Spherical normal forms for germs of parabolic line biholomorphisms

Loïc Teyssier

SPHERICAL NORMAL FORMS FOR GERMS OF PARABOLIC LINE BIHOLOMORPHISMS

LOÏC TEYSSIER

ABSTRACT. We address the inverse problem for holomorphic germs of a mapping of the complex line near a fixed point which is tangent to the identity. We provide a preferred parabolic map Δ realizing a given Birkhoff–Écalle-Voronin modulus ψ and prove its uniqueness in the functional class we introduce. The germ is the time-1 map of a Gevrey formal vector field admitting meromorphic sums on a pair of infinite sectors covering the Riemann sphere. For that reason, the analytic continuation of Δ is a multivalued map admitting finitely many branch points with finite monodromy. In particular Δ is holomorphic and injective on an open slit sphere containing 0 (the initial fixed point) and ∞, where is situated the companion parabolic point under the involution $\frac{-1}{\mathrm{Id}}$. One finds that the Birkhoff–Écalle-Voronin modulus of the parabolic germ at ∞ is the inverse $\psi^{\circ -1}$ of that at 0.

Acknowledgment. I would like to express my greatest gratitude to my skillful colleague R. SCHÄFKE, who introduced me to the joy of holomorphic fixed-point methods in dynamics.

The author is partially supported by the bilateral Hubert-Curien Cogito grant 2021-22.

1. INTRODUCTION

The classification of holomorphic univariate parabolic germs up to local change of analytic coordinate was first described by G. BIRKHOFF [Bir39] in 1939, but the manuscript has somehow been forgotten almost immediately, to be rediscovered only by the end of the century. In the meantime, J. ÉCALLE [Éca75] and S. VORONIN [Vor81] carried out this task again nearly forty years ago. In addition to building a modulus (characterizing the conjugacy class of a parabolic germ) they were able to solve the inverse problem by recognizing which similar objects came as the moduli of parabolic germs, a property G. BIRKHOFF was only able to conjecture. The latter problem is what interests us in the present paper.

1.1. **Modulus of classification and inverse problem.**

Heuristically, if Δ is a germ which is tangent to the identity and fixes $0 \in \mathbb{C}$,

$$\Delta(z) = z + \mathrm{o}(z) \in \mathrm{Diff}(\mathbb{C}, 0)_{\mathrm{Id}} \setminus \{\mathrm{Id}\},$$

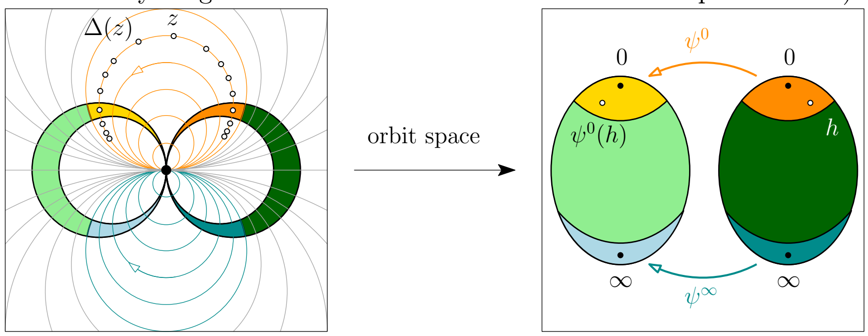
then the conformal structure of its orbit space Ω completely encodes its analytic class up to local changes of analytic coordinate near the fixed point. For the sake

Date: October 2022.
2000 *Mathematics Subject Classification.* 37F45, 37F75, 34M35, 34M50.
Key words and phrases. parabolic fixed-point, normal forms, inverse problem, parabolic renormalization.

of simplicity assume that Δ is generic, *i.e.* that it belongs to the space

$$\mathrm{Parab}_1 := \left\{z + *z^2 + \mathrm{o}\left(z^2\right)\right\} \subset \mathrm{Diff}\left(\mathbb{C}, 0\right)_{\mathrm{Id}}$$

(here and in everything which follows $*$ stands for a nonzero complex number).



It is well-known that a fundamental region for the iterative action of Δ is given by a pair of crescent-shaped regions which are identified by the dynamics near their horns, attached at the fixed point 0. More precisely, the orbit space

$$\Omega = {}^{(\mathbb{C},0)}/_{\Delta}$$

is obtained as the gluing of two spheres $\overline{\mathbb{C}}$ near 0 and ∞ by germs of a diffeomorphism $\psi^0 \in \mathrm{Diff}\left(\overline{\mathbb{C}}, 0\right)$ and $\psi^\infty \in \mathrm{Diff}\left(\overline{\mathbb{C}}, \infty\right)$, sometimes called horn maps:

$$\Omega = {}^{\overline{\mathbb{C}} \sqcup \overline{\mathbb{C}}}/_{\left(\psi^0, \psi^\infty\right)}.$$

When we endow the space ${}^{(\mathbb{C},0)}/_{\Delta}$ with the quotient topology, the two points 0 and ∞ (corresponding to the fixed point of Δ) are not separated.

One can choose conformal coordinates on the spheres so that ψ^∞ itself becomes tangent to the identity. The only degree of freedom that remains is the linear change of coordinates $h \mapsto ch$ for $c \in \mathbb{C}^\times$ applied simultaneously to both spheres. The Birkhoff–Écalle–Voronin theorem (see aforementioned bibliography) states that the map

$$\begin{aligned} \mathrm{B\acute{E}V} \ : \ {}^{\mathrm{Parab}_1}/_{\mathrm{Diff}\,(\mathbb{C},0)} &\longrightarrow {}^{\mathrm{Diff}\,(\mathbb{C},0) \times \mathrm{Diff}\,(\mathbb{C},0)_{\mathrm{Id}}}/_{\mathbb{C}^\times} \\ [\Delta] &\longmapsto \left[\left(\psi^0, \psi^\infty\right)\right] \end{aligned}$$

is well-defined and one-to-one. The Écalle–Voronin theorem states that it is also onto, answering the inverse problem.

Problem. Being given a pair $\left(\psi^0, \psi^\infty\right) \in \mathrm{Diff}\,(\mathbb{C}, 0) \times \mathrm{Diff}\,(\mathbb{C}, 0)_{\mathrm{Id}}$, to recover (synthesize) a parabolic germ Δ such that its modulus is $\mathrm{B\acute{E}V}\,(\Delta) = \left(\psi^0, \psi^\infty\right)$.

In the present article we revisit the question and provide an essentially unique representative Δ for a given pair $\left(\psi^0, \psi^\infty\right)$, what can be called a normal form. The germ Δ will be "global" on the sphere $\overline{\mathbb{C}}$ in some sense, which is why we borrow Écalle's terminology and speak about *spherical normal forms*.

1.2. **Statement of the main results.** We answer the inverse problem in the following fashion.

Synthesis Theorem. *For fixed $\mu \in \mathbb{C}$ and positive $\lambda > 0$ define the rational vector field (holomorphic near 0 and ∞), called here the **formal model**,*

$$X_0(z) := \frac{1-z^2}{1+z^2} \times \frac{\lambda z^2}{1+\mu\lambda z - z^2} \frac{\partial}{\partial z}$$

and denote by $f \mapsto X_0 \cdot f$ the associated Lie (directional) derivative on power series. Let $\psi = \left(\psi^0, \psi^\infty\right) \in \mathrm{Diff}(\mathbb{C},0) \times \mathrm{Diff}(\mathbb{C},0)_{\mathrm{Id}}$ be given and pick μ so that $\frac{\mathrm{d}\psi^0(h)}{\mathrm{d}h}(0) = \exp\left(4\pi^2\mu\right)$.

One can then find an explicit $\lambda(\psi) > 0$ such that for each $0 < \lambda \leqslant \lambda(\psi)$ there exists a unique formal power series $F \in z\mathbb{C}[[z]]$, depending on λ and satisfying all of the following properties.

(1) *The time-1 map Δ of the formal vector field, said to be in **spherical normal form**,*

$$X_F := \frac{1}{1 + X_0 \cdot F} X_0$$

*is a convergent power series at 0 (*i.e.* with positive radius of convergence).*

(2) *Δ belongs to Parab_1 with*

$$\mathrm{B\acute{E}V}(\Delta) = \left(\psi^0, \psi^\infty\right).$$

(3) *The power series F is 1-summable (see below) and its 1-sum can be realized as a pair (f^+, f^-) of functions holomorphic and bounded by 1 on the respective infinite sectors*

$$V^\pm := \left\{ z \neq 0 \ : \ |\arg(\pm z)| < \frac{5\pi}{8} \right\}$$

as in Figure 1.1.

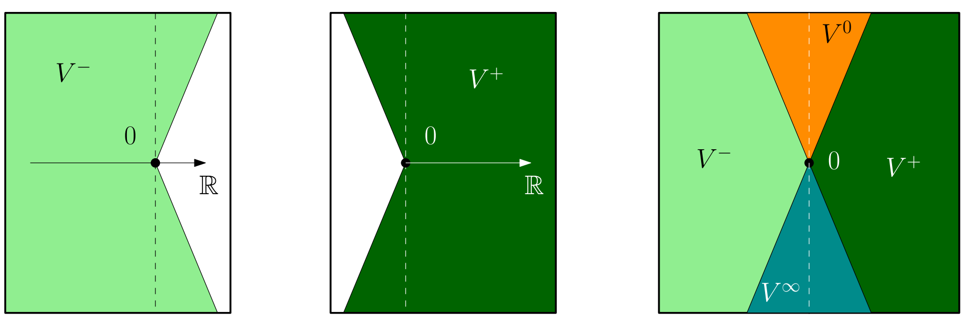


FIGURE 1.1. The infinite sectors $V^\pm$ and the components V^0, V^∞ of their intersection $V^\cap$.

A pair $f = (f^+, f^-)$ of functions holomorphic on the corresponding sector V^+ or V^- has order-1 **flat discrepancy** at 0 whenever

$$\limsup_{\substack{z \to 0 \\ z \in V^\cap}} |z| \ln \left| f^-(z) - f^+(z) \right| \in \left]-\infty, 0\right[\quad , \ V^\cap := V^+ \cap V^-.$$

It follows from a theorem of J.-P. Ramis and Y. Sibuya [LR16, Theorem ##] that there exists some $F=\sum_{n\geqslant0}F_nz^n\in\mathbb{C}\left[\left[z\right]\right]$ which is their common Gevrey-1 asymptotic expansion at 0 (in the sense of Poincaré):

$$\left(\exists C>0\right)\left(\forall N\in\mathbb{N}\right)\left(\forall z\in V^{\pm}\right)\qquad\left|f^{\pm}\left(z\right)-\sum_{n>0}^{N}F_nz^n\right|\leqslant C\left(N+1\right)!\left|z\right|^{N+1}.$$

We then say that F is **1-summable** with 1-sum f. The mapping $f\mapsto F$ is injective (because $V^{\pm}$ is wider than a half-plane). This notion is compatible with the usual arithmetic and differential operations (by taking a narrower subsector if necessary).

Remark 1.1. The notion of $*$-summability has a standard, more general definition (see *e.g.* [LR16]) allowing for more flexibility. We will not need the whole refinement of the theory as we only rely on this specific implementation of the theorem of Ramis and Sibuya.

We can exhibit a value of the upper bound $\lambda\left(\psi\right)$ as follows. There exist $\mathfrak{m}_\mu,\mathfrak{t}_\mu>0$ depending only on μ (given for instance in (4.2)) for which, if we denote by ρ^0 and ρ^∞ the respective radii of convergence of the Taylor series at 0 of ψ^0 and ψ^∞, we may define

$$\frac{1}{\ell\left(\psi\right)}:=\max\left\{1,\mathfrak{t}_\mu,2\pi+\ln\frac{\mathfrak{m}_\mu}{\min\left\{\rho^0,\rho^\infty\right\}}\right\}$$

and take

$$\lambda\left(\psi\right):=\min\left\{\ell\left(\psi\right),\frac{1}{4\sqrt{\mathfrak{m}_\mu\left|\left|\psi\right|\right|_{\ell(\psi)}}}\right\}\leqslant1$$

where

$$\left|\left|\psi\right|\right|_{\ell}:=\max_{\sharp\in\{0,\infty\}}\sup_{|h|\leqslant\mathfrak{m}_\mu\exp(2\pi-1/\ell)}\left|\frac{\mathrm{d}\log\frac{\psi^\sharp(h)}{h}}{\mathrm{d}h}\right|.$$

Remark 1.2.

(1) Any holomorphic vector field X with a double zero at 0 is the infinitesimal generator of a generic parabolic germ with modulus $\left(\mathrm{e}^{4\pi^2\mu}\mathrm{Id},\mathrm{Id}\right)$ for some $\mu\in\mathbb{C}$, see [Éca75]. In this case we say that the modulus is **trivial**. This is in particular the case for X_0. As a consequence of the uniqueness assertion above, the only normal form with convergent $F\in z\mathbb{C}\left\{z\right\}$ is the formal model itself ($F=0$).

(2) Let $X\in z^2\left(\mathbb{C}^\times+z\mathbb{C}\left[\left[z\right]\right]\right)\frac{\partial}{\partial z}$ be a formal vector field with a double zero at 0. It may happen that the formal power series

$$\Delta:=\left(\exp X\right)\cdot\mathrm{Id}$$

is convergent at 0, in which case it belongs to Parab_1 and we say that Δ is the **time-1 map** of the infinitesimal generator X. Beware that this does not imply the convergence of $\left(\exp tX\right)\cdot\mathrm{Id}$ for $t\notin\mathbb{Z}$ even for small $|t|$: this is for instance the case with the entire map $\Delta:z\mapsto\exp\left(z\right)-z$ (see [Bak62]). If the lattice of those $t\in\mathbb{C}$ for which the power series converges is not discrete then the modulus of Δ is trivial [Éca75] (*i.e.* Δ is analytically conjugate to the time-1 flow of a vector field holomorphic near 0).

(3) There might exist other representatives of $\left(\psi^0,\psi^\infty\right)$ of the form $\frac{1}{1+X_0\cdot F}X_0$ for some 1-summable F with 1-sum f on $V^\pm$. Then f must have large norm. This is likely related to what F. Loray discusses after Theorem 1 in [Lor04].

The functions $z\in V^\pm\mapsto f^\pm(z)-\frac{1-z^2}{\lambda z}+\mu\log\frac{\lambda z}{1-z^2}$ are **Fatou coordinates** for Δ (*i.e.* local conformal coordinates $(\mathbb{C},0)\to\left(\overline{\mathbb{C}},\infty\right)$ on which Δ acts as a translation by 1, see Section 3.2). Hence we parameterize the analytic classes of parabolic germs by their formal Fatou coordinates. By picking them in a class of "global" formal objects, we benefit from a rigidity which makes the parameterization injective.

