'(\zeta')\nabla'\nabla'\zeta_3 + \frac{1}{3}|\nabla'\nabla'\zeta_3|^2 \right) \\
+ \frac{1}{2}\int_\omega \frac{2}{3-2\nu}\left( \operatorname{tr}^2 e'(\zeta') - \operatorname{tr} e'(\zeta')\Delta'\zeta_3 + \frac{1}{3}(\Delta'\zeta_3)^2 \right) dx' + \\
\inf_{\overline{Q}\in L^2(\omega,\mathcal{Q}_B)} \frac{1}{2}\int_\omega \left( |\overline{Q}'|^2 + 2|\tfrac{1}{2}\zeta^{[p]}_\alpha - \overline{Q}_{\alpha 3}|^2 + (\zeta_3 - \overline{Q}_{33})^2 + \frac{1}{2-2\nu}\zeta_3^2 \right) dx' \quad (40)
\end{aligned}
$$

*where we write ${\zeta'}^{[p]} = \zeta'$ if $p = 0$ and ${\zeta'}^{[p]} \equiv 0$ if $p \in (-1, 0)$.*

Before showing the proof of Proposition 3, we need to introduce a collection of auxiliary results.

We decompose $\Omega_b$ into a finite partition of (columnar) grains so that $\Omega_b = \bigcup_j^m A_j$ up to a set of measure zero, and construct the recovery sequence for displacements and tensors on each individual grain. Then, glueing individual grains will be performed after showing that boundary layer error terms can be made as small as desired.

**Lemma 3.** *Let $\{A_j\}_{j=1}^m$ be a finite collection of domains of the form $\omega_j \times (-1,0)$, where $\omega_j \subset \omega$ are open and bounded sets. For any $\overline{Q} \in L^2(A_j, \mathcal{Q}_B)$ there exists*

1. *a sequence $(Q^\eta) \subset L^2(A_j, \mathcal{Q}_{Fr})$ of piecewise constant tensors parameterised by $\eta > 0$ such that $Q^\eta(x) \rightharpoonup \overline{Q}$ weakly as $\eta \to 0$ with $Q^\eta(x) \in \mathcal{Q}_{Fr}$ for every $\eta$ for a.e. $x \in A_j$;*
2. *a sequence $(Q^{\eta,\delta}) \subset C^\infty(A_j, \mathcal{Q}_{Fr})$ such that $Q^{\eta,\delta}(x) \rightharpoonup \overline{Q}$ weakly in $L^2(\Omega_b, \mathbb{R}^{3\times3})$ as $\eta, \delta \to 0$ with $\frac{\delta}{\eta} \to 0$ with $Q^{\eta,\delta}(x) \in \mathcal{Q}_{Fr}$ for every $\eta, \delta > 0$ $\forall x \in A_j$;*
3. *a compact set $B_j$ well contained in $A_j$ such that $Q^\eta$ and $Q^{\delta,\eta}$ are constant over $B_j$ and $\operatorname{meas}(A_j \setminus B_j) \leq \delta/\eta$, provided that $\eta \gg \delta > 0$;* [1]
4. *a constant $C > 0$ such that, for every (fixed) $\alpha > 0$*

$$\frac{1}{2}\int_{A_j\setminus B_j} \left|\varepsilon^\alpha \nabla' Q^{\eta,\delta}\right|^2 + \left|\partial_3 Q^{\eta,\delta}\right|^2 \leq C_j \frac{\delta\eta}{m\delta^2}\eta^{-2}; \tag{41}$$

5. *a piecewise-affine vector map $f(x) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that $Q^1(x) = \frac{\nabla f + \nabla^T f}{2}(x)$ where $Q^1$* [2] *is the periodic extension to $\mathbb{R}^3$ of the tensor $Q^\eta$ computed for $\eta = 1$.*

*Proof.* These are explicit constructions. See Appendix to [10]. □

**Remark 3.** *Lemma 3 revolves around a two-fold limiting process parameterized by $\delta, \eta$. The first limit (in $\eta \to 0$) identifies piece-wise constant maps approximating biaxial optic states which are constant with respect to the thickness by exhibiting fine scale optic textures (item 1 and 5). The second limit (in $\delta \to 0$) allows to smoothly interpolate such oscillating optic states by smooth transition occurring across small boundary layers whose thickness is related to the smallest material length scale of the system (item 2 and 3), namely the one associated to Frank's curvature energy, with explicit control on the total energy (item 4).*

*Proof of Proposition 3.* We introduce the recovery sequence

$$v^\varepsilon(x) := \begin{cases} v_f^\varepsilon(x) & \text{in } \Omega_f \\ v_b^{\varepsilon,\eta}(x) & \text{in } \Omega_b \end{cases}, \tag{42}$$

where

$$v_f^\varepsilon(x) := \begin{cases} \zeta_\alpha(x') - x_3\partial_\alpha\zeta_3(x') \\ \zeta_3(x') + \varepsilon^2 h^\varepsilon(x) \end{cases}, \quad (x', x_3) \in \omega \times (0,1), \quad \text{and}$$
$$v_b^{\varepsilon,\eta}(x) := v^\star(x) + \vartheta(x) w^{\varepsilon,\eta}(x), \quad (x', x_3) \in \omega \times (-1, 0) \tag{43}$$

where $v^\star := \zeta_i(x')(x_3{+}1)$ is the affine extension (along $x_3$) of the film displacement and $\zeta_3 \in H^2(\omega), \zeta' \in H^1(\omega, \mathbb{R}^2)$ and $h_\varepsilon$, $\vartheta$, $f^{\varepsilon,\eta}, w^{\varepsilon,\eta}$ are defined in the following sections. The recovery sequence is three dimensional and accounts for mechanical reduction and the emergence of optic textures at two different length scales $\eta, \varepsilon$ in $\Omega_b$. Displacements are continuous across the interface so that $v^\varepsilon \in H^1(\Omega, \mathbb{R}^3)$ for every $\varepsilon$. In the film, the displacement profile entails a vanishing shear, whilst the converging term $h^\varepsilon$

[1] how do i link this to $\vartheta$?
[2] this $\overline{Q}$ instead ?

is introduced to satisfy optimality between the in-plane and the out-of-plane deformations. Within the nematic bonding layer, in order to recover boundary conditions and interface continuity, a tailored microstructure is necessary to relax the optic tensor by formation of (weakly converging) sequences of micro-scale three-dimensional deformation patterns at length scale $\eta$. Finally, tight transition layers allow to smoothly accommodate these rapidly varying optic domains. For the reader's convenience we split the discussion, treating film layer and bonding layer separately.

**Γ-limsup film.** We start with the rescaled energy, defining the recovery sequence in the film

$$v_f^\varepsilon(x) := \begin{cases} \zeta_\alpha(x') - x_3\partial_\alpha\zeta_3(x') \\ \zeta_3(x') + \varepsilon^2 h^\varepsilon(x) \end{cases}, \quad \zeta_3 \in H^2(\omega), \quad \zeta_\alpha \in H^1(\omega). \tag{44}$$

Here, we choose $h^\varepsilon \in C_c^\infty(\Omega_f)$ such that $\partial_3 h_\varepsilon(x) \to \frac{-1}{2-2\nu} e_{\alpha\alpha}(u) = \frac{-1}{2-2\nu}\left(\partial_\alpha\zeta_\alpha - \partial_{\alpha\alpha}\zeta_3 x_3\right)$ strongly in $L^2(\Omega_b)$ as $\varepsilon \to 0$ in order to satisfy optimality of transverse strains. The associated strain components are

$$e_{\alpha\beta}(v^\varepsilon) = e_{\alpha\beta}(\zeta') - x_3\partial_{\alpha\beta}\zeta_3, \quad \varepsilon^{-2}e_{33}(v^\varepsilon) = \partial_3 h_\varepsilon, \quad \varepsilon^{-1}e_{\alpha 3}(v^\varepsilon) = \varepsilon\partial_\alpha h_\varepsilon, \tag{45}$$

note the cancellation in the shear term which allows to approximate vanishing shear deformations, for $\varepsilon \to 0$. Plugging into $J_\varepsilon^p$, passing to the limit using the characterization of $h_\varepsilon$, and computing the exact integral along the vertical coordinate, leads to

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{2}\int_\omega\int_0^1 &\left(|e'(\zeta')|^2 - 2x_3 e'(\zeta)\nabla'\nabla'\zeta_3 + x_3^2|\nabla'\nabla'\zeta_3|^2 + (\partial_3 h_\varepsilon)^2 + \frac{\varepsilon^2}{2}|\nabla' h_\varepsilon|^2\right) dx \\ &+ \frac{1}{2}\int_\omega\int_0^1 \frac{1}{1-2\nu}\left(\operatorname{tr} e'(\zeta') - x_3\Delta\zeta_3 + \partial_3 h_\varepsilon\right)^2 dx \\ &= \frac{1}{2}\int_\omega\left(|e'(\zeta')|^2 - e'(\zeta')\nabla'\nabla'\zeta_3 + \frac{1}{3}|\nabla'\nabla'\zeta_3|^2\right) dx' \\ &+ \frac{1}{2}\int_\omega \frac{2}{3-2\nu}\left(\operatorname{tr}^2 e'(\zeta') - \operatorname{tr} e'(\zeta')\Delta\zeta_3 + \frac{1}{3}(\Delta\zeta_3)^2\right) dx'. \end{aligned} \tag{46}$$

This gives us the asymptotic film energy, a common contribution to both the thick-nematic and thin-nematic regimes. Here the unknown is the displacement at the interface $\omega \times \{0\}$.

**Γ-limsup nematic layer.** In the active layer, the strategy consists in finding an upper bound to the two-variable energy $\tilde{J}_\varepsilon^p$ (which is turn an upper bound to $J_\varepsilon^p$). We evaluate along a recovery sequence that combines mechanical displacement oscillations with an ad-hoc optic texture of (smooth) nematic tensors, both introduced in the sequel.

We target a piecewise constant $\bar{Q} \in (\omega, \mathcal{Q}_B)$ by constructing a recovery sequence tailored to account for the dimension reduction in the elastic regime as well as for the optical relaxation thorough a martensite-like microstructure on a collection of grains. There, we approximate our relaxed target biaxial optic tensor by a weakly converging oscillating sequence. The key elements for the construction of the optic sequence draw heavily from [10] and are recalled in Lemma 3. The careful estimates of error terms and boundary layers require lengthy calculations which we omit, referring the interested reader to [10] for explicit details. We treat the regime $p \in (-1, 0)$ as a particular characterised by the decoupling of membrane deformations from bending modes.

The recovery sequence for displacements in the nematic layer can be written (cf. (43)) as follows

$$v_b^{\varepsilon,\eta}(x) = v^*(x) + \vartheta(x) w^{\varepsilon,\eta}(x) \tag{47}$$

where $v^*(x)$ is a target affine limit displacement and $\vartheta(x) \in C^\infty(A_j) : A_j \mapsto [0,1]$ is a smooth three-dimensional cutoff function which, in each grain, is used to recover homogeneous displacements at the grain boundary. We choose $\vartheta \equiv 1$ on a compact set well contained in $A_j$ at distance $\rho$ from its boundary and can always assume $|\nabla\vartheta| \leq \rho^{-1}$.

The oscillating sequence $w^{\varepsilon,\eta}$ reads

$$f^{\varepsilon,\eta} := \begin{cases} \eta f_\alpha(x'/(\varepsilon\eta), x_3/(\eta\varepsilon^{-p})) \\ \eta\varepsilon^{-p} f_3(x'/(\eta\varepsilon), x_3/(\eta\varepsilon^{-p})) \end{cases} \tag{48}$$

where $f$ is the vector field defined in Lemma 3, properly rescaled to account for the thin film scaling. Notice the difference in frequency of oscillations between in-plane and the out-of-plane displacements. We thus define

$$w^{\varepsilon,\eta} := f^{\varepsilon,\eta} - \bar{Q}\left(x'/\varepsilon, x_3/\varepsilon^{-p}\right)^T, \tag{49}$$

Observe that, by construction, $|w^{\varepsilon,\eta}| \leq \eta$ uniformly in $x$ and $\varepsilon$. Furthermore, $v^{\varepsilon,\eta}$ matches the displacement of the film at the interface $\omega \times \{0\}$ ensuring the necessary continuity. Recalling the definition of scaled strains introduced in (19), we can compute $\hat{\kappa}^\varepsilon(v^{\varepsilon,\eta})$ term by term. Scaled strains of the target displacement $v^*$ read

$$\hat{\kappa}^\varepsilon(v^*) = \begin{pmatrix} \varepsilon e'(\zeta') & \frac{1}{2}\left(\varepsilon^{p+1}(x_3+1)\partial_\alpha\zeta_3 + \varepsilon^{-p}\zeta_\alpha\right) \\ \text{sym} & \zeta_3 \end{pmatrix}. \tag{50}$$

As expected, depending on the value of $p$, either both in-plane and transverse components of displacements, or only transverse displacements contribute in the limit.