The power series F is global in the sense that its 1-sum $f=(f^+,f^-)$ exists on a pair of sectors whose union covers $\mathbb{C}^\times$. Moreover $\Delta|_{V^\pm}$ is the time-1 map of the 1-sum of its formal infinitesimal generator

$$X^\pm:=\frac{1}{1+X_0\cdot f^\pm}X_0$$

which is a meromorphic vector field on $V^\pm$. A quick computation tells us that the formal model X_0 is invariant by the involution

$$\begin{aligned}\sigma\ :\ \overline{\mathbb{C}}&\longrightarrow\overline{\mathbb{C}}\\ z&\mapsto\frac{-1}{z}\end{aligned}$$

fixing $\pm\mathrm{i}$, while the sectors V^+ and V^- are swapped, *i.e.*

$$\sigma\left(V^\pm\right)=V^\mp.$$

A nice feature of the synthesis is that f not only defines a parabolic germ near 0 but it also creates one at ∞, corresponding to the companion dynamics induced there by $X^\pm$, since (Section 4.3):

$$f^\pm\circ\sigma=-f^\mp. \tag{1.1}$$

This is why we call these dynamical systems *spherical* normal forms, a term borrowed from J. Écalle [Eca05] as discussed in Section 1.2.2. We can describe completely the dynamics of Δ and its monodromy as a multivalued map over the whole Riemann sphere $\overline{\mathbb{C}}$.

Globalization Theorem. *Assume $\mu\neq0$.*

(1) *If $\lambda<\min\left\{\frac{1}{2|\mu|}\ ,\ \frac{1}{12\,800}\right\}$ then $X^\pm$ has exactly three (simple) poles $\{p_{-\mathrm{i}},p_{\mathrm{i}},p_\pm\}$ which are $\mathrm{O}\left(\sqrt{\lambda}\right)$-close to $-\mathrm{i}$, i and ±1 respectively. In particular, this means that the pole $p_{\pm\mathrm{i}}$ is shared by X^- and X^+.*

(2) *(See Figure 1.2.) To each one of the four poles $p\in\{p_{-\mathrm{i}},p_{\mathrm{i}},p_-,p_+\}$ is attached a unique pair of $\mathrm{O}\left(\sqrt{\lambda}\right)$-close points $\{z_p,w_p\}$ lying at the intersection of the stable manifolds through p of X_f^+ and X_f^-, mapped onto p by the time-1 map of $X_f^\pm$. In other words, the analytic continuation of Δ to $\overline{\mathbb{C}}$ provides a multivalued map with ramification points at each z_p and w_p, to which it extends continuously by $\Delta(z_p)=\Delta(w_p)=p$. A domain of*

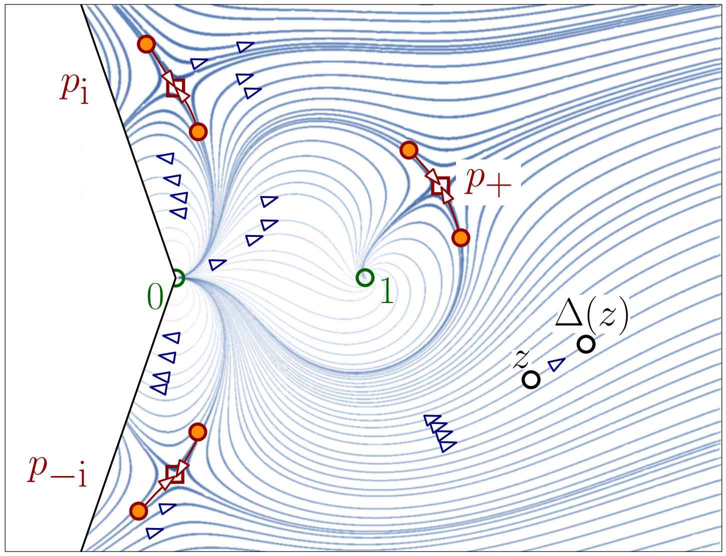


FIGURE 1.2. Foliation induced by the real-time flow of a typical sectorial vector field X^+ with the highlighted 6 ramification points z_p, w_p (orange spots) of its time-1 map Δ. The poles $p_{-\mathrm{i}},\ p_{\mathrm{i}},\ p_+$ of X^+ are figured by red squares.

holomorphy for Δ is given for instance by the slit sphere

$$\mathcal{D} := \overline{\mathbb{C}} \backslash \bigcup_{p \text{ pole}} \gamma_p,$$

where γ_p is the arc of stable manifold passing through p of X_f^+ or X_f^- (choose one) linking z_p and w_p. In particular the radius of convergence about 0 of Δ is $1 + \mathrm{O}\left(\sqrt{\lambda}\right)$.

(3) *The multivalued continuation of Δ to $\overline{\mathbb{C}}$ has 8 branch points. The monodromy around any one of the 8 points $\{z_p,\ w_p\ :\ p \in \{p_{-\mathrm{i}}, p_{\mathrm{i}}, p_-, p_+\}\}$ is involutive.*

(4) *$\Delta|_{\mathcal{D}}$ is holomorphic and injective. It possesses four fixed-points 0, ∞ and ± 1. The first two are parabolic while Δ admits a linearizable dynamics at ± 1 of multiplier $\exp\left(\mp\frac{1}{\mu}\right)$. The dynamics at infinity $\sigma^*\Delta = \sigma \circ \Delta \circ \sigma$ belongs to Parab_1 and*

$$\mathrm{B\acute{E}V}\left(\sigma^*\Delta\right) = \mathrm{B\acute{E}V}\left(\Delta\right)^{\circ -1}.$$

In the case $\mu = 0$, the result still holds except that the pole $z_\pm$ near ± 1 cancels out the stationary point at ± 1. There only remain the parabolic fixed-points of Δ and its ramification locus coming from $\{-\mathrm{i}, \mathrm{i}\}$.

It is remarkable that a twin fixed-point is automatically produced at ∞, especially because the modulus there is given by the reciprocal modulus. This has a simple explanation, however: the involution σ permutes the sectors V^+ and V^- while it does not change the infinitesimal generators much, therefore the identification at the horns is the same while taking place in the reverse direction.

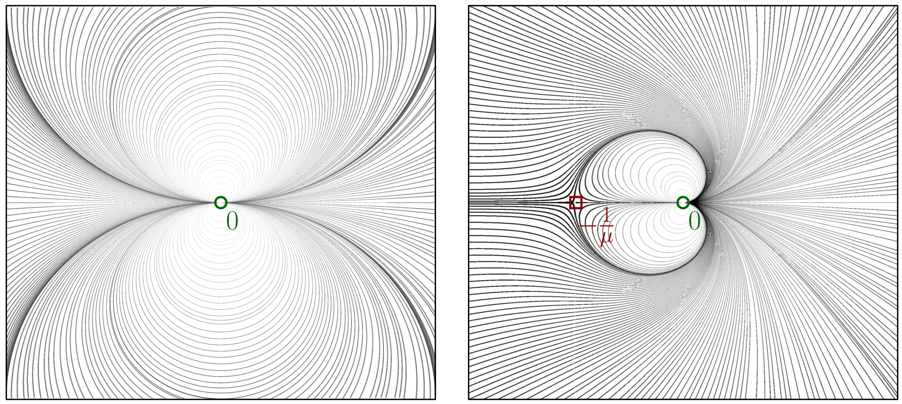


FIGURE 1.3. Foliation induced by the real-time flow of $\frac{x^2}{1+\mu x}\frac{\partial}{\partial x}$ (left $\mu := 0$, right $\mu := 2$). There is one double stationary point at 0 (green circle), yielding a parabolic germ for the time-1 map, and one saddle point (red square) corresponding to the pole $-\frac{1}{\mu}$.

Another remarkable feature is that Δ has very simple dynamics: it is an injective, holomorphic map on a slit sphere $\mathcal{D}$, that can be forward iterated on the open set $\overline{\mathbb{C}}\backslash\gamma$ where γ is the union of the closure of the stable manifolds of $X^{\pm}$ passing through the poles. On each connected component of $\overline{\mathbb{C}}\backslash\gamma$ the dynamics of Δ converges uniformly to a fixed-point (a parabolic point or the attracting point sitting at ± 1 when $\mu \notin \mathrm{i}\mathbb{R}$). The reason behind this stable behavior is the parameter λ, which is chosen so that no chaotic remains of a global dynamics, usually arising from a frontier in the analytic continuation of the modulus [Eps93], can be seen from the normal form.

The obvious downside of the previous feature is precisely that the dynamics of Δ is too simple to readily offer an insight regarding the dynamical richness that is displayed by holomorphic iteration on compact curves. What we prove is that any such dynamics, say of a rational mapping with a parabolic basin $\mathcal{B}$, is locally conjugate by some Ψ to a spherical normal form near the parabolic fixed-point 0, but the domain of the conjugacy cannot contain the whole component $\partial\mathcal{B}$ of the Julia set. Indeed, $\Psi\left(\partial\mathcal{B}\cap(\mathbb{C},0)\right)$ can be pushed forward by the flow of X^{+} to give a compact invariant set J which is locally conformally equivalent to the fractal set $\partial\mathcal{B}$. But the one set J has no relevance to the dynamics of Δ whatsoever, for the latter has only Fatou components with real-analytic boundary, and that regularity increases as $\lambda \to 0$. In that respect, an interesting question arises: can one recover part of the original dynamics by letting λ grow (which may cause $X^{\pm}$ to sport more and more poles, for instance)?

1.2.1. *Concerning the formal model.* The family of vector fields X_0 depending on $\lambda > 0$ is obtained from the "usual" formal model $\frac{x^2}{1+\mu x}\frac{\partial}{\partial x}$ (see Figure 1.3) by performing the degree-2 pullback

$$\Pi \;:\; z \mapsto x = x\left(z\right) := \frac{\lambda z}{1-z^2}$$

(which is invertible near 0). Observe that Π is invariant under the involution σ, so that σ is a symmetry of X_0, a fact which has been already remarked. The obvious consequence is that X_0 has a stationary point at ∞ that mirrors the one it admits at 0.

Remark 1.3. The only normal form X_f that can be factored through Π (*i.e.* realized as a global sectorial perturbation of the usual formal model) is the formal model itself ($f = 0$) since we must both have $f^{\pm} \circ \sigma = f^{\mp}$ (factorization through Π) and $f^{\pm} \circ \sigma = -f^{\mp}$ as in (1.1).

The time-1 map of X_0 is a generic parabolic germ (Lemma 2.4)

$$\Delta_0\left(z\right) := \Phi^1_{X_0}\left(z\right) = z + \lambda z^2 + \lambda^2\left(1-\mu\right)z^3 + \mathrm{o}\left(z^3\right)$$

with trivial BÉV modulus. Any germ $\Delta \in \mathrm{Parab}_1$ with same μ is formally conjugate to Δ_0 (whatever λ), which explains the terminology *formal model.*

The sectors and the model X_0 are tailored in such a way that the orbital region of Δ_0, as seen from the intersection $V^{\cap}$, is a neighborhood of 0 or ∞ of size $\mathrm{O}\left(\mathfrak{m}_{\mu}\mathrm{e}^{-1/\lambda}\right)$ as $\lambda \to 0$ (*cf* Section 4.1). Gaining such a control is essential: eventually the neighborhood fits within the discs of convergence of the data $\left(\psi^0, \psi^{\infty}\right)$ and keeps lying there even after small perturbations.

Picking X_0 among all vector fields realizing the trivial pair $\left(\mathrm{e}^{4\pi^2\mu}\mathrm{Id}, \mathrm{Id}\right)$ is arguably canonical since the pullback $z \mapsto \frac{\lambda z}{1-z^2}$ is very simple and the pair of sectors (V^+, V^-) is fairly standard.

Remark 1.4. For technical reasons we also need that ∞ and 0 be mapped to 0 by the pullback. Thereby the “obvious” choice of a degree-1 pullback, like $z \mapsto \frac{\lambda z}{1-z}$, cannot be used to perform the construction presented here. Apart from this restriction, one can use pretty much any other pullback, provided the “sectors” are tailored conveniently.

Example 1.5. We refer to Figure 1.4. Assume here that $\mu := 0$ so that

$$X_0\left(z\right) = \frac{\lambda z^2}{1+z^2}\frac{\partial}{\partial z}$$

has only stationary (double) points at 0 and ∞, as well as two simple poles at $\pm\mathrm{i}$ with residue $\pm\frac{\lambda \mathrm{i}}{2}$. These poles spawn the ramification locus of Δ_0, as can be seen from a direct integration of $\dot{z} = X_0\left(z\right)$ at time 1 with initial value z_*:

$$1 = \int_{z_*}^{\Delta_0(z_*)} \frac{1+z^2}{\lambda z^2}\mathrm{d}z$$

which yields

$$z_*\Delta_0\left(z_*\right)^2 - \left(z_*^2 + \lambda z_* - 1\right)\Delta_0\left(z_*\right) - z_* = 0.$$

Hence Δ_0 is algebraic and its ramification points are the four points z for which:

$$\left(z^2 + \lambda z - 1\right)^2 + 4z^2 = 0.$$

Notice that Δ_0 admits a limit at each one of these points z, and $\Delta_0\left(z\right)$ lies within the polar locus $\{\pm\mathrm{i}\}$ of X_0. Therefore Δ_0 extends as a multivalued function over the domain $\overline{\mathbb{C}}\backslash\left\{z^2 + \lambda z = 1\right\}$ (which contains 0) with involutive monodromy around any one of the ramification points given by the action of σ, which is moreover sent to a pole of X_0 by the time-1 map. More details can be found in Lemma 2.7.

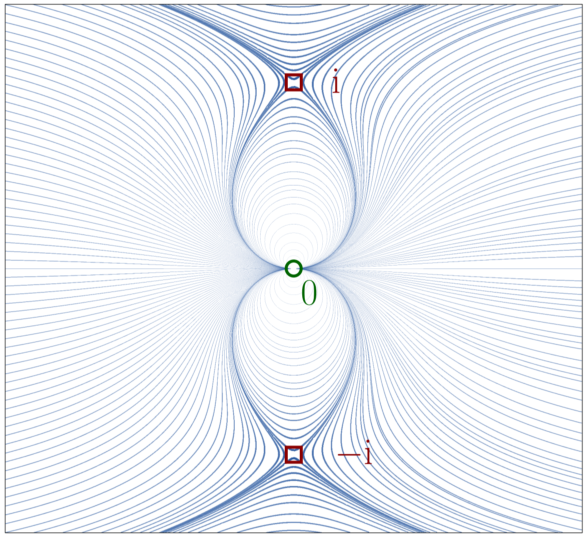


FIGURE 1.4. Foliation induced by the real-time flow of X_0 for $\mu := 0$, revealing the double stationary point (green circle) and two simple poles (red squares).

Remark 1.6. When $\mu \neq 0$ the mapping Δ_0 cannot be algebraic (see [Éca75]). It seems safe to conjecture that the only algebraic normal form Δ is Δ_0 when $\mu = 0$.

1.2.2. *Link with Écalle canonical synthesis.* In [Eca05] J. ÉCALLE revisits the inverse problem by performing the «canonical-spherical synthesis» using the framework of resurgent functions and alien/mould calculus (for details see [Éca85]) that he developed in the wake of his original paper [Éca75]. In that setting the modulus is not built from dynamical considerations, as was first done by G. BIRKHOFF, himself and S. VORONIN (the viewpoint taken here), but as coefficients $(A_n)_{n\in\mathbb{Z}}$ coming from the associated «resurgence equation» and carried by the discrete set of singularities of the analytic continuation of the Borel transform of the formal normalization $\Delta \widehat{\sim} \Delta_0$. The link between the pair $\left(\psi^0, \psi^\infty\right)$ and the collection $(A_n)_n$ is described in [Eca05, Section 5.1].

To perform the canonical-spherical synthesis J. ÉCALLE introduced a (usually large) parameter he called the **twist** $c > 0$, corresponding here to $\frac{1}{\lambda}$, and a family of resurgent monomials which served as building blocks for his synthesis. It turns out that the normal forms produced here are very likely the objects obtained by J. ÉCALLE, as witnessed for instance by the definition of the Cauchy-like kernel [Eca05, Sections 5.1 and 10.3] which is pretty much the one involved in the present construction (compare Definition 4.12).

Canonical-Spherical Synthesis Conjecture. *Given data* $\psi := \left(\psi^0, \psi^\infty\right) \in \mathrm{Diff}\,(\mathbb{C},0) \times \mathrm{Diff}\,(\mathbb{C},0)_{\mathrm{Id}}$ *and* $0 < \lambda \leqslant \lambda\,(\psi)$, *define the twist* $c := {}^1/_\lambda$. *The normal form* Δ *given by the Synthesis Theorem is then the parabolic germ coming from Écalle's canonical-spherical synthesis.*

If it were true this would allow us to invoke the effective methods provided by [Eca05] in the present dynamical context. Conversely, let us discuss in which sense the approach given here completes and sheds a dynamical light on Écalle's synthesis for parabolic germs. To quote J. ÉCALLE (the reader should be aware that he works in the variable $\frac{1}{z}$):

«*As already pointed out, our twisted monomials have much the same behavior at both poles of the Riemann sphere. The exact correspondence has just been described [...] using the so-called antipodal involution: in terms of the objects being produced, this means that canonical object synthesis automatically generates two objects: the "true" object, local at ∞ and with exactly the prescribed invariants, and a "mirror reflection", local at 0 and with closely related invariants. Depending on the nature of the [...] invariants (verification or non-verification of an "overlapping condition"), these two objects may or may not link up under analytic continuation on the Riemann sphere.*»

It could happen that both synthesis processes produce different objects, but what is sure is that the objects built in this paper match very precisely the above expectations and comments. In our setting the «mirror reflection» comes from the degree-2 pullback $\Pi\ :\ z\mapsto\frac{\lambda z}{1-z^{2}}$ invariant by the symmetry σ. The Synthesis Theorem gives quantitative bounds and a uniqueness statement that are not readily available from [Eca05], while the Globalization Theorem simply tells that the normal form is defined and injective on almost all of $\mathbb{C}$. Although it is true that the Taylor series of Δ at 0 and that of its companion near ∞, have a radius of convergence of order $1+\mathrm{O}\left(\sqrt{\lambda}\right)$, both germs are *unconditionally* obtained by analytic continuation one from the other, provided of course that $0<\lambda\leqslant\lambda\left(\psi\right)$. In our synthesis the "closely related" moduli are in fact mutually reciprocal.

1.2.3. *General parabolic and rationally indifferent germs.* Even though in this article we address in detail only the case of germs in Parab_{1}, our construction can be adapted in a straightforward manner to the space Parab_{k} consisting in those germs in the form $\Delta\left(z\right)=z+*z^{k+1}+\mathrm{o}\left(z^{k+1}\right)$ for some $k\in\mathbb{Z}_{>0}$. The inverse problem then regards the realization of collections $\left(\psi_{j}^{0},\psi_{j}^{\infty}\right)_{j\in\mathbb{Z}/k\mathbb{Z}}$, and in that case every ingredient must be considered pulled-back by the branched-covering $z\mapsto z^{k}$. Namely:

- the formal model $X_{0,k}$ is given by the pull-back of $\frac{x^{k+1}}{1+\mu x^{k}}\frac{\partial}{\partial x}$ by the mapping $\Pi_{k}\ :\ z\mapsto\frac{\lambda z}{\left(1-z^{2k}\right)^{1/k}}$ which is invariant by the $2k$-periodic homography $\sigma_{k}:=\frac{\mathrm{e}^{\mathrm{i}\pi/k}}{\mathrm{Id}}$;
- the normal form is $\frac{1}{1+X_{0,k}\cdot F}X_{0,k}$ with a k-summable $F\in z\mathbb{C}\left[\left[z\right]\right]$;
- F admits bounded k-sums on infinite sectors $V_{j}^{\pm}:=\left\{ z\neq0\ :\ \left|\arg\left(\pm z^{k}\right)\right|<\frac{5\pi}{8}\right\}$, $j\in{}^{\mathbb{Z}}/k\mathbb{Z}$, which are circularly permuted by σ_{k}.

The general case of a rationally indifferent germ $\Delta\left(z\right)=\mathrm{e}^{2\mathrm{i}\pi p/q}z+\mathrm{o}\left(z\right)$ with coprime integers p and q, not conjugate to a rational rotation, can also be covered as usual by taking a kq ramified covering $x\mapsto x^{kq}$ of the usual formal model and sectors.

1.3. **Parabolic renormalization.** The parabolic renormalization is the map

$$\begin{aligned}\mathcal{R}\ :\ \mathrm{Parab}_{1} & \longrightarrow\mathrm{Parab}\\ \Delta & \longmapsto\mathrm{B\acute{E}V}\left(\Delta\right)^{\infty}\end{aligned}$$

which to a generic parabolic germ associates the ∞-component of its invariant. It may happen that $\mathcal{R}(\Delta)$ is itself generic, in which case $\mathcal{R}$ can be iterated. The fixed-points of $\mathcal{R}$ play an important role in iterative complex dynamics; here we describe those in spherical normal form. As far as I know a statement of the kind was first communicated privately by R. Schäfke.

Parabolic Renormalization Corollary. *Take an arbitrary $\psi^0 \in \mathrm{Diff}(\mathbb{C},0)$. There exists a bound $\widehat{\lambda}(\psi^0) > 0$ such that, for all $0 < \lambda \leqslant \widehat{\lambda}(\psi^0)$ there exists a unique normal form Δ satisfying*

$$\mathrm{B\acute{E}V}(\Delta) = (\psi^0, \Delta).$$

Again the bound $\widehat{\lambda}$ can be made explicit, we refer to Corollary 6.3.

1.4. **Real case.** If $\Delta \in \mathrm{Parab}_1$ is real (that is, commutes with the complex conjugation $\overline{\bullet}$) then $\mu \in \mathbb{R}$ and its modulus $\mathrm{B\acute{E}V}(\Delta) = (\psi^0, \psi^\infty)$ satisfies the identity:

$$(\square) \qquad (\forall h \in (\mathbb{C},0)) \qquad \overline{\psi^0\left(\overline{h}\right)} = \frac{\mathrm{e}^{4\pi^2\mu}}{\psi^\infty\left(\frac{1}{h}\right)}.$$

Real Synthesis Corollary. *Take $\mu \in \mathbb{R}$, a pair (ψ^0, ψ^∞) and $\lambda > 0$ as in the Synthesis Theorem. If moreover the identity $(\square)$ holds then the synthesized normal form Δ is real.*

1.5. **Structure of the article and summary of the proofs.** One of the main concerns of this paper is to provide as explicit bounds as possible, which explains why so much effort went into technical lemmas. The main tools are basic calculus / complex analysis, and the work is necessary in any case to obtain uniform bounds with respect to λ.

All the constructions will take place within the space of pairs $f = (f^+, f^-)$ with 1-flat difference in $V^\cap$ and bounded 1-type, both at 0 and ∞, as discussed at the beginning of the introduction:

$$\mathcal{S} := \left\{ f \in \mathcal{S}\left(V^+\right) \times \mathcal{S}\left(V^-\right) \ : \ \limsup_{z^{\pm 1} \to 0} |z|^{\pm 1} \ln\left|f^-(z) - f^+(z)\right| \leqslant -6 \text{ for } z \in V^\cap \right\},$$

where

$$\mathcal{S}\left(V^\pm\right) := \left\{ f^\pm \begin{array}{c} \text{holomorphic on } V^\pm \\ \text{continuous on } \overline{V^\pm} \end{array} : \begin{array}{c} f^\pm(0) = 0 \\ f^\pm(\infty) = 0 \end{array} , \ ||f^\pm|| := \sup_{z \in V^\pm} \left|f^\pm(z)\right| < \infty \right\}$$

is equipped with the canonical product Banach norm. We denote by $\mathcal{B}$ its unit ball.

Remark 1.7. Actually the uniform bound -6 on the 1-type of $f^- - f^+$ can be sharpened as $-\frac{6}{\lambda}$ (see (3.5)), *i.e.* the difference becomes flatter and flatter as λ tends to 0, although we do not need that fact.

(1) In Section 2 we give some basic material about vector fields and their time-1 maps, focusing more particularly on the local dynamics near their poles and zeroes and how they stitch together to organize the global dynamics of the model X_0. This will allow us to conjugate

$$X_f^\pm := \frac{1}{1 + X_0 \cdot f^\pm} X_0$$

to the model X_0 over the better part of $V^\pm$ (effective bounds are described in Section 5.2). This sectorial mapping takes the form $\Phi_{X_0}^{f^\pm}$ given by the flow of X_0 at time $f^{\pm}$. The dynamics of the formal model Δ_0, studied in Section 2.3, is then pulled-back by $\Psi^{\pm}$ to give sectorial counterparts for the time-1 map of $X_f^{\pm}$.

(2) Coupled to Ramis-Sibuya theorem, the above preliminary part allows one to reduce the problem of finding a parabolic germ Δ with prescribed modulus to that of finding $(f^+, f^-) \in \mathcal{S}$ satisfying some (non-linear) Cousin problem $(\star)$ below; see Proposition 3.3 in Section 3.

(3) We can then tackle the main result (Section 4). We begin with fixing a formal class $\mu \in \mathbb{C}$ in Parab_1. Then ψ^0 must be tangent to the linear map $\mathrm{e}^{4\pi^2\mu}\mathrm{Id}$. Being given $\left(\psi^0, \psi^\infty\right) = \left(\mathrm{Id}\exp\left(4\pi^2\mu + \varphi^0\right), \mathrm{Id}\exp\varphi^\infty\right)$ with $\varphi^\sharp$ holomorphic near $\sharp \in \{0, \infty\}$ we must find $f^{\pm}$ on $V^{\pm}$ such that

$$(\star)\qquad \begin{cases} f^- - f^+ & = \varphi^0 \circ H_f^+ \qquad \text{on } V^0 \\ f^- - f^+ & = \varphi^\infty \circ H_f^+ \qquad \text{on } V^\infty \end{cases}$$

where $\left(V^\sharp\right)_{\sharp\in\{0,\infty\}}$ are the two components of $V^\cap$ and H_f^+ is a primitive first-integral of the sectorial time-1 map Δ^+ of X_f^+ (in other words, H_f^+ provide preferred orbital coordinates over the sector V^+). Condition $(\star)$ implies that Δ^+ coincides with Δ^- in $V^\cap$ and thus is the restriction to V^+ of a single $\Delta \in \mathrm{Parab}_1$.

The non-linear Cousin problem $(\star)$ is solved using a fixed-point method involving a classical Cauchy-Heine integral transform. This technique was already employed in [ST15] (and [RT0p]) to build normal forms for convergent saddle-node foliations in the complex plane (and their unfoldings). More precisely, we build a map (Proposition 4.11):

$$\mathrm{CH}^\varphi \ : f \longmapsto \Lambda(f)$$

such that $\Lambda(f)^- - \Lambda(f)^+ = \varphi \circ H_f^+$. Therefore $\mathrm{CH}^\varphi(f) = f$ if and only if, $(\star)$ holds. We prove (Proposition 4.13) that CH^φ eventually stabilizes the unit ball $\mathcal{B} \subset \mathcal{S}$ (for λ and ε small enough). The smaller λ the more contracting CH^φ, so that eventually a unique fixed-point of CH^φ exists in that ball (Corollary 4.14).

(4) In Section 5.1, as is discussed above, we establish the Globalization Theorem making use of the involution σ and the property that for the fixed-point f of CH^φ we have

$$f^{\pm} \circ \sigma = -f^{\mp}.$$

Prior to that we describe the dynamical properties of Δ in Section 5.3 using *a priori* bounds on the flow of $X^{\pm}$.

(5) The fixed-points of the parabolic renormalization are also obtained as a holomorphic fixed-point in Section 6.1. Let $\mathrm{Synth}_\lambda \ : \ \psi \mapsto \Delta$ be the synthesis map built previously and consider for given $\psi^0 \in \mathrm{Diff}(\mathbb{C}, 0)$ the iteration $\Delta_0 := \mathrm{Id}$ and $\Delta_{n+1} := \mathrm{Synth}_\lambda\left(\psi^0, \Delta_n\right)$. It so happens that Synth_λ cannot shrink too much the radius of convergence of Δ_n, which always remains above $\frac{3}{16}$. By taking λ smaller we prove next that Synth_λ is a contracting self-map of some closed ball of the Banach space $\mathrm{Holo_c}\left(\frac{3}{16}\mathbb{D}\right)$.

(6) The real setting is explored in Section 6.2 (Real Synthesis Corollary). It merely consists in checking that every step leading to the fixed-point of CH^φ preserves realness, which is straightforward.

1.6. **Notations.**

- $\dot{z}=\frac{\mathrm{d}z}{\mathrm{d}t}$ stands for differentiation with respect to the variable $t\in\mathbb{C}$.
- For a topological space $\mathcal{M}$ and a point $p\in\mathcal{M}$ we use $(\mathcal{M},p)$ to stand for a (sufficiently small) neighborhood of p in $\mathcal{M}$.
- The set $\overline{U}$ is the topological closure of $U\subset\mathcal{M}$.
- We use the notation $\mathbb{D}$ for the standard open unit disk in $\mathbb{C}$.
- $\overline{\mathbb{C}}$ stands for the compactification of the complex line $\mathbb{C}$ as the Riemann sphere.
- The n^{th} iterate of a diffeomorphism Δ is written $\Delta^{\circ n}$ for $n\in\mathbb{Z}$
- Φ_X^t designates the flow at time t of the vector field X.
- If $\Psi\ :\ U\to V$ is locally biholomorphic, we recall its action by conjugacy (change of coordinate)
  on diffeomorphisms:
  $$\Psi^*\Delta:=\Psi^{\circ-1}\circ\Delta\circ\Psi$$
  on vector fields:
  $$\Psi^*X:=\mathrm{D}\Psi^{\circ-1}\left(X\circ\Psi\right).$$
  If Δ is the time-1 map of X then $\Psi^*\Delta$ is the time-1 map of Ψ^*X.