Similarly, scaled strains associated to the optic contribution read

$$\hat{\kappa}^\varepsilon(\vartheta w^{\varepsilon,\eta}) = \begin{pmatrix} \varepsilon\nabla'\vartheta \otimes_s w' & \frac{1}{2}\left(\varepsilon^{p+1}\nabla'\vartheta w_3 + \varepsilon^{-p}\partial_3\vartheta w'\right) \\ \text{sym} & \partial_3\vartheta w_3 \end{pmatrix} + \vartheta\hat{\kappa}^\varepsilon(w^{\varepsilon,\eta}) \tag{51}$$

where the last summand is equal to $\vartheta(Q^{\eta,\delta} - \bar{Q})$ by construction.

We now show that the recovery sequence just built is optimal on a generic grain $A_j$ by splitting the energy integral in a bulk and a boundary layer contribution. Indeed, considering a compact set $B_j^\rho = B_j \cap \{x : \text{dist}(x, \partial A_j) > \rho\}$ well contained in $A_j$ where $B_j$ is the set introduced in Lemma 3-3, constructed in such a way that the gap between $A_j$ and $B_j^\rho$ is of order $\rho + \delta/\eta$.

In this fashion, $B_j^\rho$ is the smallest compact set where simultaneously $\vartheta$ and $Q^{\eta,\delta}$ are constant. The gap between $A_j$ and $B_j^\rho$ is made up of $1/\eta^2$ tubular domains (whose area is $\delta\eta$) plus a region of width $\rho$ along the boundary of $A_j$ The gap between $B_j$ and $A_j$ is of the order $\eta/\delta$.

Considering the optic layer energy (15), we can now compute the grain energy contribution (that is, the localised energy on $A_j$) along the recovery sequence $(v^{\varepsilon,\eta}, Q^{\eta,\delta})$ by isolating the bulk term and estimating the residual energy of boundary layers. We write

$$J_\varepsilon(v^{\varepsilon,\eta},Q^{\eta,\delta};A_j)=\underbrace{\frac{1}{2}\int_{B_j^\rho}|\kappa(v^{\varepsilon,\eta})-Q^{\eta,\delta}|^2+\frac{1}{1-2\nu}\operatorname{tr}^2\kappa(v^{\varepsilon,\eta})dx}_{(A)}$$

$$\underbrace{\frac{1}{2}\int_{A_j\setminus B_j^\rho}\left|\kappa(v^*)+\vartheta\kappa(w^{\varepsilon,\eta})+\begin{pmatrix}\varepsilon\nabla'\vartheta\otimes_s(w^{\varepsilon,\eta})' & \frac{1}{2}\varepsilon^{p+1}\nabla'\vartheta w_3^{\varepsilon,\eta}+\varepsilon^{-p}\partial_3\vartheta(w^{\varepsilon,\eta})' \\ \text{sym} & \partial_3\vartheta w_3^{\varepsilon,\eta}\end{pmatrix}-Q^{\eta,\delta}\right|^2dx}_{(B)}$$

$$+\underbrace{\frac{1}{2}\int_{A_j\setminus B_j^\rho}\frac{1}{1-2\nu}\operatorname{tr}^2(\kappa(v^*)+\varepsilon\nabla'\vartheta\otimes_s(w^{\varepsilon,\eta})'+\partial_3\vartheta w_3^{\varepsilon,\eta}e_3\otimes e_3)dx}_{(C)}+\underbrace{\frac{1}{2}\delta_\varepsilon^2\int_{A_j\setminus B_j^\rho}\left|\varepsilon^{p+1}\nabla'Q^{\eta,\delta}\right|^2+\left|\partial_3Q^{\eta,\delta}\right|^2dx}_{(D)}.$$

Using some algebra, we obtain

$$(B)+(C)\le M\Big(\underbrace{\int_{A_j\setminus B_j^\rho}|\kappa(v^*)|^2+\operatorname{tr}^2(\kappa(v^*))dx}_{(E1)}+\underbrace{\int_{A_j\setminus B_j^\rho}\left|\vartheta\kappa(w^{\varepsilon,\eta})-Q^{\eta,\delta}\right|^2dx}_{(E2)}\Big)$$

$$+M\underbrace{\int_{A_j\setminus B_j^\rho}\left[\left|\begin{pmatrix}\varepsilon\nabla'\vartheta\otimes_s(w^{\varepsilon,\eta})' & \frac{1}{2}\varepsilon^{p+1}\nabla'\vartheta w_3^{\varepsilon,\eta}+\varepsilon^{-p}\partial_3\vartheta(w^{\varepsilon,\eta})' \\ \text{sym} & \partial_3\vartheta w_3^{\varepsilon,\eta}\end{pmatrix}\right|^2+\left(\operatorname{tr}^2(\varepsilon\nabla'\vartheta\otimes_s(w^{\varepsilon,\eta})')+(\partial_3\vartheta w_3^{\varepsilon,\eta})^2\right)\right]dx}_{(E3)}$$

where $M>0$ is a constant which may differ from line to line. First, because the integrands are bounded and the measure of domain of integration has been estimated above, we have the bound

$$(E1)+(E2)\le C\left(\rho+\frac{\delta}{\eta}\right). \tag{52}$$

Indeed, the transition region where $0\le\vartheta<1$ is of size $\rho$ while the total measure of the region where the optical tensor $Q^{\eta,\delta}$ differs from the strain tensor $\kappa(w^{\varepsilon,\eta})$ is $\delta/\eta$. For the cross term, using Schwartz' inequality and the fact that $|w^{\varepsilon,\eta}|\le\eta$, we have

$$(E3)\le C\int_{A_j\setminus B_j^\rho}|\nabla\vartheta|^2|w^{\varepsilon,\eta}|^2\le C\frac{\eta^2}{\rho}. \tag{53}$$

Finally, in light of Lemma 3-item 3, we have

$$(D)\le C\frac{\delta_\varepsilon^2}{\delta\eta}. \tag{54}$$

Using the grain estimates (52), (53), and (54), we sum over the entire partition $\{A_j\}$ to reconstruct the three-dimensional limiting energy of the active layer along the recovery sequence, namely

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} J_b^\varepsilon(v^{\varepsilon,\eta}, Q^{\eta,\delta};\Omega_b) &= \sum_{j=1}^m \limsup_{\varepsilon\to 0} J_\varepsilon(v^{\varepsilon,\eta}, Q^{\eta,\delta}; A_j)\\
&= \limsup_{\varepsilon\to 0}\frac{1}{2}\int_{B^\rho}\left(\left|\kappa^\varepsilon(v^\star)-\overline{Q}\right|^2+\frac{1}{1-2\nu}\operatorname{tr}^2(\kappa^\varepsilon(v^\star))\right)+C_1\frac{\delta_\varepsilon^2}{\delta\eta}+C_2\left(\rho+\frac{\delta}{\eta}\right)+C_3\frac{\eta^2}{\rho}\\
&\le \frac{1}{2}\int_\omega\left(|\overline{Q}'|^2+2|\tfrac{1}{2}\zeta_\alpha^{[p]}-\overline{Q}_{\alpha 3}|^2+(\zeta_3-\overline{Q}_{33})^2+\frac{1}{2-2\nu}\zeta_3^2\right)+C_4\rho, \quad (55)
\end{aligned}
$$

where $B^\rho=\cup_j B_j^\rho$ and the $C_i$s are positive constants for $i=1,...,4$. In the last line we have computed the limit as $\varepsilon\to 0$ choosing a diagonal sequence $\eta=\eta(\varepsilon)$ and $\delta=\delta(\varepsilon)$ such that $\frac{\eta(\varepsilon)}{\varepsilon}\to 0$ and $\frac{\delta(\varepsilon)}{\varepsilon}\to 0$ as $\varepsilon\to 0$ for fixed $\rho>0$ and extended the integration domain from $B^\rho$ to $\Omega_b$ owing to the non-negativity of the local (additive) energy. We finally pass to the limit two-dimensional domain $\omega$ using the columnar structure of the integration domains along the recovery sequence. Here, we use $\zeta_\alpha^{[p]}$ as a short-hand notation for $\zeta_\alpha^{[p]}\equiv\zeta_\alpha$ if $p=0$ and $\zeta_\alpha^{[p]}\equiv 0$ if $p\in(-1,0)$. Because $\rho$ is fixed and arbitrary the last contribution may be made arbitrarily small. Finally, we are able to integrate over $x_3$ and read the results separately

$$
\limsup_{\varepsilon\to 0} J_b^\varepsilon(v^{\varepsilon,\eta}, Q^{\eta,\delta};\Omega_b)\le\frac{1}{2}\int_\omega\left(|\bar{Q}'|^2+2|\bar{Q}_{\alpha 3}|^2+(\zeta_3-\bar{Q}_{33})^2+\frac{1}{2-2\nu}\zeta_3^2\right)dx', \quad \text{if } p\in(-1,0) \quad (56)
$$

and

$$
\limsup_{\varepsilon\to 0} J_b^\varepsilon(v^{\varepsilon,\eta}, Q^{\eta,\delta};\Omega_b)\le\frac{1}{2}\int_\omega\left(|\bar{Q}'|^2+2|\frac{1}{2}\zeta_\alpha-\bar{Q}_{\alpha 3}|^2+(\zeta_3-\bar{Q}_{33})^2+\frac{1}{2-2\nu}\zeta_3^2\right)dx', \quad \text{if } p=0 \quad (57)
$$

Now, replacing a general $\overline{Q}$ with the argmin $\overline{\overline{Q}}\in L^2(\omega,\mathcal{Q}_B)$ (which is unique, owing to the convexity and compactness of $\mathcal{Q}_B$) of the right-hand-side of (56) or (57), respectively, using the characterisation $\int_\omega\left(|\overline{\overline{Q'}}|^2+2|\frac{\zeta_\alpha^{[p]}}{2}-\overline{\overline{Q}}_{\alpha 3}|^2+(\zeta_3-\overline{\overline{Q}}_{33})^2\right)dx'=\int_\omega \operatorname{dist}^2(\mathsf{K}((\zeta_1^{[p]},\zeta_2^{[p]}),\zeta_3),\mathcal{Q}_B)dx'$ and summing up film and nematic layer contributions, we obtain

$$
\begin{aligned}
\Gamma\text{-}\limsup_{\varepsilon\to 0} J_\varepsilon^p(u)&\le\limsup_{\varepsilon\to 0}\left(\inf_{\overline{Q}\in L^2(\omega,\mathcal{Q}_B)} J_b^\varepsilon(v^{\varepsilon,\eta},\overline{Q};\Omega_b)+J_b^\varepsilon(v^{\varepsilon,\eta};\Omega_f)\right)\\
&\le\limsup_{\varepsilon\to 0}\left(J_b^\varepsilon(v^{\varepsilon,\eta},Q^{\eta,\delta};\Omega_b)+J_b^\varepsilon(v^{\varepsilon,\eta};\Omega_f)\right)\\
&\le\frac{1}{2}\int_\omega\left[|e'(\zeta')|^2-e'(\zeta')\nabla'\nabla'\zeta_3+\frac{1}{3}|\nabla'\nabla'\zeta_3|^2+\frac{2}{3-2\nu}\left(\operatorname{tr}^2 e'(\zeta')-e\operatorname{tr}e'(\zeta')\Delta\zeta_3+\frac{1}{3}(\Delta\zeta_3)^2\right)\right]dx'\\
&\quad+\frac{1}{2}\int_\omega\left[\operatorname{dist}^2\left(\mathsf{K}\left((\zeta^{[p]})',\zeta_3\right),\mathcal{Q}_B\right)+\frac{1}{1-2\nu}\zeta_3^2\right]dx'. \quad (58)
\end{aligned}
$$

□

### 3.3.1 Decoupled representation for shear-free plates $(-1<p<0)$

In order to read the result in the thick plate regime $(-1<p<0)$ we perform a change of variable to decouple membrane from flexural deformations. Indeed, the peculiar structure of limit KL-displacements (cf. (27)) can be further exploited in the case at hand, where the nematic foundation is active only against transverse displacements, to represent the effective energy as a function on the traces of displacements at the mid-section of the film.