2. Sectorial normalization

We recall that the functional space $\mathcal{S}$ is defined in Section 1.5 above. The main purpose of this section is to investigate in some details the dynamics of

$$\begin{aligned}X^{\pm}&:=\frac{1}{1+X_0\cdot f^{\pm}}X_0\quad,\ f\in\mathcal{S},\ ||f||\leqslant1\\ \Delta^{\pm}&:=\Phi_{X^{\pm}}^1\in\mathrm{Parab}_1\end{aligned}$$

which we deduce from two ingredients:

- the dynamics of the rational vector field
  $$X_0\left(z\right):=\frac{1-z^2}{1+z^2}\times\frac{\lambda z^2}{1+\lambda\mu z-z^2}\frac{\partial}{\partial z}\tag{2.1}$$
  studied in Section 2.3;
- the sectorial normalization between $X^{\pm}$ and X_0 obtained in Section 2.4, that is the existence of biholomorphisms on $V^{\pm}\setminus\{\text{small neighborhhood of the poles}\}$ conjugating the dynamics of both vector fields.

The dynamics of a meromorphic vector field X on some domain U is organized by its singular locus, which splits between the **stationary points** and the **poles**

$$\begin{aligned}\mathrm{Zero}\left(X\right)&:=\left\{z\in U\ :\ X\left(z\right)=0\right\}\\ \mathrm{Pole}\left(X\right)&:=\left\{z\in U\ :\ X\left(z\right)=\infty\right\}.\end{aligned}$$

Since $f^{\pm}$ is holomorphic on $V^{\pm}$ we can never have $\frac{1}{1+X_0\cdot f^{\pm}}=0$ except at poles of X_0, hence

$$\mathrm{Zero}\,\left(X^+\right)\cup\mathrm{Zero}\,\left(X^-\right)=\mathrm{Zero}\,(X_0).$$

It is also true that $z\in\mathrm{Pole}\left(X^{\pm}\right)$ if and only if:

- $z\in\mathrm{Pole}\left(X_0\right)$ and $\left(f^{\pm}\right)'\left(z\right)=0$;
- or $z\notin\mathrm{Pole}\left(X_0\right)$ and $1+\left(X_0\cdot f^{\pm}\right)\left(z\right)=0$.

We begin with fixing some notations and recall the important Lie formula regarding holomorphic vector fields X, then proceed to a short description of the local behavior of the trajectories near singularities and finally explain how the global dynamics of rational X is woven from these local parts.

2.1. **Basic vector fields material.**

2.1.1. *Flow and time-1 map.* Let $U\subset\overline{\mathbb{C}}$ be a domain and $X=R\frac{\partial}{\partial z}$ be a holomorphic vector field on U (that is, R is a holomorphic function on U). For any $z_{*}\in U$ the differential system $\dot{z}=X\left(z\right)$ has a unique local solution $t\in\left(\mathbb{C},0\right)\mapsto z\left(t\right)$ with $z\left(0\right)=z_{*}$, giving rise to a germ of a holomorphic mapping

$$\begin{aligned}\Phi_{X}\ :\ (\mathbb{C},0)\times U & \longrightarrow U\\ (t,z_{*}) & \longmapsto\Phi_{X}^{t}\left(z_{*}\right):=z\left(t\right)\end{aligned}$$

(to be more correct, the neighborhood $\left(\mathbb{C},0\right)$ depends on z_{*} but its size is locally uniform). The **flow** of X is the maximal multivalued analytic continuation of Φ_{X}. It is probably best understood as the reciprocal of the time function

$$(z_{*},\phi)\in U^{2}\longmapsto\left(z_{*},\int_{z_{*}}^{\phi}\frac{\mathrm{d}z}{R\left(z\right)}\right)\in U\times\overline{\mathbb{C}},\tag{2.2}$$

obtained by rewriting $\dot{z}=X\left(z\right)$ as $\mathrm{d}t=\frac{\mathrm{d}z}{R(z)}$ and integrating the **time form**

$$\tau_{X}:=\frac{\mathrm{d}z}{R}.\tag{2.3}$$

The analytic continuation of the time function is explicit, at least when R is rational.

Definition 2.1. We call $\Delta\ :\ z\mapsto\Phi_{X}^{1}\left(z\right)$ the **time-1 map** of X. As a multivalued mapping over U, for any path $\gamma\ :\ [0,1]\to U$ satisfying

$$1=\int_{\gamma}\tau_{X}$$

a determination of Δ satisfies the identity

$$\Delta\left(\gamma\left(0\right)\right)=\gamma\left(1\right),$$

with the special case $\Delta\left(z\right)=z$ if $X\left(z\right)=0$.

Remark 2.2. For $t\in\mathbb{C}^{\times}$ the time-t map Φ_{X}^{t} is obtained as the time-1 map of $\frac{1}{t}X$.

2.1.2. *Lie's formula.* One also encounters the notation $\exp\left(tX\right)$ to stand for Φ_{X}^{t}. This can be understood in the light of Lie's formula relating the action of the operator $\exp\left(tX\right)$ and of the flow mapping. Pick a function $g\in\mathrm{Holo}\left(U\right)$ and some $z\in U$. For any $t\in\left(\mathbb{C},0\right)$ one has

$$\begin{aligned}g\circ\Phi_{X}^{t}\left(z\right) & =\sum_{n=0}^{+\infty}\frac{t^{n}}{n!}\left(X\cdot^{n}g\right)\left(z\right)\\ & =:\left(\exp\left(tX\right)\cdot g\right)\left(z\right)\end{aligned}\tag{2.4}$$

where the iterated Lie's derivative is defined by

$$X\cdot^{n}g:=\underbrace{X\cdot X\cdot\left(\cdots\right)\cdot X}_{n\text{ times}}\cdot g.$$

2.1.3. *Long-time behavior.* In this paper we only use the real-time flow, *i.e.* for $t \in (\mathbb{R}, 0)$, and designate by $t_{\max}(z_*) \in]0, +\infty]$ the maximal time of existence of the forward trajectory $t \in [0, t_{\max}(z_*)[\mapsto \Phi_X^t(z_*)$. Without going into too much details, a consequence of Bendixson-Poincaré theorem and of Cauchy formula applied to (2.2) is the following classification of asymptotic behavior for the real flow of a meromorphic vector field X on some domain U. Starting from $z_* \notin \mathrm{Pole}(X)$ the forward trajectory $t \geqslant 0 \mapsto z(t)$ matches one of the mutually exclusive outcomes:

(1) either $t_{\max}(z_*) = \infty$, exactly in the following situations:
 center case z is periodic (non-constant), in which case neighboring trajectories also are (with same period) and at least one stationary point of X of center type is enclosed by the trajectory;
 equilibrium case $\lim_{t \to +\infty} z(t) \in \mathrm{Zero}(X)$;
(2) or $t_{\max}(z_*) < \infty$, exactly in the following situations:
 escape case $\lim_{t \to t_{\max}(z_*)} z(t) \in \partial U$;
 separation case $\lim_{t \to t_{\max}(z_*)} z(t) \in \mathrm{Pole}(X)$.

2.2. **Local behavior near singularities.**

2.2.1. *Dynamics near a simple zero.* Let $p \in \overline{\mathbb{C}}$ be a simple stationary point of a meromorphic vector field $X = R\frac{\partial}{\partial z}$, and denote by $\alpha \neq 0$ the derivative of R at this point. One can find a conformal coordinate $z = \Psi(w)$ near p such that X takes the form

$$\psi^* X = W := \alpha w \frac{\partial}{\partial w} \ , \ w \in (\mathbb{C}, 0) .$$

A direct integration describes the local behavior of the dynamics of the linear vector field around a simple zero:

$$\Phi_W^t(w) = w \mathrm{e}^{\alpha t}.$$

In particular the time-1 map of X is linearizable near its fixed-point p, which can be of three types:

- attractive ($\Re(\alpha) < 0$) or repulsive ($\Re(\alpha) > 0$) in the equilibrium case; then there exists a domain $U := \left(\overline{\mathbb{C}}, p\right)$ such that any trajectory crossing ∂U converges asymptotically to p in forward or backward time (a basin of attraction/repulsion);
- but it can also be a center case ($\Re(\alpha) = 0$).

Example 2.3. When $\mu \neq 0$ the formal model X_0 admits a simple stationary point at ± 1 with linear part $\alpha = -1/\mu$.

2.2.2. *Dynamics near a double zero.* Take now a double zero of X which, for the sake of convenience, we locate at 0 and define the constants $a \neq 0$ and b by

$$X(z) = \left(az^2 + bz^3 + \mathrm{O}\left(z^4\right)\right) \frac{\partial}{\partial z}.$$

Lemma 2.4. *For all $a \in \mathbb{C}^\times$ and $b \in \mathbb{C}$ we have $\Phi_X^1 \in \mathrm{Parab}_1$. More precisely the 3-jet of the mapping at 0 is given by:*

$$\Phi_X^1(z) = z + az^2 + \left(a^2 + b\right) z^3 + \mathrm{O}\left(z^4\right).$$

This germ is linearly conjugate to $w \mapsto w + w^2 + \left(1 + \frac{b}{a^2}\right) w^3 + \mathrm{o}(w)$ through the transform $w := az$, hence the formal invariant of Φ_X^1 is $-b/a^2$.

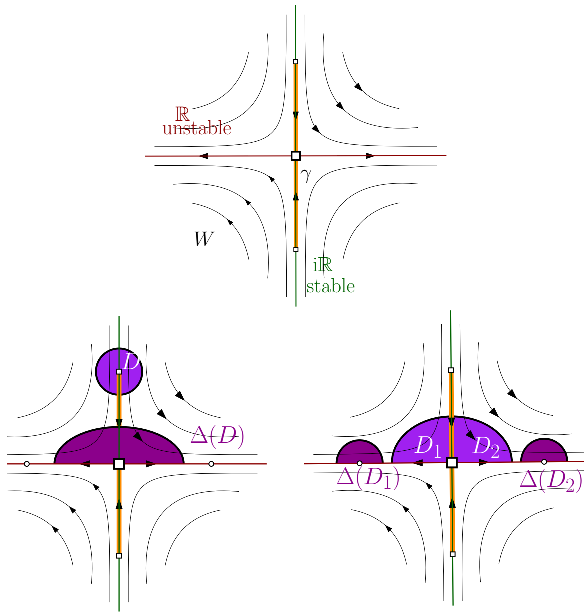


FIGURE 2.1. Saddle dynamics near a simple pole of a vector field W and induced slicing dynamics of its time-1 map Δ. The latter is holomorphic on the complement of the arc γ, included in the stable manifold and whose endpoints are sent to the pole in time 1.

Proof. Using Lie formula we obtain

$$\Phi_X^1(z) = z + \left(az^2 + bz^3\right) + a^2z^3 + \mathrm{O}\left(z^4\right).$$

The rest is straightforward algebra, but for the fact that the formal invariant of $w \mapsto w + w^2 + cw^3 + \mathrm{O}\left(w^4\right)$ is $1 - c$. □

Example 2.5. In the case of X_0 we have

$$\begin{aligned} X_0(z) &= \lambda z^2 - \lambda^2 \mu z^3 + \mathrm{O}\left(z^4\right) \\ \Delta_0(z) &= z + \lambda z^2 + \lambda^2 (1-\mu) z^3 + \mathrm{O}\left(z^4\right) \in \mathrm{Parab}_1 \end{aligned}$$

with formal invariant μ independently on λ.

Remark 2.6. There exists a domain $U := \left(\overline{\mathbb{C}}, p\right)$ such that any trajectory crossing ∂U converges asymptotically to p in forward- or backward-time (a parabolic basin).

2.2.3. *Dynamics near a simple pole (separation case).* Consider here the case where $X(p) = \infty$ is a simple pole. One can find a conformal coordinate $z = \Psi(w)$ near p such that X takes the form

$$\psi^* X = \frac{1}{w}\frac{\partial}{\partial w}\ , \ w \in (\mathbb{C}, 0).$$

A direct integration describes the local behavior of the dynamics of the linear vector field around a simple pole.

Lemma 2.7. *Let* $W := \frac{1}{w}\frac{\partial}{\partial w}$. *We refer to Figure 2.1 for an illustration of the statement to come.*

(1) *The foliation induced by W has a saddle-point at* 0 *with stable manifold* $\mathrm{i}\mathbb{R}$ *and unstable manifold* $\mathbb{R}$.
(2) *The time-1 map of* W *is defined and holomorphic on* $\mathbb{C}\backslash\gamma$ *with* $\gamma:=\mathrm{i}\sqrt{2}\,[-1,1]$, *where it evaluates to*

$$\Phi_W^1(w)=\pm\sqrt{2+w^2}.$$

(3) *The analytic continuation of the above map as a multivalued function can be realized over* $\mathbb{C}\backslash\left\{\pm\mathrm{i}\sqrt{2}\right\}$ *with monodromy* $\mathbb{Z}/2\mathbb{Z}$.

Example 2.8. The vector field X_0 admits two symmetric simple poles $\pm\mathrm{i}$, fixed by σ, as well as two additional simple poles

$$z_\pm:=\frac{\lambda\mu\pm\sqrt{\lambda^2\mu^2+4}}{2},$$

swapped by σ, provided $\mu\neq\pm{}^{2\mathrm{i}}/_{\lambda}$ (otherwise it is a double pole $z_+=z_-=\mu\in\mathrm{i}\mathbb{R}$) and $\mu\neq0$ (else the «pole» ±1 cancels out the «zero» of X_0 located at ±1).

Near $\pm\mathrm{i}$ the vector field X_0 is conjugate to Φ_W^1 and this conjugacy sends σ to the involution $w\mapsto-w$, so that the monodromy of Δ_0 around the ramification points attached to $\pm\mathrm{i}$ is σ (the ramification is induced by the change of variable Π). The corresponding local involution near $z_\pm$ has a different nature and comes from the involution $\nu\neq\mathrm{Id}$ of the usual formal model $\frac{x^2}{1+\mu x}\frac{\partial}{\partial x}$ near the pole $-\frac{1}{\mu}$, which solves

$$\mu\log x-\frac{1}{x}=\mu\log\nu-\frac{1}{\nu}$$

(observe indeed that $t\mapsto\mu\log t-\frac{1}{t}$ has a critical point at $-\frac{1}{\mu}$).