*Proof of Theorem 2.* Proposition 2 (Γ-liminf inequality) and Proposition 3 (Γ-limsup inequality) show that, for $u \in KL$, we have

$$J^-(u) = \int_\omega \frac{1}{2}\left(|e'(\zeta')|^2 - e'(\zeta')\nabla'\nabla'\zeta_3 + \frac{1}{3}|\nabla'\nabla'\zeta_3|^2\right) dx' \\ + \frac{1}{2}\int_\omega \frac{2}{3-2\nu}\left(\mathrm{tr}^2\, e'(\zeta') - \mathrm{tr}\, e'(\zeta')\Delta'\zeta_3 + \frac{1}{3}(\Delta'\zeta_3)^2\right) dx' + \frac{1}{2}\int_\omega \mathrm{dist}^2\left(\mathsf{K}\left(0,\zeta_3\right), \mathcal{Q}_B\right) dx'. \quad (59)$$

To write the Γ-limit result in $KL^\sharp$ we replace

$$\zeta'(x') = \zeta'_\sharp(x') + \frac{1}{2}\nabla'\zeta_3(x'), \qquad \text{a.e. } x' \in \omega, \quad (60)$$

so that, after straightforward algebraic manipulations, (59) yields

$$J^-(u) = \frac{1}{2}\int_\omega \left(|e'(\zeta'_\sharp)|^2 + \frac{1}{12}|\nabla'\nabla'\zeta_3|^2\right) dx' \\ + \frac{1}{2}\int_\omega \frac{2}{3-2\nu}\left(\mathrm{tr}^2\, e'(\zeta'_\sharp) + \frac{1}{12}(\Delta'\zeta_3)^2\right) dx' + \frac{1}{2}\int_\omega \mathrm{dist}^2\left(\mathsf{K}\left(0,\zeta_3\right), \mathcal{Q}_B\right) dx', \quad (61)$$

for $u \in KL^\sharp$ and therefore Theorem 2 is proven. □

# 4 Actuation

In this section we analyse the asymptotic models of nematic elastomer bilayers in the thin and thick plate regimes, where the LC order tensor is assumed to be constant. Since the LC orientation (as well as order states) are frozen but can be controlled by means of external forces and boundary conditions. We refer to such a class of boundary value problems as Actuation because tuning of the order tensor $Q$ leads to spontaneous shape morphing. From the mathematical standpoint, we deal with the limit as $\varepsilon \to 0$ and $\delta \to \infty$, corresponding to the processes of structural relaxation for constant $Q$ with no optic relaxation. Additionally, we require $\delta^2\varepsilon^{p+2} \to \infty$. This corresponds to the limit regime of thin elastic foundations of small size. This way, we describe the modeling of small NLCE units which are building blocks of structures with heterogeneously patterned LC orientations, a proxy to non-isometric origami or optically active membranes [23], [22]. As a paradigm for externally-controlled shape morphing, we perform the analysis of NLCE bilayers under an external electric field.

In presence of an electric field, the complete form of energy as introduced in (15) is

$$
\begin{aligned}
J^p_\varepsilon(v,Q,\varphi) &= \tilde{J}^\varepsilon_f(v) + J^\varepsilon_b(v,Q) - J^\varepsilon_{ele}(Q,\varphi) = \\
&\frac{1}{2}\int_{\Omega_f} \left(|e_{\alpha\beta}(v)|^2 + (\varepsilon^{-2}e_{33}(v))^2 + 2|\varepsilon^{-1}e_{\alpha 3}(v)|^2\right)dx + \frac{1}{1-2\nu}\left((e_{\alpha\alpha}(v)) + \varepsilon^{-2}e_{33}(v)\right)^2 dx \\
&+ \frac{1}{2}\int_{\Omega_b}\left(|\varepsilon e_{\alpha\beta}(v) - Q_{\alpha\beta}|^2 + (e_{33}(v) - Q_{33})^2\right)dx \\
&+ \frac{1}{2}\int_{\Omega_b} 2\left|\frac{1}{2}\left(\varepsilon^{p+1}\partial_\alpha v_3 + \varepsilon^{-p}\partial_3 v_\alpha\right) - Q_{\alpha 3}\right|^2 dx + \frac{1}{1-2\nu}\left(\varepsilon e_{\alpha\alpha}(v) + e_{33}(v)\right)^2 dx \\
&+ \frac{1}{2}\int_{\Omega_b}\delta_\varepsilon^2\left(\varepsilon^{p+2}|\nabla' Q|^2 + |\partial_3 Q|^2\right)dx - \frac{1}{2}\int_{\Omega_b}(\nabla^\varepsilon\varphi)^T \mathsf{D}(Q)\nabla^\varepsilon\varphi dx, \quad (62)
\end{aligned}
$$

where we have used the concise notation $\nabla^\varepsilon(\cdot) := \left(\nabla'(\cdot), \frac{1}{\varepsilon^{p+1}}\partial_3(\cdot)\right)$ to indicate the scaled gradient of a scalar function.

**Dielectric tensor.** To characterise the dielectric tensor explicitly, we write

$$
\epsilon_0 \mathsf{D}(Q) := \epsilon_o\left(\frac{2\epsilon_\perp + \epsilon_{||}}{3}I + (\epsilon_{||} - \epsilon_\perp)Q\right). \quad (63)
$$

Constants appearing in (63) (including $\epsilon_0 > 0$) are defined in Table (1) and represent dielectric parameters of the nematic liquid crystal. The main point here is that, for every $Q \in \mathcal{Q}_X$, with $X = Fr, U$ or $B$, $\mathsf{D}(Q)$ is a symmetric positive definite matrix. Consequently, there exists a constant $C > 0$ such that

$$
\frac{1}{C}|\xi|^2 \le \xi^T \mathsf{D}(Q)\xi \le C|\xi|^2, \qquad \forall \xi \in \mathbb{R}^3. \quad (64)
$$

As a direct consequence of (64), $\varphi \mapsto -J^\varepsilon_{ele}(\cdot, Q)$ is a concave (and non-positive) functional and therefore the total energy is not bounded below. Before proceeding with the analysis of (62), we elucidate on the admissible space of electrostatic potentials we envision in our experiments.

**Remark 4.** *We assign boundary conditions for the electrostatic potential in the following manner. We define a function $\varphi_0 \in H^1(\Omega_b)$ such that $\partial_3\varphi_0 = 0$ a.e. in $\omega \times (-1,0)$. Then, define a subset $\partial_D\omega \subset \partial\omega$ with $\mathcal{H}^1(\partial_D\omega) > 0$ and define $\partial_D\Omega := \partial_D\omega \times [-1,0]$. We take $\varphi \in H^1(\Omega_b)$ equal to $\varphi_0$ on $\partial_D\Omega$ (in the sense of traces) and we say $\varphi - \varphi_0 \in H^1_D(\Omega_b)$ where*

$$
H^1_D(\Omega_b) := \{f \in H^1(\Omega_b), f = 0 \text{ on } \partial_D\Omega\}. \quad (65)
$$

For fixed $\varepsilon$ and $\delta_\varepsilon > 0$ analysis of critical points of (62) is pursued in [9]. We summarise here the result.

**Proposition 4.** *Fix $Q \in L^2(\Omega_b, \mathcal{Q}_X)$ where $X$ stands for either $Fr$, $U$ or $B$. Let $\varepsilon > 0$ and fixed. Let $\mathsf{D}(Q)$ as defined in (63) Let $\varphi_o$ as in Remark 4. First, there exists a unique solution to*

$$
\min_{\varphi \in H^1_D(\Omega_b) + \varphi_o} \int_{\Omega_b} (\nabla^\varepsilon\varphi)^T \mathsf{D}(Q)\nabla^\varepsilon\varphi dx. \quad (66)
$$

*Equivalently, the minimiser of (66) is the (unique) solution to (full, 3D) Gauss Law*

$$
-\operatorname{div}_\varepsilon\left(\mathsf{D}(Q)\nabla^\varepsilon\varphi\right) = 0 \quad \text{in } H^{-1}(\Omega_b), \quad (67)
$$

*where* $\mathrm{div}_\varepsilon = (\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{1}{\varepsilon^{p+1}}\frac{\partial}{\partial x_3})$.

*Second. Label* $\varphi_Q$ *the solution to* (66) *for the given* $Q \in L^2(\Omega_b, \mathcal{Q}_X)$. *Take a sequence* $\{Q_k\} \subset L^2(\Omega_b, \mathcal{Q}_X)$ *such that* $Q_k \to Q$ *strongly in* $L^2(\Omega_b, \mathcal{Q}_X)$. *Then,*

$$\varphi_{Q_k} \to \varphi_Q \text{ strongly in } H^1(\Omega_b), \tag{68}$$

*where* $\varphi_{Q_k}$ *is the solution to* (66) *when* $Q$ *is replaced by* $Q_k$. *Third,*

$$J^\varepsilon_{ele}(Q_k, \varphi_{Q_k}) = \frac{1}{2}\int_{\Omega_b} (\nabla^\varepsilon \varphi_{Q_k})^T \mathsf{D}(Q_k)\nabla^\varepsilon \varphi_{Q_k} dx \to \frac{1}{2}\int_{\Omega_b} (\nabla^\varepsilon \varphi_Q)^T \mathsf{D}(Q)\nabla^\varepsilon \varphi_Q dx = J^\varepsilon_{ele}(Q, \varphi_Q), \tag{69}$$

*as* $k \to \infty$.

**Remark 5.** *Precisely,* $\varphi_Q$ *is defined as an operator mapping* $L^2(\Omega_b, \mathcal{Q}_X) \mapsto H^1(\Omega_b)$. *In this sense* (68) *is a statement regarding the continuity of such operator with respect to the strong* $L^2(\Omega_b, \mathbb{R}^{3\times 3})$ *topology of order tensors. With some abuse of notation, we adopt the same symbol to indicate both the abstract operator* $\varphi_Q : L^2(\Omega_b, \mathcal{Q}_X) \to H^1(\Omega_b)$ *as well as the function obtained when mapping a fixed* $Q$ *with the mapping* $\varphi_Q$.

*Proof of Proposition 4.* It is enough to see that, for fixed $Q \in L^2(\Omega_b, \mathcal{Q}_X)$, $\varphi \to J^\varepsilon_{ele}(Q, \cdot)$ is coercive thanks to (64) and Poincaré inequality. Thanks to (63), $\varphi \to J^\varepsilon_{ele}(Q, \cdot)$ is strictly-convex and hence weakly lower semicontinuous. Therefore by applying the direct method of the calculus of variations we have (66) is indeed attained as a minimum with a unique minimiser. Characterization of the unique minimiser of (66) as the unique solution to (67) is a classical result for elliptic integrals. Then, (68) and (69) are also standard. A proof is contained in [9], Proposition 2.2. □

**Proposition 5** (Theorem 2.1, [9])**.** *Let* $J^p_\varepsilon(v, Q, \varphi)$ *as in* (62) *and* $\varphi_0$ *as in Remark* (4) *where* $\varepsilon, \delta_\varepsilon > 0$ *are fixed. Then,* $(u^*, Q^*, \varphi^*)$ *is a min-max point of* $J^p_\varepsilon(v, Q, \varphi)$ *that is*

$$J^p_\varepsilon(u^*, Q^*, \varphi^*) = \min_{u\in\mathcal{V}, Q\in H^1(\Omega_b, \mathcal{Q}_X)} \max_{\varphi\in H^1_D(\Omega_b)+\varphi_0} J^p_\varepsilon(u, Q, \varphi), \tag{70}$$

*if and only if* $(u^*, Q^*, \varphi^*)$ *is a solution to*

$$\min\left\{J^p_\varepsilon(u, Q, \varphi_Q) : \quad u \in \mathcal{V}, Q \in H^1(\Omega_b, \mathcal{Q}_X)\right\}, \tag{71}$$

*where* $\varphi_Q \in H^1_D(\Omega_b) + \varphi_0$ *solves*

$$-\mathrm{div}_\varepsilon\left(\mathsf{D}(Q)\nabla^\varepsilon \varphi\right) = 0 \quad \text{ in } H^{-1}(\Omega_b). \tag{72}$$

*Proof.* Consider (70) first, Proposition 4 shows that the maximum problem in (70) has a unique solution, for given $Q \in L^2(\Omega_b, \mathcal{Q}_X)$, denoted by $\varphi_Q$. Thanks to ellipticity (64), one has

$$\max_{\varphi\in H^1_D(\Omega_b)+\varphi_o} -J^\varepsilon_{ele}(Q, \varphi) \geq -\frac{C}{\varepsilon^{2p+2}}\|\nabla\varphi_o\|^2_{L^2(\Omega_b)}. \tag{73}$$

Thanks to continuity of $Q \to J^\varepsilon_{ele}(Q, \varphi_Q)$ in the strong $L^2(\Omega_b, \mathbb{R}^{3\times 3})$ topology (69) and the boundedness from below (73) it follows that

$$J^p_\varepsilon(u, Q, \varphi_Q) = \max_{\varphi\in H^1_D(\Omega_b)+\varphi_0} J^p_\varepsilon(u, Q, \varphi), \tag{74}$$

which is now a functional in the two variables $(u, Q)$, is coercive and lower semicontinuous in the weak-$H^1(\Omega, \mathbb{R}^3)$ topology for $u$ and weak-$H^1(\Omega_b, \mathbb{R}^{3\times 3})$ topology for $Q$. Therefore the claim follows with $\varphi^* := \varphi_{Q^*}$. To show (71) coincides with (70), observe the unique solution $\max_{\varphi \in H^1_D(\Omega_b)+\varphi_0} J^p_\varepsilon(u, Q, \varphi)$ is characterised by (72) as shown in Proposition 4, see Eqs. (66) and (67).

□

Our strategy is as follows. First, we compute the Γ-limits of the electrostatic work functional for general materials and conclude using a general stability property of Γ-convergence to the situation at hand. In computing our Γ-convergence Theorem we identify a class of dielectric materials which we call "regular" (that is, non-singular). This is the only class of materials for which we are able to compute the limiting functional. However, the regular character of the dielectric matrix is a consequence of the strong convergence of optic tensors and the continuity of the dielectric matrix, thus the regularity requirement is not restrictive in our context. We start with a lemma.

**Lemma 4.** *Let $\{\mathsf{D}_k\} \subset L^2(\Omega_b, \mathbb{R}^{3\times 3})$ with $\mathsf{D}_k$ symmetric, uniformly bounded and positive definite (that is, $\mathsf{D}_k = \mathsf{D}_k^T$ and $\frac{1}{C}|\xi|^2 \le \xi^T \mathsf{D}_k \xi \le C|\xi|^2$ for every $\xi \in \mathbb{R}^3$, for some $C > 0$) and $\mathsf{D}_k \to \mathsf{D}$ strongly in $L^2(\Omega_b, \mathbb{R}^{3\times 3})$. Let $\{f_k\} \subset L^2(\Omega_b)$ with $f_k \rightharpoonup f$ weakly in $L^2(\Omega_b, \mathbb{R}^3)$. Then*

$$\int_{\Omega_b} f^T \mathsf{D} f \, dx \le \liminf_{k\to\infty} \int_{\Omega_b} f_k^T \mathsf{D}_k f_k \, dx. \tag{75}$$

*Proof.* Observe,

$$\int_{\Omega_b} (f_k - f)^T \mathsf{D}_k (f_k - f) dx \ge \frac{1}{C} \|f_k - f\|^2_{L^2(\Omega_b, \mathbb{R}^3)} \ge 0 \quad \forall k \in \mathbb{N}. \tag{76}$$

Then, notice that, for each $\vartheta \in L^2(\Omega_b, \mathbb{R}^3)$,

$$\lim_{k\to\infty} \int_{\Omega_b} \vartheta^T \mathsf{D}_k f_k dx = \int_{\Omega_b} \vartheta^T \mathsf{D} f dx. \tag{77}$$

To prove (77) observe $|\vartheta^T \mathsf{D}_k| \le M|\vartheta|$ for some $M > 0$ and $\vartheta^T \mathsf{D}_k \to \vartheta^T \mathsf{D}$ pointwise almost everywhere in $\Omega_b$ (up to a subsequence here not relabelled) and therefore $\vartheta^T \mathsf{D}_k \to \vartheta^T \mathsf{D}$ strongly in $L^2(\Omega_b, \mathbb{R}^3)$ for the Lebesgue Dominated Convergence Theorem. Hence (77) follows. Expanding (76) we have

$$\liminf_{k\to\infty} \int_{\Omega_b} f_k^T \mathsf{D}_k f_k dx \ge \lim_{k\to\infty} 2 \int_{\Omega_b} f^T \mathsf{D}_k f_k dx - \lim_{k\to\infty} \int_{\Omega_b} f^T \mathsf{D}_k f dx. \tag{78}$$

Thanks to 77 (replacing $\vartheta$ with $f$) we have the claim. □

In Section 3 we deal with dimension reduction problems for materials with uniform elastic constants, that is, $\mathsf{E}$ and $\nu$ do not depend on $x$. A similar assumption for dielectric coefficients is restrictive in that the dielectric tensor depends linearly on the variable $Q$ which is, in general, $x$-dependent. To this end, we introduce a class of nearly homogeneous material, that is, materials whose dielectric tensor -although not constant over $\Omega_b$- lies in a neighbourhood of its average controlled by the layer thickness.

**Definition 1** (Nearly transversely homogeneous dielectric tensor)**.** *Let $\mathsf{D}_\varepsilon \subset L^\infty(\Omega_b, \mathbb{R}^{3\times 3})$ for every $\varepsilon$ and symmetric and positive definite uniformly in $\varepsilon$, that is there exists a universal constant $C > 0$ such that*

$$\frac{1}{C}|\xi|^2 \le \xi^T \mathsf{D}_\varepsilon(x)\xi \le C|\xi|^2, \qquad \forall \xi \in \mathbb{R}^3; \text{ for a.e. } x \in \Omega_b; \forall \varepsilon > 0. \tag{79}$$

*We define a nearly transversely homogeneous dielectric tensor a matrix such that*

$$\mathsf{D}_\varepsilon(x_1,x_2,x_3) = \overline{\mathsf{D}}(x_1,x_2) + z_\varepsilon \mathsf{D}^{\sim}(x_1,x_2,x_3) \tag{80}$$

*where* $\overline{\mathsf{D}}(x_1,x_2) : \int_{-1}^{0} \mathsf{D}_\varepsilon(x_1,x_2,x_3)dx_3$, $z_\varepsilon \mathsf{D}^{\sim}(x_1,x_2,x_3) := \mathsf{D}_\varepsilon(x_1,x_2,x_3) - \overline{\mathsf{D}}(x_1,x_2)$, $\|\mathsf{D}^{\sim}\|_{L^\infty(\Omega_b,\mathbb{R}^{3\times3})} \leq M$ *where* $M$ *does not depend on* $\varepsilon$ *and* $z_\varepsilon \to 0$ *when* $\varepsilon \to 0$.

Intuitively, nearly homogeneous materials defined above generalise materials with homogeneous properties in the following sense. Over a thin layer of thickness $\varepsilon$ (that is, the geometrical dimension which is asymptotically small) we admit oscillations $x_3 \to \mathsf{D}(x',\cdot)$ $\varepsilon$-close to a constant matrix. Such an assumption is designed to rule out the presence of additional length scales smaller than $\varepsilon$, in which case, on top of the dimension-reduction relaxation, further homogenisation of dielectric material constants in direction $x_3$ might occur. We will show that the behaviour of the dielectric tensor for our dimension reduction problem responds precisely to our assumption (80). Therefore, focussing on the class of nearly homogeneous materials is not a restriction. Indeed, for nematic elastomers in the actuation configuration, hypothesis (80) is shown to be a consequence of the *topology* for the admissible minimising sequences of order tensors and not a true *material* restriction.

From the functional point of view, observe that near-homogeneity is an assumption on the strong convergence of dielectric tensors in the sense that, for matrices specified in (80) we have

$$\mathsf{D}_\varepsilon(x_1,x_2,x_3) \to \overline{\mathsf{D}}(x_1,x_2) \text{ strongly in } L^2(\Omega_b) \text{ as } \varepsilon \to 0. \tag{81}$$

(and vice-versa). Importantly, the same does not hold for the weak convergence of matrices. Indeed,

$$\mathsf{D}_\varepsilon(x_1,x_2,x_3) \rightharpoonup \overline{\mathsf{D}}(x_1,x_2) \text{ weakly in } L^2(\Omega_b) \text{ as } \varepsilon \to 0 \tag{82}$$

does not imply (80).

## 4.1 Convergence of the electrostatic work for nearly transversely homogeneous dielectric tensors

**Lemma 5.** *Let* $\varphi_0$ *as in Remark 4 and* $\mathsf{D}_\varepsilon(x)$ *and* $\overline{\mathsf{D}}$ *as in Definition* (1).

$$I_\varepsilon(\varphi) := \frac{1}{2}\begin{cases} \int_{\Omega_b} (\nabla^\varepsilon\varphi)^T \mathsf{D}_\varepsilon(x)\nabla^\varepsilon\varphi dx & \text{if } \varphi \in H^1_D(\Omega_b)+\varphi_0 \\ +\infty & \text{otherwise in } L^2(\Omega_b) \end{cases} \tag{83}$$

*Then, the Gamma-limit of* $I_\varepsilon$ *in the strong* $L^2(\Omega_b)$ *topology as* $\varepsilon \to 0$ *is*

$$I_0(\overline{\varphi}) := \begin{cases} \frac{1}{2}\int_{\Omega_b} (\nabla'\overline{\varphi})^T \overline{\mathsf{B}}(x')\nabla'\overline{\varphi}\, Bx' & \text{if } \overline{\varphi} \in H^1_D(\omega)+\varphi_0 \\ +\infty & \text{otherwise in } L^2(\omega) \end{cases} \tag{84}$$

*where*

$$\overline{\mathsf{B}}(x') = \begin{pmatrix} \overline{\mathsf{d}}_{11} - \dfrac{\overline{\mathsf{d}}_{13}^2}{\overline{\mathsf{d}}_{33}} & \overline{\mathsf{d}}_{12} - \dfrac{\overline{\mathsf{d}}_{13}\,\overline{\mathsf{d}}_{23}}{\overline{\mathsf{d}}_{33}} \\ \overline{\mathsf{d}}_{12} - \dfrac{\overline{\mathsf{d}}_{13}\,\overline{\mathsf{d}}_{23}}{\overline{\mathsf{d}}_{33}} & \overline{\mathsf{d}}_{22} - \dfrac{\overline{\mathsf{d}}_{23}^2}{\overline{\mathsf{d}}_{33}} \end{pmatrix}(x') = \overline{\mathsf{D}'}(x') + \overline{\mathsf{B}}_{sh}(x'), \tag{85}$$

*and* $\overline{\mathsf{d}}_{ij} = \overline{\mathsf{d}}_{ij}(x')$ *are components of* $\overline{\mathsf{D}}(x')$, $\overline{\mathsf{D}'}(x')$ *is the top-left* $2\times2$ *submatrix of* $\overline{\mathsf{D}}(x')$ *and* $\overline{\mathsf{B}}_{sh} = -\frac{1}{\overline{\mathsf{d}}_{33}}\overline{\mathsf{d}}_{\alpha3}\overline{\mathsf{d}}_{\beta3}$.

*Proof.* We prove the statement in three steps, first, we show compactness of minimising sequences, second, we show the lower bound inequality, third we prove the upper bound inequality.