Remark 2.9. The case of a k^{th}-order pole $\frac{1}{w^k}\frac{\partial}{\partial w}$ is similar, with time-1 map $\Phi_W^1(w)=\left(k+1+w^{k+1}\right)^{1/k+1}$, giving rise to $k+1$ ramification points and $k+1$ determinations of the time-1 map. The integer k is a topological invariant.

2.3. **Global dynamics of the formal model.** We explain why/how we choose the formal model X_0 and the sector $V^\pm$ in Section 4.1. For now, let us present the global features of the dynamics of X_0 on $\overline{\mathbb{C}}$ for fixed $\lambda>0$ and $\mu\in\mathbb{C}$. We explain how the local dynamics near the singular set of X we described above stitch together by studying the real-analytic foliation of the sphere $\overline{\mathbb{C}}$ induced by the real-time flow (Figure 2.2).

Remark 2.10.

(1) Whether $\mu=0$ or not leads to different dynamics, since the simpler $X_0(z)=\frac{\lambda z^2}{1+z^2}\frac{\partial}{\partial z}$ has less poles and zeroes. The dynamics of this particular vector field has been studied in Example 1.5, so we allow ourselves to assume that $\mu\neq0$ whenever it leads to a more straightforward exposition.
(2) In order to avoid other zero/pole cancellations we must in addition suppose that $\mu\neq\pm{}^{2\mathrm{i}}/_{\lambda}$ (the latter is ensured whenever $\lambda<\frac{1}{2|\mu|}$). Under these assumptions, the rational vector field has three stationary points located at 0 (double) and ±1 (simple), as well as four simple poles located at $z_\pm:=\frac{\lambda\mu\pm\sqrt{\lambda^2\mu^2+4}}{2}$ and $\pm\mathrm{i}$.

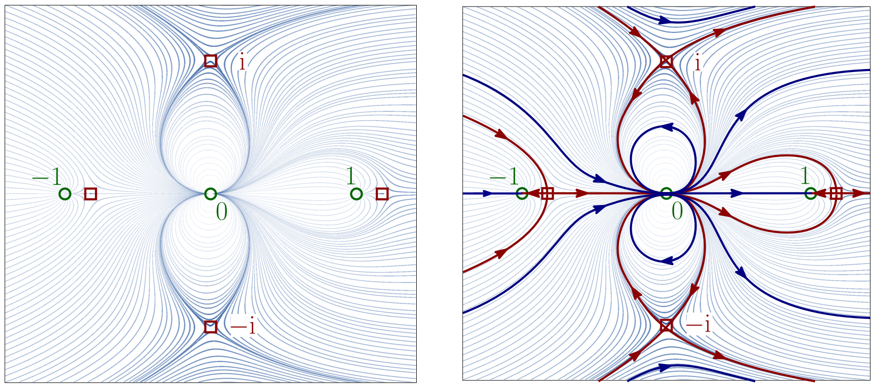


FIGURE 2.2. Foliation induced by the real-time flow of X_0 (left) for $\lambda := \frac{1}{2}$ and $\mu := \frac{1}{2}$, revealing the three stationary points (green circles) and four poles (red squares). On the right, the spinal (blue) and separatrix (red) graphs are depicted.

2.3.1. *Spinal graph.* After the pioneering work of S. SMALE and *al* starting at the beginning of the 1980's to study general numerical Newton-like schemes for solving polynomial equations [Sma81, STW88], the foliations induced by holomorphic polynomial vector fields X has been thoroughly studied in the generic case by A. DOUADY, F. ESTRADA and P. SENTENAC in an unpublished manuscript [DES05]. Then B. BRANNER and K. DIAS [BD10] completed the task for every polynomial vector field, and in his thesis J. TOMASINI studied some rational vector fields. All these considerations result in the following statement: the topological class of X (up to orientation-preserving homeomorphism) is completely classified by its combinatorial dynamics. Roughly speaking, it is encoded as the spinal graph of X, the way poles and stationary points of the vector field are connected by the closure of maximal trajectories.

Lemma 2.11. *Let X be a rational vector field. The **separatrix graph** $Sep(X)$ of X is the closure of the union of all stable and unstable manifolds passing through the poles of X. Then $\overline{\mathbb{C}}\backslash Sep(X)$ consists in finitely many X-invariant connected components. Any two trajectories in the same component are:*

- *either both periodic with same period;*
- *or they link the same pair of stationary points $q \to p$ of X.*

Proof. Assume that $t \mapsto z(t)$ is a periodic trajectory. Cauchy formula indicates that nearby trajectories are also periodic with same period. A straightforward connectedness argument allows us to conclude that every other trajectory in the component is periodic with same period.

Assume now that $t \mapsto z(t)$ is neither a center nor a separatrix. Then z goes towards some $p \in \mathrm{Zero}(X)$ in forward time, say, and then there exists a domain $U_p \ni p$ (be it the basin of attraction of an attractive fixed-point or a parabolic basin for a multiple zero) such that any trajectory landing at p eventually hits ∂U_p and *vice versa*. Hence z links ∂U_q to ∂U_p in finite time, and the flow-box theorem asserts this remains the case for neighboring trajectories. Again a direct connectedness argument brings the conclusion. $\square$

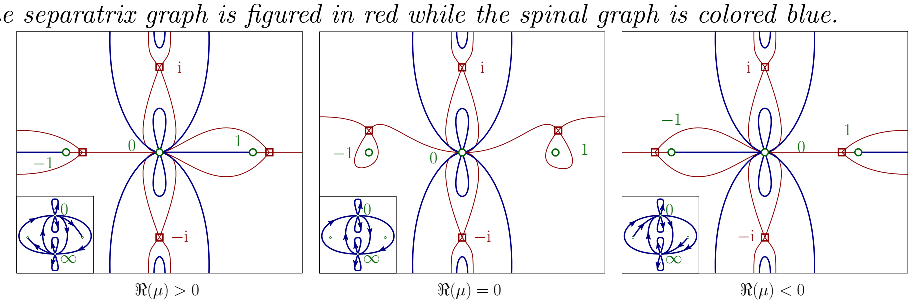
Definition 2.12. We define the **spinal graph** $\mathrm{Spine}(X)$ as the oriented graph with vertices $\mathrm{Zero}(X)$ and ordered edges corresponding to a trajectory of X linking those vertices $q \to p$, one per component of $\overline{\mathbb{C}} \backslash \mathrm{Sep}(X)$. Isolated vertices correspond to center stationary points.

Remark 2.13. Both graphs $\mathrm{Spine}(X)$ and $\mathrm{Sep}(X)$ come with a canonical, non-ambiguous geometric realization with edges as integral curves of X. For that reason we identify the combinatorial data and its canonical geometric realization as a subset of $\overline{\mathbb{C}}$.
We can equip both graphs with an oriented length, by integrating the time form (2.3) along injective subpaths γ

$$\ell_X(\gamma) := \int_\gamma \tau_X = \int_\gamma \frac{\mathrm{d}z}{X \cdot \mathrm{Id}} \in \overline{\mathbb{R}}. \tag{2.5}$$

We omit the proof of the next lemma.

Lemma 2.14. *If $\mu \neq 0$ and $\mu \neq \pm\frac{2\mathrm{i}}{\lambda}$ the graph $\mathrm{Spine}(X_0)$ belongs to the following list, according to the sign of $\Re(\mu)$. A center bifurcation occurs when $\Re(\mu) = 0$. The separatrix graph is figured in red while the spinal graph is colored blue.*



2.3.2. *Dynamics of the time-1 map* Δ_0. From the description of the qualitative dynamics of time-1 maps of rational vector fields, we work out more quantitative specifics for the model X_0.

Lemma 2.15. *Let X be a rational vector field and Δ be its time-1 map. Then Δ is a multivalued map over $\overline{\mathbb{C}}$ with branch points $z_* \in \mathrm{Sep}(X)$ at time-1 from an element of $\mathrm{Pole}(X)$, by which we mean that there exists a subpath $\gamma \subset \mathrm{Sep}(X)$ with $\gamma(0) = z_*$, $\gamma(1) \in \mathrm{Pole}(X)$ and $\ell_X(\gamma) = 1$ as in (2.5). A pole of order k gives rise to $k+1$ local determinations.*

Proof. For Δ to be locally holomorphic at $z \in U$ there must exist a path $\tau : [0,1] \longrightarrow \mathbb{C}$ with $\tau(0) = 0$ and $\tau(1) = 1$, on a neighborhood of whose image the germ $t \in (\mathbb{C},0) \mapsto \Phi_X^t(z)$ admits an analytic continuation. This is clearly the case as long as $\Phi_X^\tau(z)$ does not cross $\mathrm{Pole}(X)$. If $t \in]0,1[$ and $\tau(t) \in \mathrm{Pole}(X)$ then one can slightly deform τ, while keeping the endpoints fixed, in order to avoid the pole (the different nonequivalent choices of the deformation provide differing determinations of Δ). Hence Δ can be analytically continued around z as long as $\Phi_X^{\tau(1)}(z) = \Delta(z) \notin \mathrm{Pole}(X)$. If the trajectory issued from $z_* \in \mathbb{C}$ reaches some pole p in finite time $t_{\max}(z_*) \in \mathbb{R} > 0$ then z_* belongs to a stable manifold passing through the pole p. It follows from Lemma 2.7 that there is only k such smooth manifolds in the neighborhood of p, thus γ as described is a well-defined closed set of the sphere. Hence, whenever $z_* \notin \gamma$ we can define a locally analytic Δ near z_*, and that mapping can be extended to $\partial\gamma$ continuously by setting $\Delta(\partial\gamma) := \{p\}$. $\square$

Remark 2.16. As we did not use explicitly the rationality of X, this lemma actually holds for vector fields whose real-time trajectories do not escape from their domain of meromorphy.

Proposition 2.17. *Assume $\mu \neq 0$ and $0 < \lambda < \frac{1}{2|\mu|}$ small enough.*

1. *The time-1 map Δ_0 of X_0 is holomorphic and injective on the dense domain $\overline{\mathbb{C}} \backslash \Gamma = (\mathbb{C}, 0) \cup (\overline{\mathbb{C}}, \infty)$, where $\Gamma = \bigcup_{p \in Pole(X_0)} \gamma(p)$ is the union of 4 real-analytic, forward X_0-invariant and smooth curves passing through the poles of X_0. Each one of these curves is the arc of the stable manifold of X_0 through p joining the two points mapped to p in time 1 along X_0.*
2. *The holed-out sphere*
$$\mathcal{D}_\lambda := \overline{\mathbb{C}} \backslash \left(D_{-\mathrm{i}} \cup D_{\mathrm{i}} \cup D_{z_-} \cup D_{z_+}\right)$$
is included in $\overline{\mathbb{C}} \backslash \Gamma$, where D_p is an open disc centered at the pole p whose diameter decreases to 0 as λ does. In fact $\Phi_{X_0}^{\tau}$ is holomorphic on $\mathcal{D}_\lambda$ for any $\tau \in \overline{\mathbb{D}}$.
3. *Δ_0 extends as a multivalued function over $\mathbb{C} \backslash \bigcup_{p \in Pole(X_0)} \partial\gamma(p)$ with involutive monodromy around each one of the poles p. The endpoints $\partial\gamma(p)$ of the arc $\gamma(p)$ are mapped to p by Δ_0.*

In case $\mu = 0$ the result still holds save for the fact that the pole at $z_\pm = \pm 1$ cancels out the stationary point at ± 1. There only remain the parabolic fixed-points of Δ_0 and their ramification locus coming from the two poles $\{\pm \mathrm{i}\}$ of X_0.

Remark 2.18.

1. We give quantitative bounds on the smallness of λ and D_p in Section 5.2.
2. Whatever the value of $\lambda > 0$ and $|\tau| \leqslant 1$, the flow $\Phi_{X_0}^{\tau}$ is holomorphic on $(\mathbb{C}, 0)$ since 0 can be reached only in infinite time.
3. The complement in $\overline{\mathbb{C}}$ of the closure of all stable manifolds of X_0 through its poles is an open and dense forward X_0-invariant set, *i.e.* on which Δ_0 can be (forward-)iterated *ad lib.* It is not a neighborhood of 0 nor of ∞.

Proof.

1. The holomorphy of Δ_0 outside Γ is simply the content of Lemma 2.15. Observe next that only at a pole p can two trajectories meet in finite time. As a matter of consequence, if $\Delta_0(z_0) = \Delta_0(z_1) =: p$ then the real trajectories of X_0 issued from z_0 and z_1 must cross each other at time 1, hence p is a pole of X_0 and z_0, z_1 belong to $\partial\Gamma$. In other words Δ_0 is injective outside Γ.
2. We wish to bound the magnitude of $\Phi_{X_0}^{\tau}(z_*)$ for any z_* close to a pole p of X_0 and $\tau \in \mathbb{S}^1$. It is clear that the connected component of $\ell_{X_0}^{-1}(\mathbb{D})$ containing p shrinks to $\{p\}$ as λ goes to 0, one can then take D_p containing that component. More details are given in Section 5.2.
3. X_0 is conjugate to $\frac{1}{w}\frac{\partial}{\partial w}$ as in Lemma 2.7 on a full neighborhood of each $\gamma(p)$. For a detailed proof we refer to the proof of Proposition 5.8.

□

2.4. **Sectorial normalization of $X^{\pm}$.** Let us reformulate the previous study for variable-time flow Φ_X^f.

Proposition 2.19. *Let f be a function holomorphic on a domain $U \subset \mathbb{C}$ and X be a meromorphic vector field on U. Define the meromorphic vector field on U by*

$$X_f := \frac{1}{1 + X \cdot f} X.$$

(1) *X and X_f share the same stationary points while*

$$Pole(X_f) = Pole(X) \cap (f')^{-1}(0) \quad \cup \quad (1 + X \cdot f)^{-1}(0) \setminus Pole(X).$$

(2) *If the flow $\Psi \;:\; z \mapsto \Phi_X^{f(z)}(z)$ at time f along X is locally holomorphic around some $z \in U$ then:*
   (a)
$$\frac{X \cdot \Psi}{1 + X \cdot f} = X \circ \Psi$$
   (this particularly means that if Ψ is locally biholomorphic at $z \in U$ then $\Psi^* X = X_f$ around z);
   (b)
$$\Psi \circ \Delta_f = \Delta \circ \Psi$$
   where Δ_f is the time-1 map of X_f and Δ is that of X (this particularly means that if Ψ is locally biholomorphic at $z \in U$ then $\Psi^* \Delta = \Delta_f$ around z).