**Compactness.** Take an admissible minimising sequence $(\varphi_\varepsilon) \subset L^2(\Omega_b)$ for which uniform boundedness of the energy $I_\varepsilon(\varphi_\varepsilon) \leq C$ implies

$$\left\| \left( \nabla' \varphi_\varepsilon, \frac{1}{\varepsilon^{p+1}} \partial_3 \varphi_\varepsilon \right) \right\|^2_{L^2(\Omega_b)} \leq C; \quad \frac{1}{\varepsilon^{2p+2}} \|(\partial_3 \varphi_\varepsilon)\|^2_{L^2(\Omega_b)} \leq C', \tag{86}$$

which yields, thanks to Poincaré's inequality, that

$$\varphi_\varepsilon \rightharpoonup \overline{\varphi} \text{ weakly in } H^1(\Omega_b); \partial_3 \varphi_\varepsilon \to 0 \text{ strongly in } L^2(\Omega_b); \frac{1}{\varepsilon^{p+1}} \partial_3 \varphi_\varepsilon \rightharpoonup c \text{ weakly in } L^2(\Omega_b). \tag{87}$$

This identifies the limit space

$$H^1_D(\omega) := \{\overline{\varphi} \in H^1(\omega), \overline{\varphi} = 0 \text{ on } \partial_D \omega\}. \tag{88}$$

**Gamma-liminf inequality.** It is enough to consider sequences making the functional finite and uniformly bounded in $\varepsilon$. We write

$$\begin{aligned} C \geq \liminf_{\varepsilon \to 0} I_\varepsilon(\varphi_\varepsilon) = \liminf_{\varepsilon \to 0} \frac{1}{2} \int_{\Omega_b} (\nabla^\varepsilon \varphi)^T \, \mathsf{D}_\varepsilon(x) \nabla^\varepsilon \varphi dx \\ \geq \frac{1}{2} \int_{\Omega_b} (\nabla' \overline{\varphi}, c)^T \, \overline{\mathsf{D}}(x') \, (\nabla' \overline{\varphi}, c) \, dx \geq \frac{1}{2} \int_\omega (\nabla' \overline{\varphi}, \overline{c})^T \, \overline{\mathsf{D}}(x') \, (\nabla' \overline{\varphi}, \overline{c}) \, dx, \end{aligned} \tag{89}$$

where $\overline{\mathsf{D}}(x')$ is the average of $\mathsf{D}(x)$ over the height; $\overline{\varphi}$ and $c$ are the weak limits introduced above. We remark that the inequality above holds due to lower semicontinuity thanks to Lemma (4) because $\mathsf{D}_\varepsilon$ converges strongly to $\overline{\mathsf{D}}$ in $L^2(\Omega_b)$ according to Definition (1). The last inequality above follows from Jensen's inequality, where the only function which possibly depends on $x_3$ is $c$. Here $\overline{c}$ is the average of $c$ over $x_3$. Then,

$$\int_\omega (\nabla' \overline{\varphi}, \overline{c})^T \, \overline{\mathsf{D}}(x') \, (\nabla' \overline{\varphi}, \overline{c}) \, dx \geq \int_\omega (\nabla' \overline{\varphi}, \overline{c}^*)^T \, \overline{\mathsf{D}}(x') \, (\nabla' \overline{\varphi}, \overline{c}^*) \, dx = \int_\omega (\nabla' \overline{\varphi})^T \, \overline{\mathsf{B}}(x') \, (\nabla' \overline{\varphi}) \, dx \tag{90}$$

where

$$\overline{c}^* = - \frac{\overline{\mathsf{d}}^{13} \partial_1 \overline{\varphi} + \overline{\mathsf{d}}^{23} \partial_2 \overline{\varphi}}{\overline{\mathsf{d}}^{33}} (x') \tag{91}$$

has been obtained by pointwise minimisation of the transverse term in the integrand of (90). A substitution in the expression of the energy returns the final statement in (90).

**Gamma-limsup.** Consider a general $\widehat{\varphi} \in H^1_D(\omega) + \varphi_0$, Take $\widehat{\varphi}_{\varepsilon,\eta} = \overline{\varphi} + \varepsilon^{p+1} \overline{c}^*(x_3 + 1) * \rho_\eta$ where $\overline{c}^*$ is defined in (91). Here $\rho_\eta$ is the standard mollifier in $C^\infty_c(\omega)$. Notice that with this choice

$\varepsilon^{p+1}\overline{c}^*(x_3+1)*\rho_\eta \in C_c^\infty(\omega)\cap C^\infty(\Omega_b)$ so $\widehat{\varphi}_{\varepsilon,\eta}$ satisfies prescribed boundary conditions and $\widehat{\varphi}_{\varepsilon,\eta}\to\overline{\varphi}$ strongly in $L^2(\Omega_b)$ as $\varepsilon\to 0$, for a fixed $\eta>0$. Plugging $\widehat{\varphi}_{\varepsilon,\eta}$ into $I_\varepsilon(\cdot)$ we have

$$I_\varepsilon(\widehat{\varphi}_{\varepsilon,\eta}) = \frac{1}{2}\int_{\Omega_b} (\nabla^\varepsilon\widehat{\varphi}_{\varepsilon,\eta})^T\,\mathsf{D}_\varepsilon(x)\nabla^\varepsilon\widehat{\varphi}_{\varepsilon,\eta}dx = \frac{1}{2}\int_{\Omega_b}\left(\nabla'\overline{\varphi},\overline{c}^**\rho_\eta\right)^T\mathsf{D}_\varepsilon(x)\left(\nabla'\overline{\varphi},\overline{c}^**\rho_\eta\right)dx +$$
$$\frac{1}{2}\int_{\Omega_b} 2\left(\nabla'\overline{\varphi},\overline{c}^**\rho_\eta\right)^T\mathsf{D}_\varepsilon(x)\left(\varepsilon^{p+1}(x_3+1)\nabla'(\overline{c}^**\rho_\eta),0\right)dx$$
$$+\frac{1}{2}\int_{\Omega_b}\left(\varepsilon^{p+1}(x_3+1)\nabla'(\overline{c}^**\rho_\eta),0\right)^T\mathsf{D}_\varepsilon(x)\left(\varepsilon^{p+1}(x_3+1)\nabla'(\overline{c}^**\rho_\eta),0\right)dx. \quad (92)$$

We now discuss the three summands appearing on the right-hand side of (92). First, observe

$$\int_{\Omega_b}\left(\nabla'\overline{\varphi},\overline{c}^**\rho_\eta\right)^T\mathsf{D}_\varepsilon(x)\left(\nabla'\overline{\varphi},\overline{c}^**\rho_\eta\right)dx \to \int_\omega\left(\nabla'\overline{\varphi},\overline{c}^*\right)^T\overline{\mathsf{D}}(x')\left(\nabla'\overline{\varphi},\overline{c}^*\right)dx$$
$$= \int_\omega\left(\nabla'\overline{\varphi}\right)^T\overline{\mathsf{B}}(x')\nabla'\overline{\varphi}dx, \quad (93)$$

as both $\eta,\varepsilon\to 0$ since $\overline{c}^**\rho_\eta\to\overline{c}^*$ strongly in $L^2(\omega)$, $\mathsf{D}_\varepsilon\to\overline{\mathsf{D}}(x')$ strongly in $L^2(\Omega_b)$ with $\mathsf{D}_\varepsilon(x)$ uniformly bounded for every $\varepsilon$. Second, observe,

$$|2\int_{\Omega_b}\left(\nabla'\overline{\varphi},\overline{c}^**\rho_\eta\right)^T\mathsf{D}_\varepsilon(x)\left(\varepsilon^{p+1}(x_3+1)\nabla'(\overline{c}^**\rho_\eta),0\right)dx| \le$$
$$M\varepsilon^{p+1}\|\nabla'\overline{\varphi},\overline{c}^**\rho_\eta\|_{L^2(\omega)}\|\nabla'(\overline{c}^**\rho_\eta),0\|_{L^2(\omega)} \le M\varepsilon^{p+1}\|\overline{c}^**\nabla'\rho_\eta\|_{L^2(\omega)} \le O(\eta), \quad (94)$$

for $\varepsilon=\varepsilon(\eta)$. Finally, consider

$$\int_{\Omega_b}\left(\varepsilon^{p+1}(x_3+1)\nabla'(\overline{c}^**\rho_\eta),0\right)^T\mathsf{D}_\varepsilon(x)\left(\varepsilon^{p+1}(x_3+1)\nabla'(\overline{c}^**\rho_\eta),0\right)dx \le \varepsilon^{2p+2}M\|\nabla'(\overline{c}^**\rho_\eta)\|^2_{L^2(\omega,\mathbb{R}^2)}dx \quad (95)$$

Observe that $|\nabla'(\overline{c}^**\rho_\eta)| = |\overline{c}^**\nabla'\rho_\eta| \le M\eta^{-2}$ and therefore for fixed $\eta>0$ there exists $\varepsilon=\varepsilon(\eta)$ small enough such that

$$\int_{\Omega_b}\left(\varepsilon(\eta)^{p+1}(x_3+1)\nabla'(\overline{c}^**\rho_\eta),0\right)^T\mathsf{D}_{\varepsilon(\eta)}(x)\left(\varepsilon(\eta)^{p+1}(x_3+1)\nabla'(\overline{c}^**\rho_\eta),0\right)dx \le O(\eta). \quad (96)$$

Thus one can take the sequence $\widehat{\varphi}_{\varepsilon(\eta),\eta} = \overline{\varphi}+\varepsilon(\eta)^{p+1}\overline{c}^*(x_3+1)*\rho_\eta$ to read the result. □

**Remark 6.** *Because of the ellipticity of the three-dimensional matrix* $\mathsf{D}$*, the effective matrix* $\overline{\mathsf{B}}$ *defined by Equation* (85) *is, in particular, symmetric and positive definite.*

## 4.2 Continuity of electrostatic work

**Lemma 6.** *Let* $\varphi_0$ *as in Remark 4,* $\overline{Q}$ *constant in* $\Omega_b$ *and a sequence* $\{Q_k\}\subset H^1(\Omega_b,\mathcal{Q}_X)$ *of uniformly bounded order tensors. Define, for* $\varepsilon>0$ *and* $k\in\mathbb{N}$

$$I_{k,\varepsilon}(\varphi) := \begin{cases} \frac{1}{2}\int_{\Omega_b}\left(\nabla^\varepsilon\varphi\right)^T\mathsf{D}(Q_k)\nabla^\varepsilon\varphi dx & \text{in } H^1_D(\Omega_b)+\varphi_0 \\ +\infty & \text{otherwise in } L^2(\Omega_b)\end{cases} \quad (97)$$

*Let $Q_k \to \overline{Q}$ strongly in $L^2(\Omega_b, \mathbb{R}^{3\times 3})$ as $k \to \infty$. Then, the $\Gamma$-limit of $I_{k,\varepsilon}$ in the strong $L^2(\Omega_b)$ topology as $\varepsilon \to 0$ and $k \to \infty$ is*

$$I_{\infty,0}(\overline{\varphi}) := \begin{cases} \dfrac{1}{2}\displaystyle\int_{\Omega_b} (\nabla'\overline{\varphi})^T\, \overline{\mathsf{B}}(\overline{Q})\nabla'\overline{\varphi}dx' & \text{in } H^1_D(\omega)+\varphi_0 \\ +\infty & \text{otherwise in } L^2(\Omega_b), \end{cases} \tag{98}$$

*where*

$$\overline{\mathsf{B}}(\overline{Q}) = \begin{pmatrix} \mathsf{d}_{11}(\overline{Q}) - \dfrac{\mathsf{d}^2_{13}(\overline{Q})}{\mathsf{d}_{33}} & \mathsf{d}^2_{12}(\overline{Q}) - \dfrac{\mathsf{d}_{13}(\overline{Q})\mathsf{d}_{23}(\overline{Q})}{\mathsf{d}_{33}} \\ \mathsf{d}_{12}(\overline{Q}) - \dfrac{\mathsf{d}_{13}(\overline{Q})\mathsf{d}_{23}(\overline{Q})}{\mathsf{d}_{33}(\overline{Q})} & \mathsf{d}_{22}(\overline{Q}) - \dfrac{\mathsf{d}^2_{23}(\overline{Q})}{\mathsf{d}_{33}(\overline{Q})} \end{pmatrix}. \tag{99}$$

*Also, denoting by $\operatorname{div}_\varepsilon = (\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{1}{\varepsilon^{p+1}}\frac{\partial}{\partial x_3})$ the rescaled divergence and by $\varphi_{Q_k,\varepsilon}$ the solution to (3D) Gauss equation*

$$\varphi \in H^1_D(\Omega_b)+\varphi_0: \qquad -\operatorname{div}_\varepsilon\left(\mathsf{D}(Q_k)\nabla^\varepsilon\varphi\right) = 0 \text{ in } H^{-1}(\Omega_b), \tag{100}$$

*we have*

$$\varphi_{Q_k,\varepsilon} \to \overline{\varphi}_{\overline{Q}} \text{ strongly in } H^1(\Omega_b), \tag{101}$$

*with $\overline{\varphi}_{\overline{Q}} \in H^1(\Omega_b)$ such that $\partial_3\overline{\varphi}_{\overline{Q}} = 0$ in $(-1,0)$ (equivalently, $\overline{\varphi}_{\overline{Q}} \in H^1(\omega)$ constantly extended along $x_3$) and*

$$\frac{1}{\varepsilon^{p+1}}\partial_3\varphi_{Q_k,\varepsilon} \to -\frac{\mathsf{d}_{13}(\overline{Q})\partial_1\overline{\varphi}_{\overline{Q}} + \mathsf{d}_{23}(\overline{Q})\partial_2\overline{\varphi}_{\overline{Q}}}{\mathsf{d}_{33}(\overline{Q})} \text{ strongly in } L^2(\Omega_b), \tag{102}$$

*where and $\mathsf{d}_{ij}(Q)$ are components of the matrix $\mathsf{D}(Q)$ and $\overline{\varphi}_{\overline{Q}}$ is a solution to (2D) Gauss Law*

$$\overline{\varphi} \in H^1_D(\omega)+\varphi_0: \qquad -\operatorname{div}'(\overline{\mathsf{B}}(\overline{Q})\nabla'\overline{\varphi}) = 0 \quad \text{in } H^{-1}(\omega), \tag{103}$$

*with $\overline{\mathsf{B}}$ according to (85). Additionally,*

$$\min_{\varphi\in H^1_D(\Omega_b)+\varphi_o}\int_{\Omega_b}(\nabla^\varepsilon\varphi)^T\,\mathsf{D}(Q_k)\,(\nabla^\varepsilon\varphi)\,dx = \int_{\Omega_b}(\nabla^\varepsilon\varphi_{Q_k,\varepsilon})^T\,\mathsf{D}(Q_k)\nabla^\varepsilon\varphi_{Q_k,\varepsilon}dx$$
$$\to \int_\omega \nabla\overline{\varphi}^T_{\overline{Q}}\overline{\mathsf{B}}(\overline{Q})\nabla\overline{\varphi}_{\overline{Q}}dx' = \min_{\varphi\in H^1_D(\omega)+\varphi_o}\int_\omega \nabla'\overline{\varphi}^T\overline{\mathsf{B}}(\overline{Q})\nabla'\overline{\varphi}dx', \tag{104}$$

*as $k\to\infty$ and $\varepsilon \to 0$.*

*Proof.* For fixed $k$, there exists $\varepsilon = \varepsilon(k)$ such that $\mathsf{D}(Q_k)$ is a nearly transversely homogeneous dielectric matrix. Therefore, Theorem 4.1 applies verbatim to the convergence of (97) to (98) in the limit $k\to\infty, \varepsilon = \varepsilon(k) \to 0$. Consequently, (104) follows directly from the convergence of the minimum and minimiser of (97) to the minimum and minimiser of (98). We are left with showing (101) and (102). First, from (104) one has that

$$\int_{\Omega_b}\left(\nabla'\varphi_{Q_k,\varepsilon(k)}, \frac{1}{\varepsilon(k)^{p+1}}\partial_3\varphi_{Q_k,\varepsilon(k)}\right)^T \mathsf{D}(Q_k)\left(\nabla'\varphi_{Q_k,\varepsilon(k)}, \frac{1}{\varepsilon(k)^{p+1}}\partial_3\varphi_{Q_k,\varepsilon(k)}\right)dx, \leq C \tag{105}$$

and by equi-coercivity there exists $\overline{\varphi}^* \in H^1(\Omega_b)$ such that, as $k \to \infty$,

$$\varphi_{Q_k,\varepsilon(k)} \rightharpoonup \overline{\varphi}^* \text{ weakly in } H^1(\omega), \quad \partial_3\varphi_{Q_k,\varepsilon(k)} \to 0 \text{ strongly in } L^2(\Omega_b),$$
$$\frac{1}{\varepsilon(k)^{p+1}}\partial_3\varphi_{Q_k,\varepsilon(k)} \rightharpoonup c^* \text{ weakly in } L^2(\Omega_b) \quad (106)$$

up to a subsequence here not relabelled. Thanks to the Fundamental Theorem of Γ-convergence, such a sub-sequence converges to the minimiser $\overline{\varphi}_{\overline{Q}}$ of the right hand side of (104) in the sense specified by the first two terms in (106). This uniquely determines $\overline{\varphi}^* \equiv \overline{\varphi}_{\overline{Q}}$. We notice that, since the solution to both the $\varepsilon, k$-dependent problem and the Γ-limits are unique due to strict-convexity, the convergence is indeed recovered for the entire $k$-sequence and it is not necessary to pass to subsequences.

In order to identify $c^*$, we derive the associated Euler equations and pass to the limit, exploiting convergences established so far. Consider a generic test function $\vartheta \in H^1(\omega) + \varphi_0$. We have

$$\int_{\Omega_b} (\nabla'\varphi_{Q_k,\varepsilon(k)}, \frac{1}{\varepsilon(k)^{p+1}}\partial_3\varphi_{Q_k,\varepsilon(k)})^T \mathsf{D}(Q_k)(\nabla'\vartheta, 0)dx = 0, \quad \forall\vartheta \in H^1(\omega) + \varphi_0, \quad (107)$$

and in the limit $k \to \infty$

$$\int_{\Omega_b} (\nabla'\overline{\varphi}_{\overline{Q}}, c^*)^T \overline{\mathsf{D}}(\overline{Q})(\nabla'\vartheta, 0)dx = 0, \quad \forall\vartheta \in H^1(\omega) + \varphi_0. \quad (108)$$

(Notice we have replaced $\varphi^*$ with $\overline{\varphi}_{\overline{Q}}$ above as they are identified thanks to Γ-convergence.) Additionally, $\overline{\varphi}_{\overline{Q}}$ is such that

$$\int_{\omega} (\nabla'\overline{\varphi}_{\overline{Q}})^T \overline{\mathsf{B}}(\overline{Q})\nabla'\vartheta dx = 0, \quad \forall\vartheta \in H^1(\omega) + \varphi_0 \quad (109)$$

by minimality, and we can map the integral to $\Omega_b$ by a constant extension of its argument along $x_3$. Observe that we have the identity

$$\int_{\Omega_b} (\nabla'\overline{\varphi}_{\overline{Q}}, \tilde{c})^T \overline{\mathsf{D}}(\overline{Q})(\nabla'\vartheta, 0)dx \equiv \int_{\Omega_b} (\nabla'\overline{\varphi}_{\overline{Q}})^T \overline{\mathsf{B}}(\overline{Q})(\nabla'\vartheta)dx = 0 \quad \forall\vartheta \in H^1(\omega) + \varphi_0 \quad (110)$$

where we have written the right hand side of (102) as $\tilde{c}$. Therefore (108) and (110) coincide, and the last property in (106) follows with $\tilde{c} \equiv c^*$. Finally, to pass from the weak convergence to the strong convergence we consider again (104). Upon replacing $\mathsf{D}(Q_k)$ with $D(\overline{Q})$ in the second integral in (104) we obtain, as $k \to \infty$,

$$\int_{\Omega_b} \left(\nabla^{\varepsilon(k)}\varphi_{Q_k,\varepsilon(k)}\right)^T \mathsf{D}(\overline{Q})\nabla^{\varepsilon(k)}\varphi_{Q_k,\varepsilon(k)}dx \to \int_{\Omega_b} (\nabla\overline{\varphi}_{\overline{Q}}, c^*)^T \mathsf{D}(\overline{Q})(\nabla\overline{\varphi}_{\overline{Q}}, c^*)dx \quad (111)$$

and, in turn,

$$\int_{\Omega_b} \left(\nabla'\varphi_{Q_k,\varepsilon(k)} - \nabla\overline{\varphi}_{\overline{Q}}, \frac{1}{\varepsilon(k)^{p+1}}\partial_3\varphi_{Q_k,\varepsilon(k)} - c^*\right)^T \overline{\mathsf{D}}(\overline{Q}) \left(\nabla'\varphi_{Q_k,\varepsilon(k)} - \nabla\overline{\varphi}_{\overline{Q}}, \frac{1}{\varepsilon(k)^{p+1}}\partial_3\varphi_{Q_k,\varepsilon(k)} - c^*\right) dx \to 0.$$

By using the estimate for elliptic dielectric matrices (64) we have

$$\left\|\nabla'\varphi_{Q_k,\varepsilon(k)} - \nabla\overline{\varphi}_{\overline{Q}}, \frac{1}{\varepsilon(k)^{p+1}}\partial_3\varphi_{Q_k,\varepsilon(k)} - c^*\right\|^2_{L^2(\Omega_b)} \to 0, \quad \text{as } k \to \infty, \quad (112)$$

and (102) is proven.

□

The strong $L^2(\Omega_b, \mathbb{R}^{3\times3})$-convergence of order tensors is key in ensuring the strong $H^1$-convergence of the electrostatic potential solving Gauss equation and, in turn, the global Γ-convergence result for nematic foundations with an applied electric field. This is a problem of G-closure ([13]) of elliptic operators under strong convergence of elliptic coefficients. An outstanding open problem is the characterization of the G-closure for elliptic operators of the form $-\operatorname{div}(\mathsf{D}(Q_k)\nabla\cdot)$ under the weak $L^2(\Omega_b, \mathbb{R}^{3\times3})$-convergence of order tensors.

**Remark 7** (Opto-electric effects in bilayer structures)**.** *In the wake of relaxation induced by the dimension reduction over $x_3$, the limit system is described by an effective matrix of relaxed dielectric parameters $\overline{\mathsf{B}}$ (85). Analogoulsy as for (85), we can decompose $\overline{\mathsf{B}}(\overline{Q}) = \overline{\mathsf{D}}'(\overline{Q}) + \mathsf{B}s_{sh}(\overline{Q})$ where we have denoted by $\mathsf{D}'$ the upper-left $2\times 2$ submatrix of $\overline{\mathsf{B}}(\overline{Q})$, and by $\overline{\mathsf{B}}_{sh}$ a matrix constructed only with shear terms, namely,*

$$\overline{\mathsf{B}}' = \begin{pmatrix} \overline{\mathsf{d}}_{11} & \overline{\mathsf{d}}_{12} \\ \overline{\mathsf{d}}_{12} & \overline{\mathsf{d}}_{22} \end{pmatrix}(\overline{Q}), \qquad \overline{\mathsf{B}}_{sh} = -\frac{1}{\overline{\mathsf{d}}_{33}} \begin{pmatrix} \overline{\mathsf{d}}_{13}^{\,2} & \overline{\mathsf{d}}_{13}\overline{\mathsf{d}}_{23} \\ \overline{\mathsf{d}}_{13}\overline{\mathsf{d}}_{23} & \overline{\mathsf{d}}_{23}^{\,2} \end{pmatrix}(\overline{Q}). \tag{113}$$

*The former of the matrices is the dielectric tensor matrix that describes purely planar electric fields $\varphi = \varphi(x')$ albeit in 3D structures which cannot relax via dimension reduction. The matrix $\overline{\mathsf{B}}$ coincides with* (113) *if and only if $\overline{\mathsf{d}}_{13} = \overline{\mathsf{d}}_{23} = 0$. Recalling that the plane components of the dielectric vector are linear with respect to the $\alpha 3$ components of the order tensor, namely $\overline{\mathsf{d}}_{\alpha 3} \propto q_{\alpha 3}$, this circumstance occurs when the optical axis or order states induced by the liquid crystal molecules are either planar in the $x_1, x_2$ plane or antiplanar, i.e., parallel in the $x_3$ direction. This setting identifies a plane dielectric state, which is the electric analogue of mechanical plane stress conditions. All other states involving sheared out-of-plane states induce a relaxation of the dielectric matrix.*

## 4.3 Convergence of elastic energies with rigid $Q$

In this section we discuss the dimension reduction problem for nematic bi-layer with rigid order tensor, analysing mechanical energies in the variables $(u, Q)$. Indeed, unlike in the relaxation section 3, we do not perform an initial minimization over allowed order tensors $Q$ which is treated as an independent variable. This will allow us to discuss parametric problems which are relevant for applications (Paragraph 4.5.2). The energy functional (here $X$ stands for either $Fr, U$, or $B$) reads

$$J^p_\varepsilon(u,Q) := \begin{cases} \tilde{J}^p_\varepsilon(u,Q,0) & \text{if } (u,Q) \in \mathcal{V} \times H^1(\Omega_b, \mathcal{Q}_X) \\ +\infty & \text{otherwise in } L^2(\Omega,\mathbb{R}^3) \times H^1(\Omega,\mathbb{R}^{3\times3}). \end{cases} \tag{114}$$

where $\tilde{J}^p_\varepsilon(u,Q,0)$ reads as in (62).

We start with $p = 0$. Below and in the remainder of this section, we introduce parametrised sequences $\delta_{\varepsilon_j} \equiv \delta_j \to \infty$ and $\varepsilon_j \to 0$ (with $\delta_j^2 \varepsilon_j^{p+2} \to \infty$) indexed by $\mathbb{N} \ni j \to \infty$, adopting the short-hand notation $u_j$ instead of $u_{\delta_j,\varepsilon_j}$ and $Q_j$ instead of $Q_{\delta_j,\varepsilon_j}$.