(3) *Ψ is locally holomorphic at all $z \in U$ except maybe for $z \in Pole(X_f)$. In particular Ψ is holomorphic on a neighborhood of $Zero(X)$ and $\Psi|_{Zero(X)} =$* Id.

Proof.

(1) If $p \in \mathrm{Pole}(X)$ then $X_f(p) = \frac{1}{f'(p)} \frac{\partial}{\partial z}$, else a pole in X_f can only come from a zero of $1 + X \cdot f$.

(2)
   (a) Using Lie's formula $\Psi = \sum_{n=0}^{\infty} \frac{f^n}{n!} X \cdot^n \mathrm{Id}$ we derive formally

$$\begin{aligned} X_f \cdot \Psi &= \frac{1}{1 + X \cdot f} X \cdot \sum_{n=0}^{\infty} \frac{f^n}{n!} X \cdot^n \mathrm{Id} \\ &= \frac{1}{1 + X \cdot f} \left(\sum_{n=0}^{\infty} \frac{f^n}{n!} X \cdot^{n+1} \mathrm{Id} + (X \cdot f) \frac{f^{n-1}}{(n-1)!} X \cdot^n \mathrm{Id} \right) \\ &= \sum_{n=0}^{\infty} \frac{f^n}{n!} X \cdot^{n+1} \mathrm{Id} \\ &= X \circ \Psi. \end{aligned}$$

   (b) The Lie formula again implies that

$$\Psi \circ \Delta_f = \Psi \circ \Phi_{X_f}^1 = \sum_{n=0}^{\infty} \frac{1}{n!} X_f \cdot^n \Psi$$

   but a direct recursion yields $X_f \cdot^n \Psi = (X \cdot^n \mathrm{Id}) \circ \Psi$.

(3) See Proposition 2.17 and subsequent remark.

□

Remark 2.20. Item 2. remains true for formal $f \in z\mathbb{C}[[z]]$ and formal $X \in z\mathbb{C}[[z]]\frac{\partial}{\partial z}$ (in that case Ψ is invertible as a formal transform at 0).

Following the previous discussion we define for $f \in \mathcal{S}$, with $||f|| \leqslant 1$,

$$\Psi^{\pm} \ : \ z \longmapsto \Phi_{X_0}^{f^{\pm}(z)}(z) .$$

Whenever $\Psi^{\pm}$ is locally invertible we have

$$\left(\Psi^{\pm}\right)^* X_0 = X^{\pm}$$

by Proposition 2.19 1 with $X := X_0$.

Lemma 2.21. *Invoking the notations of Lemma 2.17, the mapping $\Psi^{\pm}$ is locally biholomorphic on the holed-out domain $V^{\pm} \cap \mathcal{D}_{\lambda}$ and therefore conjugate X_0 with $X^{\pm} = X_{f^{\pm}}$.*

Proof. Because $|f^{\pm}(z)| \leqslant 1$ for all $z \in V^{\pm}$ the construction of $\mathcal{D}_{\lambda}$ guarantees that $\Phi_{X_0}^{f^{\pm}}$ is well-defined and holomorphic on $V^{\pm} \cap \mathcal{D}_{\lambda}$. But the poles of X_0 lie outside $\mathcal{D}_{\lambda}$ therefore the conclusion follows from Proposition 2.19 3. $\square$

From this Lemma we deduce that the dynamics of $X^{\pm}$ (governed by its «sectorial spinal graph») is very close to that of $X_0|_{V^{\pm}}$. A more quantitative analysis is conducted in Sections 5 and 6.1.

3. Reduction

The proof of the next proposition, which reduces the problem of finding Δ to that of finding the pair (f^+, f^-) in orbit space, is performed in Section 3.3. The definition of the functional space $\mathcal{S}$ is given in Definition 4.6 and the orbit space of the model vector field X_0 is studied in details in Section 3.1 below.

Definition 3.1.

(1) We say that a holomorphic $L \ : \ U \to \mathbb{C}$ is a **first-integral** of Δ when

$$L \circ \Delta = L.$$

(2) A function $H \neq 0$ holomorphic on V is a **primitive function** of X whenever

$$X \cdot H = 2\mathrm{i}\pi H.$$

Primitive functions H (actually unique up to a multiplicative constant) play a central role in the present study, because the Cousin problem $(\star)$ below comes from the structure of the ring of first-integrals of the map Δ. When U is a component of $V^{\cap}$, the primitive function H turns out to be a functional generator of the ring (Corollary 3.8): every first-integral L of Δ factors holomorphically as $\phi \circ H$. The latter property can be reworded equivalently as: H gives a preferred coordinate on the space of orbits of Δ.

Remark 3.2. A primitive function H of X is indeed a first-integral of Δ (Lie formula):

$$H \circ \Delta = \sum_{n=0}^{+\infty} \frac{1}{n!} X \cdot^n H = \left(\sum_{n=0}^{+\infty} \frac{(2\mathrm{i}\pi)^n}{n!} \right) H = H,$$

hence any function of the form $\phi \circ H$ is a first-integral of Δ.

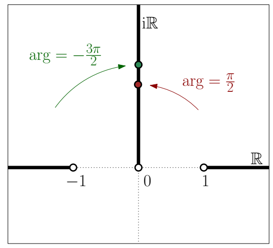


FIGURE 3.1. Branch-cut scheme of H_0 (cuts along fat lines).

Proposition 3.3. *Take $f \in \mathcal{S}$ and fix $\lambda > 0$. Let*

$$X^{\pm} := \frac{1}{1 + X_0 \cdot f^{\pm}} X_0$$

and $\Delta^{\pm}$ be the time-1 map of $X^{\pm}$. We write Δ_0 for the time-1 map of X_0.

(1)
  (a) *$\Delta^{\pm}$ is holomorphic and bounded on $V^{\pm} \cap (\mathbb{C}, 0)$ and $\Delta^{\pm}(z) = \Delta_0(z) + \mathrm{o}(z^3)$ near $0 \in \overline{V^{\pm}}$.*
  (b) *Its space of orbits over $V^{\pm} \cap (\mathbb{C}, 0)$ is canonically given by the range of the primitive function $H^{\pm} := H_0 \exp(2\mathrm{i}\pi f^{\pm})$, where*

$$H_0(z) := \left(\frac{\lambda z}{1 - z^2}\right)^{2\mathrm{i}\pi\mu} \exp\left(-2\mathrm{i}\pi \frac{1 - z^2}{\lambda z}\right)$$

  is a primitive function of X_0. Moreover

$$V^{\pm} \cap (\mathbb{C}, 0)/_{\Delta^{\pm}} \simeq H^{\pm}\left(V^{\pm} \cap (\mathbb{C}, 0)\right) = \mathbb{C}^{\times}.$$

  (c) *For $\sharp \in \{0, \infty\}$ one has*

$$\lim_{\substack{z \to 0 \\ z \in V^{\sharp}}} H^{\pm}(z) = \sharp,$$

  so that $H^{\pm}\left(\overline{V^{\sharp}} \cap (\mathbb{C}, 0)\right) = \left(\overline{\mathbb{C}}, \sharp\right)$.

(2) *The following properties are equivalent.*
  (a) *$\Delta^{\pm}$ is the restriction to $V^{\pm} \cap (\mathbb{C}, 0)$ of a parabolic germ in* Parab_1*;*
  (b) *$\Delta^{+} = \Delta^{-}$ on $V^{\cap} \cap (\mathbb{C}, 0)$;*
  (c) *the difference $f^{-} - f^{+}$ is a first-integral of Δ^{+};*
  (d) *there exists $\varphi^{\sharp} \in \mathrm{Holo}\left(\overline{\mathbb{C}}, \sharp\right)$, $\sharp \in \{0, \infty\}$, such that*

$$(\star) \qquad \begin{cases} f^{-} - f^{+} & = \varphi^{0} \circ H_f^{+} \qquad \text{on } V^{0} \cap (\mathbb{C}, 0) \\ f^{-} - f^{+} & = \varphi^{\infty} \circ H_f^{-} \qquad \text{on } V^{\infty} \cap (\mathbb{C}, 0) \end{cases}.$$

(3) *Assume here that $(\star)$ holds, so that $\Delta^{\pm} = \Delta \in \mathrm{Parab}_1$.*
  (a) *The modulus of Δ is $\mathrm{B\acute{E}V}(\Delta) = \left(\mathrm{Id}\exp\left(4\pi^2\mu + \varphi^{0}\right), \mathrm{Id}\exp\varphi^{\infty}\right)$.*
  (b) *(f^{+}, f^{-}) is the 1-sum of a formal power series $F \in z\mathbb{C}[[z]]$ and Δ is the time-1 map of $\frac{1}{1+X_0 \cdot F} X_0$.*

3.1. **Primitive function and space of orbits near** 0. Define, by analytic continuation, the multivalued function over $\mathbb{C}\backslash\{0,\pm1\}$

$$H_0(z) := \exp(2\mathrm{i}\pi t(z)) = \left(\frac{\lambda z}{1-z^2}\right)^{2\mathrm{i}\pi\mu}\exp\left(-2\mathrm{i}\pi\frac{1-z^2}{\lambda z}\right),$$

with branch-cuts and determination indicated in Figure 3.1. By construction it is a primitive function of X_0:

$$X_0\cdot H_0 = 2\mathrm{i}\pi H_0$$

and the real-flow of X_0 corresponds to level curves of $|H_0|$.

Definition 3.4. The **cut sector** $\widehat{V}$ is the complement

$$\widehat{V}^{\pm} := V^{\pm}\backslash\pm\mathbb{R}_{\geqslant1}$$

on which H_0 is given the holomorphic determination induced by Figure 3.1 and written $H_0|_{\widehat{V}}$. Notice that

$$\widehat{V}^{-}\cap\widehat{V}^{+} = V^{-}\cap V^{+} = V^{\cap} = V^{0}\cup V^{\infty}.$$

For every $z\in\widehat{V}$ such that $\Delta_0(z)\in\widehat{V}$ one has

$$H_0(\Delta_0(z)) = H_0(z).$$

Moreover, the orbits of $\Delta_0|_{\widehat{V}}$ are in 2-to-1 correspondence with level sets of $H_0|_{\widehat{V}}$, but the correspondence becomes 1-to-1 on $(\mathbb{C},0)$. As $z\to0$ in $V^{\pm}$ we have

$$H_0(z)\sim_0(\lambda z)^{2\mathrm{i}\pi\mu}\,\mathrm{e}^{-2\mathrm{i}\pi/(\lambda z)}$$

hence

$$\begin{cases}|H_0(z)| & = \mathrm{O}\left(\mathrm{e}^{-\frac{1}{|*z|}}\right)\text{ as }z\xrightarrow[V^0]{}0\\ \left|\frac{1}{H_0(z)}\right| & = \mathrm{O}\left(\mathrm{e}^{-\frac{1}{|*z|}}\right)\text{ as }z\xrightarrow[V^\infty]{}0\end{cases}. \tag{3.1}$$

This particularly shows that $H_0\left(\widehat{V}^{\pm}\right) = \mathbb{C}^{\times}$: the sectorial space of orbits near 0 of Δ_0 is a doubly-punctured sphere,

$$V^{\pm}\cap(\mathbb{C},0)/_{\Delta_0}\simeq\mathbb{C}^{\times}, \tag{3.2}$$

the whole sphere $\overline{\mathbb{C}}$ being obtained by taking the closure $\overline{V^{\sharp}}\cap(\mathbb{C},0)$ for $\sharp\in\{0,\infty\}$.

Remark 3.5. Similarly, as $z\to\infty$ in $V^{\pm}$ we have

$$H_0(z)\sim_\infty(-z/\lambda)^{-2\mathrm{i}\pi\mu}\,\mathrm{e}^{2\mathrm{i}\pi z/\lambda}$$

so that

$$\begin{cases}|H_0(z)| & = \mathrm{O}\left(\mathrm{e}^{-|*z|}\right)\text{ as }z\xrightarrow[V^0]{}\infty\\ \left|\frac{1}{H_0(z)}\right| & = \mathrm{O}\left(\mathrm{e}^{-|*z|}\right)\text{ as }z\xrightarrow[V^\infty]{}\infty\end{cases}. \tag{3.3}$$

The (direct) monodromy of H_0 around 0 is generated by the linear map

$$h\mapsto\mathrm{e}^{-4\pi^2\mu}h.$$

Therefore, for the choice of the determination and branch-cuts we made, we have

$$\begin{cases}H_0|_{V^-} = \mathrm{e}^{4\pi^2\mu}H_0|_{V^+} =: \psi^0\circ H_0|_{V^+} & \text{on } V^0\\ H_0|_{V^-} = H_0|_{V^+} =: \psi^\infty\circ H_0|_{V^+} & \text{on } V^\infty\end{cases}, \tag{3.4}$$

which gives us a representation of the space of orbits of $\Delta_0|_{(\mathbb{C},0)}$ as

$$(\mathbb{C},0)/_{\Delta_0} \simeq \overline{\mathbb{C}} \sqcup \overline{\mathbb{C}}/_{\left(\psi^0,\psi^\infty\right)},$$

where $\overline{\mathbb{C}} \sqcup \overline{\mathbb{C}}$ stands for the abstract, disjoint union of two copies of $\overline{\mathbb{C}}$.

3.2. **Sectorial orbit space and first-integrals.** Here we are particularly interested in the structure of the space of orbits of the sectorial time-1 maps $\Delta^\pm$ of $X^\pm$. We wish to prove that these perturbations of X_0 still possess the properties underlined in Section 3.1 for the model, which is obtained *via* the sectorial normalization (Section 2.4).

Example 3.6. A **Fatou coordinate** of X is a locally biholomorphic mapping Ψ such that

$$\Psi^* \frac{\partial}{\partial z} = X.$$

This condition is equivalent to $X \cdot \Psi = 1$. Obviously the Abel equation $\Psi^*\Delta = \mathrm{Id}+1$ holds whenever the time-1 map is defined, and we also speak of a Fatou coordinate for Δ.

Let $H^\pm := H_0 \exp\left(2\mathrm{i}\pi f^\pm\right)$ for $f \in \mathcal{S}$. By construction $H^\pm$ is a primitive function of $X^\pm$:

$$\begin{aligned} X^\pm \cdot H^\pm &= \frac{1}{1+X_0\cdot f^\pm} X_0 \cdot H^\pm = \frac{1}{1+X_0 \cdot f^\pm}\left(X_0\cdot H_0 + 2\mathrm{i}\pi H_0 X_0 \cdot f^\pm\right)\exp\left(2\mathrm{i}\pi f^\pm\right) \\ &= 2\mathrm{i}\pi H^\pm. \end{aligned}$$

Hence $\frac{1}{2\mathrm{i}\pi}\log H^\pm$ is a (sectorial) Fatou coordinate of $X^\pm$ on $\widehat{V}^\pm$.

3.2.1. *Sectorial orbit space.* According to Section 2.4 we have near $(\mathbb{C},0)$

$$X^\pm = \left(\Phi_{X_0}^{f^\pm}\right)^* X_0$$

and the sectorial dynamics of $X^\pm$ (*resp.* of $\Delta^\pm$) are conjugate to that of $X_0|_{V^\pm}$ (*resp.* of $\Delta_0|_{V^\pm}$). Because $X_0(0) = 0$ the mapping $\Phi_{X_0}^{f^\pm}$ is tangent to the identity near 0. We deduce at once the following result from the study performed above Section 2.3 by pull-back. In particular, observe that the primitive function $H^\pm$ of $X^\pm$ (which provides the preferred coordinate on the orbit space of Δ) is the pull-back of $H_0|_{V^\pm}$ by $\Phi_{X_0}^{f^\pm}$ according to Lie's formula (2.4):

$$H_0 \circ \Phi_{X_0}^{f^\pm} = \sum_{n=0}^{\infty} \frac{\left(f^\pm\right)^n}{n!} X_0 \cdot^n H_0 = H_0 \sum_{n=0}^{\infty} \frac{\left(2\mathrm{i}\pi f^\pm\right)^n}{n!} = H^\pm.$$

In that sense $\Delta^\pm$ and $\Delta_0|_{V^\pm}$ share the same canonical orbital coordinate, a fact we summarize below.

Lemma 3.7. *Let $f^\pm$ be holomorphic on $V^\pm$ with continuous extension to $\overline{\mathbb{C}}$. Define $X^\pm := \frac{1}{1+X_0\cdot f^\pm}X_0$ and $H^\pm := H_0 \times \exp\left(2\mathrm{i}\pi f^\pm\right)$. There exists $\mathcal{V} := (\mathbb{C},0)$ such that the following assertions hold.*

1. *The time-1 map $\Delta^\pm$ of $X^\pm$ is holomorphic and injective on $V^\pm \cap \mathcal{V}$. Moreover $\Delta^\pm(z) = \Delta_0(z) + \mathrm{o}\left(z^3\right)$ near 0 in $V^\pm$.*
2. *$H^\pm\left(V^\pm \cap \mathcal{V}\right) = \mathbb{C}^\times$ and $H^\pm\left(\overline{V^\sharp} \cap \mathcal{V}\right) = \left(\overline{\mathbb{C}},\sharp\right)$ for $\sharp \in \{0,\infty\}$.*

(3) *There exists a bijection between orbits of* $\Delta^{\pm}$ *on* $V^{\pm}\cap\mathcal{V}$ *and level sets of* $H^{\pm}|_{V^{\pm}\cap\mathcal{V}}$.

3.2.2. *Sectorial primitive first-integrals.*

Corollary 3.8. *Let* U *be an open subsector of* $V^{\pm}$ *attached to* 0. *The primitive function* $H^{\pm}$ *of* X *is a primitive first-integral of* $\Delta^{\pm}$, *in the sense that any first-integral* L *of* $\Delta^{\pm}$ *on* U *factors as*

$$L=g\circ H^{\pm}$$

for some g *holomorphic on* $H^{\pm}(U)$. *(The converse is trivial.)*

Proof. Let L be such that $L\circ\Delta=L$. Then it induces a holomorphic function $g\ :\ h\mapsto g(h)$ on the orbit space ${}^{U\cap(\mathbb{C},0)}/_{\Delta}$. According to the previous lemma this space embeds as $H^{\pm}(U)\subset\mathbb{C}^{\times}$. Therefore we can assume that the coordinate h is given by $H^{\pm}$, *i.e.* $L=g\circ H^{\pm}$. □

3.3. **Reduction: proof of Proposition 3.3.**

3.3.1. *Item 1: sectorial dynamics and orbit space.* This is the content of Lemma 3.7, hence the item is proved.

3.3.2. *Item 2: gluing condition.*

(a)⇒(b) This is trivial.

(b)⇒(a) The condition $\Delta^{-}=\Delta^{+}$ guarantees that both sectorial functions glue to form a holomorphic germ, still called Δ, on a punctured neighborhood of 0. As Δ is bounded, Riemann's theorem on removable singularity applies: Δ extends holomorphically to $(\mathbb{C},0)$. Finally, since $\Delta_{0}(z)=z+\lambda z^{2}+\mathrm{o}\left(z^{2}\right)$ (Lemma 2.4) and Ψ fixes 0 we conclude $\Delta\in\mathrm{Parab}_{1}$.

(c)⇒(b) The time-1 map of $X^{\pm}$ is Δ^{-} with primitive first-integral $H_{f}^{-}=H_{0}\exp\left(2\mathrm{i}\pi f^{-}\right)=H_{f}^{+}\times\exp\left(2\mathrm{i}\pi\left(f^{-}-f^{+}\right)\right)$. Hence H_{f}^{-} is a first-integral of Δ^{+} as well as of Δ^{-}. Take $z\in V^{\cap}$ a point where both Δ^{-} and Δ^{+} are defined (this property holds on some bounded subsector $V^{\sharp}\cap(\mathbb{C},0)$). Then

$$\begin{aligned}H_{f}^{-}\left(\Delta^{+}(z)\right) & =H_{f}^{-}(z)\\ & =H_{f}^{-}\left(\Delta^{-}(z)\right),\end{aligned}$$

which means that $\Delta^{+}=\left(\Delta^{-}\right)^{\circ\ell}$ for some $\ell\in\mathbb{Z}$, since orbits of Δ^{-} are in 1-to-1 correspondence with level sets of H_{f}^{-}. The integer ℓ may depend on z, but Baire's category theorem together with the principle of analytic continuation assert that ℓ does actually not.
On the one hand Δ^{+} and Δ^{-} coincide with Δ_{0} up to $\mathrm{o}\left(z^{3}\right)$, while on the other hand $\Delta_{0}^{\circ\ell}(z)=z+\ell\lambda z^{2}+\mathrm{o}\left(z^{2}\right)$ for all $z\in(\mathbb{C},0)$. Thus $\ell=1$ and $\Delta^{+}=\Delta^{-}$ on $V^{\cap}$.

(b)⇒(c) The previous argument can actually be read backwards too.

(c)⇔(d) This is the content of Corollary 3.8.

3.3.3. *Item 3: realization.*

(a) By construction the Écalle–Voronin modulus of Δ satisfies

$$\begin{cases} H_f^- = \psi^0 \circ H_f^+ & \text{on } V^0 \cap (\mathbb{C},0) \\ H_f^- = \psi^\infty \circ H_f^+ & \text{on } V^\infty \cap (\mathbb{C},0) \end{cases}.$$

Replacing $H^\pm$ by $H_0 \exp(2\mathrm{i}\pi f^\pm)$, then ψ^0 by $\mathrm{Id}\exp\left(4\pi^2\mu + \varphi^0\right)$ and ψ^∞ by $\mathrm{Id}\exp(\varphi^\infty)$ brings the previous system into $(\star)$.

(b) Near 0 in V^0 the function $H^\pm$ is 1-flat according to (3.1):

$$\begin{aligned}\left|H^\pm(z)\right| &\leqslant |H_0(z)| \exp\left(2\pi\left|f^\pm(z)\right|\right) \\ &\leqslant \exp\left(2\pi - \frac{\varepsilon}{|z|}\right)\end{aligned}$$

for some choice of

$$0 < \varepsilon \leqslant 6 < \frac{2\pi\cos\frac{\pi}{8}}{\lambda} \tag{3.5}$$

since $\lambda \leqslant 1$. Because $\varphi^0(0) = 0$ the composition $\varphi^0 \circ H_f^+$ is also 1-flat at 0, with same 1-type. Therefore $f^- - f^+$ is 1-flat at 0 in V^0, with same 1-type. The same argument also applies in V^∞ (beware that φ^∞ is embodied as a convergent power series in $\frac{1}{H_f^+}$). Then Ramis–Sibuya theorem [LR16, Theorem ##] asserts exactly that (f^+, f^-) is the 1-sum of some $F \in z\mathbb{C}[[z]]$. Being a 1-sum is stable by differentiation so that $(X_0 \cdot f^+, X_0 \cdot f^-)$ is a 1-sum of $X_0 \cdot F$, with same 1-type. Hence $\left(\Phi_{X_0}^{f^+}, \Phi_{X_0}^{f^-}\right)$ is locally[1] a 1-sum of $\Psi := \Phi_{X_0}^F$ (according to Lemma 2.19), from which follows that (Δ^+, Δ^-) is a 1-sum of

$$\Phi_X^1 = \Psi^* \Delta_0.$$

Since $\Delta^+ = \Delta^-$ is a convergent power series at 0, it can only mean that $\Delta = \Phi_X^1$.

4. Synthesis

We begin with fixing a formal class $\mu \in \mathbb{C}$ in Parab_1 (then ψ^0 must be tangent to the linear map $\mathrm{e}^{4\pi^2\mu}\mathrm{Id}$) and pick an analytic data

$$\begin{aligned} \psi^0 \ : \ \left(\overline{\mathbb{C}},0\right) &\longrightarrow \left(\overline{\mathbb{C}},0\right) \\ h &\longmapsto h\exp\left(4\pi^2\mu + \varphi^0(h)\right) \quad , \ \varphi^0(0) = 0 \ , \\ \psi^\infty \ : \ \left(\overline{\mathbb{C}},\infty\right) &\longrightarrow \left(\overline{\mathbb{C}},\infty\right) \\ h &\longmapsto h\exp\left(\varphi^\infty(h)\right) \quad , \ \varphi^\infty(\infty) = 0 \ . \end{aligned}$$

We wish to incarnate the abstract variable h as a concrete coordinate on a sectorial orbit space of a germ $\Delta \in \mathrm{Parab}_1$. In order to perform this task, we seek a pair of functions (H^+, H^-)

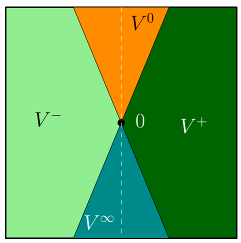
$$H^\pm \ : \ V^\pm \longrightarrow \mathbb{C} \subset \overline{\mathbb{C}}$$

[1] Here the notion of 1-sum differs from that stated in the introduction in that we only have the holomorphy of $\Psi^\pm$ on a small sectorial region near 0. This is sufficient, though.

whose range is biholomorphic to ${}^{V^{\pm}}/_{\Delta}$ and such that $\left(\psi^{0},\psi^{\infty}\right)$ gives the transition between H^{+} and H^{-}:

$$H^{-}=\psi^{\sharp}\circ H^{+}\quad\text{ on }V^{\sharp}\text{ for }\sharp\in\left\{0,\infty\right\}.$$

We recall that the two connected components of the intersection $V^{\cap}$ are sectors defined by



$$V^{0}:=\left\{z\;:\;\Im\left(z\right)>0\right\}\cap V^{\cap},$$
$$V^{\infty}:=\left\{z\;:\;\Im\left(z\right)<0\right\}\cap V^{\cap}.$$

Starting from a pair

$$\varphi:=\left(\varphi^{0},\varphi^{\infty}\right),$$

the idea is to obtain $H^{\pm}$ as a perturbation

$$H^{\pm}:=H_{0}\times\exp\left(2\mathrm{i}\pi f^{\pm}\right)\quad,\;f^{\pm}\in\mathcal{S}$$

of a primitive first-integral of the time-1 map Δ_{0} of the formal model X_{0} (see Section 2.3). According to Proposition 3.3 we need to solve the non-linear Cousin problem

$$(\star)\qquad\begin{cases}f^{-}-f^{+} & =\varphi^{0}\circ H_{f}^{+}\qquad\text{on }V^{0}\cap\left(\mathbb{C},0\right)\\ f^{-}-f^{+} & =\varphi^{\infty}\circ H_{f}^{+}\qquad\text{on }V^{\infty}\cap\left(\mathbb{C},0\right)\end{cases}.$$

We introduce the Cauchy-Heine operator CH^{φ} in Section 4.3, then prove it admits a unique fixed point in the unit ball of $\mathcal{S}$ (Section 4.4). The latter is the sought solution of $(\star)$.

Remark 4.1. The proofs of all the technical lemmas involved below are to be found in Section 4.5.

4.1. **Choice of the formal model and of the sectors.** Before even considering solving the Cousin problem $(\star)$, it should be well-posed. In particular the compositions $\varphi^{0}\circ H_{f}^{+}$ and $\varphi^{\infty}\circ H_{f}^{-}$ must make sense in the respective intersections V^{0} and V^{∞}. This condition is not technical: for a genuine parabolic germ Δ a horn map is defined at least on the maximal domain of orbits that are sent by the local dynamics from one end of the sector V^{+} to the corresponding end of V^{-}.

Roughly speaking, this orbital domain is the range of the corresponding sectorial first-integral H_{f}^{+} of the respective intersection's component V^{0} or V^{∞}. In order to be able to gain control on its size, we impose the technical restriction

$$\left|\left|H_{f}^{+}\right|\right|_{V^{0}}:=\sup_{z\in V^{0}}\left|H_{f}^{+}\left(z\right)\right|<\rho^{0}$$
$$\left|\left|\frac{1}{H_{f}^{+}}\right|\right|_{V^{\infty}}<\rho^{\infty}$$

where ρ^0 and ρ^∞ are the radius of convergence of φ^0 and φ^∞, the former as a power series in $h \in (\mathbb{C}, 0)$ and the latter in $\frac{1}{h} \in (\mathbb{C}, 0)$.

The natural idea that comes up is to use the 1-flatness of sectorial first-integrals at 0: on a small pair of sectors near 0 the above condition can be easily enforced. One then synthesizes Δ on $(\mathbb{C}, 0)$ in her preferred fashion (invoking Ahlfors-Bers theorem or directly by a holomorphic fixed-point). Treading on that path involves technical complications and other shortcomings, least of all the complete lack of a control on the form of the resulting germ Δ. This in turn prevents any foreseeable statement about the uniqueness of Δ.

To address the uniqueness question one is therefore led to synthesize a *global* object Δ. But one has to pay a price for it: any given model first-integral has a given range Ω^0 over V^0 which cannot fit within the domain of *every* germ φ^0. Admittedly one could contract the coordinate h by a linear map, but this would automatically increase the observed size of the orbital domain Ω^∞ near ∞. Hence only rare functions with $\rho^0\rho^\infty$ bounded from below by the «size» of Ω^0 and Ω^∞ can be realized as a perturbation of this first-integral.

Remark 4.2. In particular when the data is unilateral (in Écalle terminology, meaning $\varphi^0 = 0$ or $\varphi^\infty = 0$) it is possible to play the rescaling game on the orbits sphere and obtain a realization without caring about the parameter λ. The same holds if one of the component of φ is entire.