**Theorem 3.** *Let $J^p_\varepsilon(u,Q)$ as in* (114) *where $X$ stands for either $Fr, U$ or $B$. For $p = 0$ we have*

$$\Gamma\text{-}\lim_{j\to+\infty} J^0_\varepsilon(u,Q) = J^0(u,Q) \tag{115}$$

*in the product of the strong $L^2(\Omega_f, \mathbb{R}^3)$ with the strong $H^1(\Omega_b, \mathbb{R}^{3\times3})$ topologies where*

$$J^0(\overline{u}, \overline{Q}) = \begin{cases} j^0(\zeta', \zeta_3, \overline{Q}) & \text{in } H^1(\omega,\mathbb{R}^2) \times H^2(\omega) \times \mathcal{Q}_X \\ +\infty & \text{otherwise in } L^2(\omega,\mathbb{R}^2) \times L^2(\omega) \times L^2(\omega,\mathbb{R}^{3\times3}), \end{cases} \tag{116}$$

*and the limit energy density reads*

$$j^0(\zeta', \zeta_3, Q) = \frac{1}{2}\int_\omega \left[|e_{\alpha\beta}(\zeta)|^2 - e_{\alpha\beta}(\zeta)\partial_{\alpha\beta}\zeta_3 + \frac{1}{3}|\nabla^2\zeta_3|^2 + \frac{2}{3-2\nu}\left((\partial_\alpha\zeta_\alpha)^2 + \partial_\alpha\zeta_\alpha\partial_{\beta\beta}\zeta_3 + \frac{1}{3}(\partial_{\alpha\alpha}\zeta_3)^2\right)\right] dx'$$
$$+ \frac{1}{2}\int_\omega \left[|\bar{Q}'|^2 + 2|\frac{1}{2}\zeta_\alpha - \bar{Q}_{\alpha 3}|^2 + (\zeta_3 - \bar{Q}_{33})^2 + \frac{1}{1-2\nu}\zeta_3^2\right] dx' \tag{117}$$

*Proof.* We discuss compactness of sequences, liminf inequality and limsup inequality separately.

**Compactness.** Taking sequences $(u_j, Q_j)$ such that $J^0_\varepsilon(u_j, Q_j) \le C$, where $C$ does not depend on $\varepsilon_j$ nor $\delta_j$, shows that the limit set of displacements is given by $KL$ (see Proposition 1). Furthermore, to determine the limit set for the order tensors, observe that $\delta_j^2 \int_{\Omega_b} |\nabla Q_j|^2 dx \le I(u_j, Q_j) \le C$ implies $Q_j \to \overline{Q}$ strongly in $H^1(\Omega_b, \mathcal{Q}_X)$ where $X$ stands for either $Fr, U$ or $B$ and $\overline{Q} : \Omega_b \to \mathcal{Q}_X$ is necessarily constant in $(x_1, x_2, x_3)$.

**Liminf inequality.** The proof in the film layer is identical to the proof of Proposition 2 because the physics in the film layer do not directly depend on $\delta$ nor on $Q$.

For the nematic layer the argument of Proposition 2 can be easily adapted to the present situation. Indeed, notice that

$$\liminf_{j\to\infty} \frac{1}{2}\int_{\Omega_b} (\hat{\kappa}(u_j) - Q_j)^2 + \frac{1}{1-2\nu}\operatorname{tr}^2 \hat{\kappa}(u_j) + \frac{1}{2}\delta_j^2\left(\varepsilon^{p+2}|\nabla' Q_j|^2 + |\partial_3 Q_j|^2\right) =$$
$$\liminf_{j\to\infty} \frac{1}{2}\int_{\Omega_b} (\hat{\kappa}(u_j) - \overline{Q})^2 + \frac{1}{1-2\nu}\operatorname{tr}^2 \hat{\kappa}(u_j) \tag{118}$$

To show (118) it is enough to observe $\delta_j^2 \int_{\Omega_b}\left(\varepsilon^{p+2}|\nabla' Q_j|^2 + |\partial_3 Q_j|^2\right) \to 0$ as $j \to \infty$ and use the fact that $\left|\hat{\kappa}(u_j) - Q_j \pm \overline{Q}\right|^2 = |\hat{\kappa}(u_j) - \overline{Q}|^2 + |\overline{Q} - Q_j|^2 + 2\left(\hat{\kappa}(u_j) - \overline{Q}\right)\cdot\left(\overline{Q} - Q_j\right)$. Now observe that

$$\int_{\Omega_b} \left((\overline{Q} - Q_j)^2 + \left(\hat{\kappa}(u_j) - \overline{Q}\right)\cdot(\overline{Q} - Q_j)\right) dx \to 0 \tag{119}$$

when $Q_j \to \overline{Q}$ strongly in $L^2(\Omega_b, \mathbb{R}^{3\times 3})$ and $\kappa(u_j)$ is bounded in $L^2(\Omega_b, \mathbb{R}^{3\times 3})$ as $j \to \infty$ thanks to Lebesgue Dominated Convergence Theorem. Then, reasoning along the lines of the proof of Proposition 2 we get

$$\liminf_{j\to\infty} \frac{1}{2}\int_{\Omega_b} \left[(\hat{\kappa}(u_j) - Q_j)^2 + \frac{1}{1-2\nu}\operatorname{tr}^2 \hat{\kappa}(u_j) + \delta_j^2\left(\varepsilon^{p+2}|\nabla' Q_j|^2 + |\partial_3 Q_j|^2\right)\right] dx \ge$$
$$\liminf_{j\to\infty} \frac{1}{2}\int_{\Omega_b} \left[(\hat{\kappa}(u_j) - \overline{Q})^2 + \frac{1}{1-2\nu}\operatorname{tr}^2 \hat{\kappa}(u_j)\right] dx \ge \frac{1}{2}\int_\omega \left[|\overline{Q}|^2 + 2|\frac{1}{2}\zeta_\alpha - \overline{Q}_{\alpha 3}|^2 + (\zeta_3 - \overline{Q}_{33})^2 + \frac{1}{1-2\nu}\zeta_3^2\right] dx',$$

By summing contributions in film and active layers and taking infimum over all sequences we conclude

$$\Gamma\text{-}\liminf_{j\to\infty} J^0_\varepsilon(u, Q) \ge J_0(\overline{u}, \overline{Q}),$$

where $\overline{u} = (\zeta'(x') - x_3\partial_\alpha\zeta_3(x'), \zeta_3(x'))$, as requested.

**Limsup inequality.** The proof of Proposition (3) applies here as well. It is enough to take the trivial recovery sequence $\hat{Q}_j \equiv \overline{Q}$ constant in $\Omega_b$, without the need to introduce smooth boundary layers. $\square$

Consider now $-1 < p < 0$.

**Theorem 4.** *Let $J^p_\varepsilon(u, Q)$ as in (114) with $X$ stands for either $Fr, U$ or $B$. For $-1 < p < 0$ we have*

$$\Gamma\text{-}\lim_{j\to+\infty} J^p_\varepsilon(u, Q) = J^-(\overline{u}, \overline{Q}) \tag{120}$$

*in the product of the strong $L^2(\Omega_f, \mathbb{R}^3)$ with the strong $H^1(\Omega_b, \mathbb{R}^{3\times3})$ topologies where*

$$J^-(\overline{u}, \overline{Q}) = \begin{cases} j^-(\zeta'_\sharp, \zeta_3, Q) & \text{in} \quad H^1(\omega, \mathbb{R}^2) \times H^2(\omega) \times \mathcal{Q}_X \\ +\infty & \text{otherwise in } L^2(\omega, \mathbb{R}^3) \times H^1(\omega, \mathbb{R}^{3\times3}), \end{cases} \tag{121}$$

*with*

$$j^-(\zeta'_\sharp, \zeta_3, Q) = \frac{1}{2}\int_\omega \left[ |e'(\zeta'_\sharp)|^2 + \frac{1}{3}|\nabla^2\zeta_3|^2 + \frac{2}{3-2\nu}\left(\operatorname{tr}^2 e'(\zeta'_\sharp) + \frac{1}{3}(\Delta\zeta_3)^2\right)\right] dx' + \frac{1}{2}\int_\omega \left[ |\overline{Q}'|^2 + 2\left|\overline{Q_{\alpha 3}}\right|^2 + \frac{1}{1-2\nu}\zeta_3^2 + \left(\zeta_3 - \overline{Q_{33}}\right)^2\right] dx'. \tag{122}$$

*Proof.* Follows as in Proof of Theorem (3) with obvious modifications. $\square$

## 4.4 Convergence of mechanical energy and electrostatic work

Finally, we are in a position to discuss the global Gamma-convergence of the total energy of the system composed of elastic bending energy of the tensor $Q$, the bulk mechanical energy in the nematic layer ($J^p_b(v, Q)$), the mechanical bulk energy in film layer ($J^\varepsilon_f(v)$) and the electrostatic work stemming from an external source ($J^\varepsilon_{ele}(Q, \varphi)$). The full asymptotic result follows readily by combining the Gamma-convergence results of Theorems 3 and 4 for the mechanical energies only and by noticing that the electrostatic work is a continuous perturbation (in the sense specified by Lemma 6) of the total energy in the prescribed topology.

For the sake of conciseness we present in detail the results for the fully coupled scenario, that of thin nematics ($p = 0$), and discuss the thick nematic ($-1 < p < 0$) case, which requires simple modifications, at the end of the section.

Consider $J^p_\varepsilon$ as in (62). We define the functional

$$\mathcal{E}^p_\varepsilon(u, Q) = \begin{cases} \max\limits_{\varphi\in H^1_D(\Omega)+\varphi_0} J^p_\varepsilon(u, Q, \varphi) & \text{in } KL \times H^1(\Omega_b, \mathcal{Q}_X) \\ +\infty & \text{otherwise in } L^2(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathbb{R}^{3\times3}), \end{cases} \tag{123}$$

for which the following holds.

**Theorem 5.** *Let $\mathcal{E}^p_\varepsilon(u, Q)$ as in (123) and $J^p_\varepsilon, J^p, j^p$ as in Theorem 3 for $p = 0$. We have*

$$\Gamma\text{-}\lim_{j\to+\infty} \mathcal{E}^p_{\varepsilon_j}(u, Q) = \mathcal{E}^p(\overline{u}, \overline{Q}) \tag{124}$$

*in the strong-$L^2(\Omega_f,\mathbb{R}^3)\times$strong-$H^1(\Omega_b,\mathbb{R}^{3\times3})$ topology, where*

$$\mathcal{E}^p(\overline{u},\overline{Q}) = \begin{cases} \max\limits_{\overline{\varphi}\in H^1_D(\omega)+\varphi_0} \widetilde{\mathcal{E}}^p(\zeta',\zeta_3,\overline{Q},\overline{\varphi}) & \text{on } H^1(\omega,\mathbb{R}^2)\times H^2(\omega)\times\mathcal{Q}_X \\ +\infty & \text{otherwise in } L^2(\omega,\mathbb{R}^3)\times H^1(\omega,\mathbb{R}^{3\times3}), \end{cases} \tag{125}$$

*with*

$$\widetilde{\mathcal{E}}^p(\zeta',\zeta_3,\overline{Q},\overline{\varphi}) = j^p(\zeta',\zeta_3) - \frac{1}{2}\int_\omega \nabla'\overline{\varphi}^T\overline{\mathsf{B}}(\overline{Q})\nabla'\overline{\varphi}dx \tag{126}$$

*where $j^p$ is given in* (117) *for $p = 0$ and* (122) *for $-1 < p < 0$, $\overline{\mathsf{B}}$ is as in* (99)*, and $\overline{u} = (\zeta'(x') - x_3\partial_\alpha\zeta_3(x'),\zeta_3(x'))$.*

*Proof.* This follows from a direct application of Theorem 6 and Theorem 3. Observe

$$\mathcal{E}^p_\varepsilon(u,Q) = \left[J^p_\varepsilon(u,Q) - \inf_{\varphi\in H^1_D(\Omega_b)+\varphi_0}\frac{1}{2}\int_{\Omega_b}\nabla\varphi^T\mathsf{D}(Q)\nabla\varphi dx\right].$$

In light of Theorem 6, $Q \to max_{\varphi\in H^1_D(\Omega)+\varphi_0}[-\int_{\Omega_b}\nabla\varphi^T\mathsf{D}(Q)\nabla\varphi dx]$ is a continuous perturbation of $J^0_\varepsilon$ in the strong $L^2(\Omega_b,\mathbb{R}^{3\times3})$ topology and there follows

$$\max_{\varphi\in H^1_D(\Omega)+\varphi_0}\left[-\int_{\Omega_b}\nabla\varphi^T\mathsf{D}(Q_k)\nabla\varphi dx\right] \to \max_{\varphi\in H^1_D(\omega)+\varphi_0}\left[-\int_{\Omega_b}\nabla\overline{\varphi}^T\overline{\mathsf{D}}(\overline{Q})\nabla\overline{\varphi}dx'\right], \tag{127}$$

as $Q_k \to \overline{Q}$ strongly in $L^2(\Omega_b,\mathbb{R}^{3\times3})$ for $Q_k \in \mathcal{Q}_X$, $\forall k\in\mathbb{N}$ and $\overline{Q}:\omega\to\mathcal{Q}_X$ is constant. Therefore the claim follows by a standard property of Γ-convergence ensuring stability with respect to continuous perturbations [13].

□

## 4.5 Physical implications

We prove convergence of minima and minimisers of $\mathcal{E}^p_\varepsilon(Q,u)$. This follows from equicoercivity. We show this property follows easily. Let $Q\in H^1(\Omega,\mathcal{Q}_X)$. By minimality and (63) we have

$$\inf_{\varphi\in H^1_D(\Omega_b)+\varphi_0}\int_{\Omega_b}\nabla\varphi^T\mathsf{D}(Q)\nabla\varphi dx \le M\int_{\Omega_b}|\nabla\varphi_0|^2dx = 2C, \tag{128}$$

for some $M > 0$. Now we can write, for every $(u,Q)\in H^1(\Omega,\mathbb{R}^3)\times H^1(\Omega_b,\mathcal{Q}_X)$, where $X$ stands either for $Fr, U$ or $B$,

$$\mathcal{E}^p_\varepsilon(u,Q) = \max_{\varphi\in H^1_D}\tilde{J}^p_\varepsilon(u,Q,\varphi) \ge \tilde{J}^p_\varepsilon(u,Q,\varphi_0) = J^p_\varepsilon(u,Q) - C, \tag{129}$$

(notice constants appearing in 128 and (129) are equal) and hence equicoercivity is obtained in $H^1(\Omega,\mathcal{Q}_X)\times H^1(\Omega,\mathbb{R}^3)$ by applying Korn's and Poincaré inequality and considering that $\mathcal{Q}_X$ is a bounded set. As a direct consequence, we obtain the standard result (see [13])

**Theorem 6** (Fundamental theorem of Γ-convergence)**.** *Consider $\mathcal{E}^p_\varepsilon$ and $\mathcal{E}^p$ as defined in Theorem 5 where $p$ stands for 0. Then:*

$$\min\mathcal{E}^p = \lim_{j\to+\infty}\left(\min\mathcal{E}^p_{\varepsilon_j}\right) \qquad \text{(convergence of minima).}$$

*Then, let* $\{u_j, Q_j\} \subset L^2(\Omega, \mathbb{R}^3) \times H^1(\Omega_b, \mathbb{R}^{3\times3})$ *be a minimising sequence for* $\mathcal{E}^p_\varepsilon$ *(i.e.* $\lim_{j\to\infty} \mathcal{E}^p_{\varepsilon_j}(Q_j, u_j) = \lim_j \inf \mathcal{E}^p_{\varepsilon_j}$*). Then, up to a subsequence (not relabeled),* $u_j \rightharpoonup \overline{u}$*,* $Q_j \to \overline{Q}$ *with* $\overline{u} = (\zeta'(x') - x_3\partial_\alpha\zeta_3(x'), \zeta_3(x'))$*;* $\zeta' \in H^1(\omega, \mathbb{R}^2)$*,* $\zeta_3 \in H^2(\omega)$*, with* $\overline{Q} \in \mathcal{Q}_X$ *and constant where*

$$\mathcal{E}^p(\overline{Q}, \overline{u}) = \min \mathcal{E}^p \qquad \text{(convergence of minimum points).} \tag{130}$$

### 4.5.1 Convergence of saddle-points

Corollary 5 implies convergence for the min-max problem. We make this explicit.

**Theorem 7** (Convergence of min-max problems)**.** *Consider* $\mathcal{E}^p_\varepsilon$ *and* $\mathcal{E}^p$ *as defined in Theorem 5 with* $p = 0$*. Then we have (here* $X$ *stands either for* $Fr, U$ *or* $B$*).*

*1.* (Convergence of min-max values)

$$\min\left(\max_{\varphi\in H^1_D(\omega)+\varphi_0} \mathcal{E}^p(\overline{u}, \overline{Q}, \overline{\varphi})\right) = \lim_{j\to\infty}\left(\inf_{(u,Q)\in\mathcal{V}\times H^1(\Omega_b,\mathcal{Q}_X)} \max_{\varphi\in H^1_D(\Omega_b)+\varphi_0} J^p_\varepsilon(u, Q, \varphi)\right).$$

*This is equivalent to*

$$\min\left(\mathcal{E}^p(\overline{u}, \overline{Q}, \overline{\varphi}) \text{ sub 2D Gauss Law (103)}\right) \tag{131}$$

$$= \lim_{j\to+\infty}\left(\inf_{(u,Q)\in\mathcal{V}\times H^1(\Omega_b,\mathcal{Q}_X)} \left\{J^p_\varepsilon(u, Q, \varphi) \text{ sub 3D Gauss Law (100)}\right\}\right).$$

*Denote by* $\varphi_Q$ *the solution to the 3D Gauss equation* (100) *for some* $Q \in H^1(\Omega_b, \mathcal{Q}_X)$*. Let* $\{u_j, Q_j, \varphi_{Q_j}\} \subset \mathcal{V} \times H^1(\Omega_b, \mathcal{Q}_X) \times H^1_D(\Omega_b) + \varphi_0$ *be a min-maximising sequence for* $\{J^p_{\varepsilon_j}\}$*, i.e.*

$$\lim_{j\to+\infty} J^p_\varepsilon\big(Q_j, u_j, \varphi_{Q_j}\big) = \lim_{j\to+\infty} \inf_{(u,Q)\in\mathcal{V}\times H^1(\Omega_b,\mathcal{Q}_X)} \max_{\varphi\in H^1_D(\Omega_b)+\varphi_0} J^p_\varepsilon(Q, u, \varphi),$$

*or, equivalently,*

$$\lim_{j\to+\infty} J^p_\varepsilon\big(Q_j, u_j, \varphi_{Q_j}\big) = \lim_{j\to+\infty} \inf_{(u,Q)\in\mathcal{V}\times H^1(\Omega_b,\mathcal{Q}_X)} \Big\{J^p_\varepsilon\big(Q, u, \varphi\big) \text{ sub 3D Gauss Law (100)}\Big\}.$$

*Then, up to a subsequence (not relabelled),* $u_j \rightharpoonup u^* \in KL$ *weakly in* $H^1(\Omega_f, \mathbb{R}^3)$ *with* $u^* = ((\zeta^*)' - x_3\partial_\alpha\zeta^*_3, \zeta^*_3)$*;* $(\zeta^*)' \in H^1(\omega, \mathbb{R}^2)$*,* $\zeta^*_3 \in H^2(\omega)$*;* $Q_j \to \overline{Q}^*$ *strongly in* $H^1(\Omega_b, \mathbb{R}^{3\times3})$ *with* $\overline{Q}^* \in \mathcal{Q}_X$ *constant; and* $\varphi_{Q_j} \to \overline{\varphi}_{\overline{Q}^*}$ *strongly in* $H^1(\Omega_b)$*, as* $j \to \infty$*, and:*

*2.* (Convergence of min-max points)

$$\mathcal{E}^p(\overline{Q}^*, \overline{u}^*, \varphi_{\overline{Q}^*}) = \min\left(\max_{\overline{\varphi}\in H^1_D(\omega)+\varphi_0} \mathcal{E}^p(\overline{Q}, \overline{u}, \overline{\varphi})\right)$$

or, equivalently,

$$\mathcal{E}^p(\overline{Q}^*, \overline{u}^*, \overline{\varphi}_{\overline{Q}^*}) = \min\left(\mathcal{E}^p(\overline{Q}, \overline{u}, \overline{\varphi}) \text{ sub 2D Gauss Law (103)}\right).$$

*Proof.* The results above follow from Theorem 6 in light of Proposition 5. □

As a consequence of convergence of saddle points, we infer that the saddle structure is preserved in the limit problem, thus equilibrium in the limit system is given by min-max points

**Remark 8.** *Theorems* (6) *and* (7) *have an analogue for the regime* $-1 < p < 0$ *(not displayed here). The results for* $-1 < p < 0$ *follow with a change of variables.*

#### 4.5.2 Application to the mechanical actuation of the director

Results of the previous section still apply when minimisation over the pair $(u, Q)$ is replaced with a *parametric* minimisation over the displacement $u$ only and for a given matrix $Q \in \mathcal{Q}_X$. The following lemma describes the situation where the order tensor $Q$ is *frozen*, that is, is considered as imposed by an external field (not necessarily electric) and not subject to minimisation. This problem corresponds to determining the spontaneous deformation and shape change of bilayer structures when the liquid crystal order tensor is regarded as a load parameter. The purpose is to highlight two mechanisms. When the tensor describes perfect alignment of liquid crystal molecules with a distinguished optical axis, that is $Q \in \mathcal{Q}_{Fr}$, minimisation represents the controlled shape change of a bilayer driven by collective reorientation of molecules. Contrarily, in conceptual experiment where the $Q$ tensor is taken in the set $Q_U$ or $Q_B$, low order sates, such as optical isotropy, biaxial states, and order melting are admissible. In this case falls the thermal actuation of plates, when one controls separately the director and optical axis (represented by the eigenframe of the tensor $Q \in \mathcal{Q}_U$) and the degree of order of nematic molecules (represented by the eigenvalues of $Q \in \mathcal{Q}_U$), see [23, 22],

**Theorem 8.** *Let* $J^0(u, Q)$ *as in Theorem* (3)*. Fix* $\overline{Q} \in H^1(\omega, \mathcal{Q}_X)$*, where* $X$ *stands for either* $Fr, U$ *or* $B$*. Then there exists a unique solution to*

$$\min_{L^2(\omega,\mathbb{R}^3)} J^0(\zeta', \zeta_3, \cdot). \tag{132}$$

*Proof.* This follows from an application of the direct method in the calculus of variations. Taking a minimising sequence $(\zeta', \zeta_3)_k \in H^1(\omega, \mathbb{R}^2) \times H^2(\omega)$ for every $k \in \mathbb{N}$ such that $J^0((\zeta', \zeta_3)_k, \cdot) \le C$, it follows that

$$|e_{\alpha\beta}(\zeta)_k|^2_{L^2(\omega)} + |\nabla^2(\zeta_3)_k|^2_{L^2(\omega)} + |(\zeta_\alpha)_k|^2_{L^2(\omega)} + |(\zeta_3)_k|^2_{L^2(\omega)} \le C, \quad \forall k \in \mathbb{N}.$$

By invoking Poincaré and Korn inequalities, along the transverse direction and for the in-plane symmetrised gradient respectively, we have

$$(\zeta_\alpha)_k \rightharpoonup (\zeta_\alpha)^*, \text{ weakly in } H^1(\omega, \mathbb{R}^2), \quad (\zeta_3)_k \rightharpoonup (\zeta_3)^*, \text{ weakly in } H^2(\omega),$$

for some $(\zeta_\alpha, \zeta_3)^* \in H^1(\omega, \mathbb{R}^2) \times H^2(\omega)$. Then, by convexity, $J^0(\zeta', \zeta_3, \cdot)$ is weakly lower semicontinuous and therefore the claim follows. □

**Remark 9.** *Theorem 8 has an analogue for the regime* $-1 < p < 0$*, which is a consequence of Theorem* (4)*, whose proof follows with obvious modifications.*

**Numerical example of Fig. 2.** To illustrate the purpose of the analysis so far performed, we have presented in Figure 2 a numerical actuation experiment for a thin nematic bilayer membrane which exemplifies a nontrivial solution of a complex actuation mechanism performed on the basis of simple ingredients. We are interested in inducing out-of-plane displacements via nematic actuation, and, through coupling between membrane deformations and bending modes, possibly exert work. Consider the square domain $\omega = (0,1)^2$ clamped a the boundaries and subject to an imposed (frozen) director $Q_0 = n_0 \otimes n_0 - \frac{1}{3} I_3$ where $n_0 = (e_1 + e_3)/\sqrt{2}$, as displayed in the cartoon in Figure 2.left. Our computation refers to the fully coupled model of Theorem 3, where nematic actuation directly activates a spontaneous in-plane stretch and transverse displacements. The (unique) equilibrium configuration, cf. Theorem 8, displays a non-symmetric bending mode coupled to planar membrane deformations, in competition with homogeneous Dirichlet-type boundary conditions on $\partial\omega$. The spontaneous stretch is triggered by the strong opto-elastic strain coupling which characterises the nematic active layer in the actuation regime.

In Figure 2-right we plot the value of (the norm of) in-plane displacements $|\zeta'|$, in the deformed configuration, with a discrete colour coding for readability. Note that the explicit coupling between the in-plane and out-of-plane deformation is due to the cross term in Equation (117), resulting in an out-of-plane deflection above the reference $z = 0$ surface. In addition, because the actuator field is tilted with respect to the azimuthal axis, both shear and vertical terms of the active foundation are effective.

The numerical solution has been obtained by finite elements discretisation in the FEniCS environment [20] using PETSc as data management and linear algebra package [2, 3].

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