To drive the point home we can obtain bounds for the primitive function

$$\widehat{H}(x) := \exp\left(-\frac{2\mathrm{i}\pi}{x}\right) x^{2\mathrm{i}\pi\mu}$$

of the usual formal model $\frac{x^2}{1+\mu x}\frac{\partial}{\partial x}$.

Lemma 4.3. *Define for* $\tau \in \mathbb{C}^\times$

$$\mathfrak{h}(\tau) := \exp\left(-2\mathrm{i}\pi\left(\tau + \mu \log \tau\right)\right).$$

For every $0 < \delta < \frac{\pi}{2}$ *the following estimates hold:*

$$\left(\forall \tau \ : \ \left|\arg \tau + \frac{\pi}{2}\right| = \delta,\ \mathfrak{t} < |\tau|\right) \qquad |\mathfrak{H}(\tau)| < \mathfrak{m}$$

$$\left(\forall \tau \ : \ \left|\arg \tau - \frac{\pi}{2}\right| = \delta,\ \mathfrak{t} < |\tau|\right) \qquad |\mathfrak{H}(\tau)| > \frac{1}{\mathfrak{m}}$$

where

$$\mathfrak{m} = \mathfrak{m}_{\mu,\delta} := \exp\left(2\pi^2 \left|\Re(\mu)\right| + 2\pi \Im(\mu) \ln \frac{\left|\Im(\mu)\right|}{\mathrm{e}\cos\delta}\right) > 0$$

$$\mathfrak{t} = \mathfrak{t}_{\mu,\delta} := \max\left\{1, \frac{\ln \mathfrak{m}}{2\pi\cos\delta}\right\}.$$

We may think of τ as $\frac{1}{x}$ because $\widehat{H}(x) = \mathfrak{h}\left(\frac{1}{x}\right)$. Hence the above estimate tells us that the size (in the coordinate $\widehat{H}$) of the sectorial orbit spaces Ω_0 and Ω_∞ shrinks to 0 over smaller and smaller sectors of radius $\frac{1}{\mathfrak{t}} \to 0$. But if we want to work on unbounded sectors covering $\mathbb{C}^\times$ we cannot act on the «smallness» of the sectors anymore, we must thereby find another way to control the size of the orbits domains to accommodate a given $\left(\psi^0, \psi^\infty\right)$. This is done by performing the pullback of $\frac{x^2}{1+\mu x}\frac{\partial}{\partial x}$ by a 1-parameter family of rational maps $x := \Pi(z)$ sending

$V^{\cap}$ into sectors of size $\frac{1}{\mathfrak{t}}=\mathrm{O}(\lambda)$ for which the previous lemma gives a bound $\mathfrak{m}\sim\exp\left(-\mathrm{cst}/\lambda\right)$.

Changing the variable $z\mapsto x=\Pi(z)$ in the usual formal model $\frac{x^2}{1+\mu x}\frac{\partial}{\partial x}$ results in a vector field whose time-1 flow has a first-integral of the form $\mathfrak{H}(\tau)$ for $\tau=\tau(z):=\frac{1}{\Pi(z)}$. Namely if

$$\Pi(z)=\frac{\lambda z}{1-z^2}$$

then $\frac{x^2}{1+\mu x}\frac{\partial}{\partial x}$ is transformed into the σ-invariant vector field

$$X_0(z):=\frac{1-z^2}{1+z^2}\times\frac{\lambda z^2}{1+\mu\lambda z-z^2}\frac{\partial}{\partial z}$$

and τ into

$$\tau=\tau_\lambda(z):=\frac{1-z^2}{\lambda z}$$

yielding the first-integral

$$\begin{aligned}H_0(z)&=\widehat{H}\left(\frac{\lambda z}{1-z^2}\right)=\mathfrak{H}\left(\tau_\lambda(z)\right)\\&=\exp\left(-2\mathrm{i}\pi\frac{1-z^2}{\lambda z}-2\mathrm{i}\pi\mu\log\frac{1-z^2}{\lambda z}\right).\end{aligned}\tag{4.1}$$

Because $\tau\circ\sigma=\tau$, all the above objects are σ-invariant. We establish now that this first-integral fulfills the expected properties on the pair of unbounded sectors $V^{\pm}$. Firstly, we ensure that $\tau(V^{\cap})$ is included in an open sector of opening $\frac{\pi}{2}$ with vertex at ∞.

Lemma 4.4. *Fix $\lambda\leqslant 1$ and set $\tau=\tau_\lambda(z):=\frac{1-z^2}{\lambda z}$. Then*

$$\begin{cases}\inf_{z\in V^{\cap}}|\tau| & \geqslant\frac{1}{\lambda}\\ \sup_{z\in V^{0}}\left|\arg\tau+\frac{\pi}{2}\right| & =\frac{3\pi}{8}\\ \sup_{z\in V^{\infty}}\left|\arg\tau-\frac{\pi}{2}\right| & =\frac{3\pi}{8}\end{cases}.$$

Secondly, we adapt the bounds of Lemma 4.3 for $\delta:=\frac{3\pi}{8}$ to obtain the sought values of $\mathfrak{m}_\mu$ and $\mathfrak{t}_\mu$ (we recall that $2\cos\frac{3\pi}{8}=\sqrt{2-\sqrt{2}}$):

$$\begin{aligned}\mathfrak{m}_\mu&:=\exp\left(2\pi^2\left|\Re(\mu)\right|+2\pi\Im(\mu)\ln\frac{4\left|\Im(\mu)\right|}{\mathrm{e}\sqrt{2-\sqrt{2}}}\right)\\\mathfrak{t}_\mu&:=\max\left\{1,\frac{\ln\mathfrak{m}_\mu}{\pi\sqrt{2-\sqrt{2}}}\right\}\end{aligned}\tag{4.2}$$

as testified by the concluding result of this section.

Corollary 4.5. *Pick $0<\lambda<\frac{1}{\mathfrak{t}_\mu}$ and define*

$$\begin{aligned}R_\lambda&:=\inf_{z\in V^{\infty}}\left|H_0(z)\right|,\\r_\lambda&:=\sup_{z\in V^{0}}\left|H_0(z)\right|.\end{aligned}$$

The following estimate holds:

$$r_\lambda \leqslant \frac{\mathfrak{m}_\mu}{\exp {}^1/\lambda} \leqslant \frac{\exp {}^1/\lambda}{\mathfrak{m}_\mu} \leqslant R_\lambda.$$

In fact, one has more precisely

$$\begin{cases} |H_0(z)| < \mathfrak{m}_\mu \mathrm{e}^{-|\tau_\lambda(z)|} & \text{for } z \in V^0 \\ |H_0(z)| > \frac{1}{\mathfrak{m}_\mu} \mathrm{e}^{|\tau_\lambda(z)|} & \text{for } z \in V^\infty \end{cases}.$$

Proof. The constants $\mathfrak{m}_\mu^2$ and $\mathfrak{t}_\mu$ are related to that of Lemma 4.3 but for 2μ. Indeed, we have for $\left|\arg \frac{\tau}{\mathrm{i}}\right| < \frac{3\pi}{8}$:

$$\begin{aligned} |\mathfrak{H}(\tau)|^2 &= |\exp(-2\mathrm{i}\pi\tau)| \times |\exp(-2\mathrm{i}\pi(\tau + 2\mu\log\tau))| \\ &\geqslant \exp\left(|\tau|\,\pi\sqrt{2-\sqrt{2}}\right) \times |\exp(-2\mathrm{i}\pi(\tau + 2\mu\log\tau))| \end{aligned}$$

which becomes (since $\pi\sqrt{2-\sqrt{2}} > 2$):

$$|\mathfrak{H}(\tau)|^2 \geqslant \exp(2|\tau|)\frac{1}{\mathfrak{m}_\mu^2}$$

whenever $|\tau| > \mathfrak{t}_\mu$. The latter condition is ensured as soon as $\frac{1}{\lambda} > t_\mu$ thanks to Lemma 4.4. Under this hypothesis we finally obtain:

$$|H_0(z)| = |\mathfrak{h}(\tau)| \geqslant \frac{1}{\mathfrak{m}_\mu}\exp|\tau| \geqslant \frac{\exp {}^1/\lambda}{\mathfrak{m}_\mu}.$$

The case $\left|\arg \frac{\tau}{-\mathrm{i}}\right| < \frac{3\pi}{8}$ is similar. □

4.2. **Functions adapted to a data.** Let us translate in functions space the previous discussion.

Definition 4.6. As in the beginning of the section, we define the intersection

$$V^\cap := V^+ \cap V^- = V^0 \sqcup V^\infty.$$

(1) Let $U \subset \mathbb{C}$ be a domain. We introduce the Banach space

$$\mathrm{Holo_c}(U)$$

of holomorphic, bounded functions f on U with continuous extension to the closure $\overline{U}$, equipped with the sup norm

$$||f||_U := \sup|f(U)|.$$

(2) As a particular case we will be interested in

$$\mathcal{S}(V^\pm) := \left\{ f^\pm \in \mathrm{Holo_c}(V^\pm) \ : \ \begin{matrix} f^\pm(0) = 0 \\ f^\pm(\infty) = 0 \end{matrix} \ , \ ||f^\pm|| := \sup_{z\in V^\pm} \left|f^\pm(z)\right| < \infty \right\}.$$

(3) We define $\mathcal{S}$ as the space of pairs $f = (f^+, f^-)$ with 1-flat difference in $V^\cap$ and bounded 1-type, both at 0 and ∞:

$$\mathcal{S} := \left\{ f \in \mathcal{S}(V^+) \times \mathcal{S}(V^-) \ : \ \limsup_{z^{\pm1}\to 0} |z|^{\pm1} \ln\left|f^-(z) - f^+(z)\right| \leqslant -6 \text{ for } z \in V^\cap \right\},$$

equipped with the canonical product Banach norm. We denote by $\mathcal{B}$ its unit ball.

(4) For $f \in \mathcal{S}$ we define the associated **sectorial first-integral** $H_f = \left(H_f^+, H_f^-\right)$ given by

$$H^{\pm} := H_0 \exp\left(2\mathrm{i}\pi f^{\pm}\right).$$

(5) Let $\varphi = \left(\varphi^0, \varphi^\infty\right)$ be given. We say that (λ, f) is **adapted** to φ whenever $\overline{H_f^+\left(V^\sharp\right)}$ is included in the (open) disc of convergence of $\varphi^\sharp$ for $\sharp \in \{0, \infty\}$. We define for all $\lambda > 0$

$$\mathrm{Adapt}_\lambda\left(\varphi\right) := \left\{f \in \mathcal{S} \ : \ (\lambda, f) \text{ is adapted to } \varphi\right\}.$$

We have, with a corresponding estimate for φ^∞:

$$(\forall h \in (\mathbb{C}, 0)) \qquad \left|\varphi^0\left(h\right)\right| \leqslant |h| \left|\left|\frac{\mathrm{d}\varphi^0}{\mathrm{d}h}\right|\right|_{(\mathbb{C},0)} < +\infty. \tag{4.3}$$

Proposition 4.7. *Choose a data $\varphi = \left(\varphi^0, \varphi^\infty\right)$. The subspace $\mathrm{Adapt}_\lambda\left(\varphi\right)$ is an open set of $\mathcal{S}$ and if λ is small enough, then it contains the unit ball:*

$$\mathcal{B} \subset \mathrm{Adapt}_\lambda\left(\varphi\right).$$

More generally, being given $f \in \mathcal{S}$ it is always possible to take λ small enough to ensure that $f \in \mathrm{Adapt}_\lambda\left(\varphi\right)$. (The following remark gives quantitative bounds.)

Remark 4.8. Denote by $\rho^\sharp \in]0, +\infty]$ for $\sharp \in \{0, \infty\}$ the radius of convergence of $\varphi^\sharp$. Quantitatively the following conditions ensure that $f \in \mathrm{Adapt}_\lambda\left(\varphi\right)$

$$\lambda < \frac{1}{\mathfrak{t}_\mu}$$

$$2\pi\left|\left|f\right|\right| < \frac{1}{\lambda} + \ln\frac{\min\left\{\rho^0, \rho^\infty\right\}}{\mathfrak{m}_\mu},$$

since according to Corollary 4.5 this implies

$$\inf\left|H_f^+\left(V^\infty\right)\right| > \frac{1}{\rho^\infty}$$

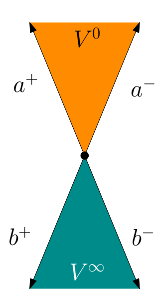
$$\sup\left|H_f^+\left(V^0\right)\right| < \rho^0.$$

In particular $\varphi^\sharp \circ H_f^+ \in \mathrm{Holo_c}\left(V^\sharp\right)$.

Proof. The mapping $f \in \mathcal{S} \mapsto H_f^+ \in \mathrm{Holo_c}\left(V^\sharp\right)$ is continuous, and for each $f \in \mathrm{Adapt}_\lambda\left(\varphi\right)$ the set $\overline{H_f^+\left(V^\sharp\right)}$ is compact. Thus $\mathrm{Adapt}_\lambda\left(\varphi\right)$ is open in $\mathcal{S}$. The remark just above precisely states that for any $r > 0$ the ball $r\mathcal{B}$ is included in $\mathrm{Adapt}_\lambda\left(\varphi\right)$ whenever

$$\frac{1}{\lambda} + \ln\frac{\min\left\{\rho^0, \rho^\infty\right\}}{\mathfrak{m}_\mu} > 2\pi r.$$

□



4.3. **Cauchy-Heine transform.** Now that H_0 and the sectors have been defined we tackle the Cousin problem itself with data $\varphi := \left(\varphi^0, \varphi^\infty\right)$ as in $(\star)$. Let us define the integral transform which is the key to the construction.

Definition 4.9. Let $\varphi := \left(\varphi^0, \varphi^\infty\right)$ be given. Assume that (λ, f) is adapted to it. Let $a^\pm$, $b^\pm$ be the outward-going half-lines making up the boundary $\partial V^\pm$ as in the side figure. We define $\Lambda_f := \left(\Lambda_f^+, \Lambda_f^-\right)$ where

$$\Lambda_f^\pm(z) := \frac{\sqrt{z}}{2\mathrm{i}\pi}\int_{a^\pm}\frac{\varphi^0\left(H_f^+(\xi)\right)}{\sqrt{\xi}\,(\xi - z)}\mathrm{d}\xi - \frac{\sqrt{z}}{2\mathrm{i}\pi}\int_{b^\pm}\frac{\varphi^\infty\left(H_f^+(\xi)\right)}{\sqrt{\xi}\,(\xi - z)}\mathrm{d}\xi.$$

Before stating anything about Λ_f we need to get convinced that it is well-defined. For the sake of example, let us deal with the $\int_a$-part. According to the identity (4.3) we may find some $C = \left|\left|\frac{\mathrm{d}\varphi}{\mathrm{d}h}\right|\right|_{H_f^+(V^\cap)} \geqslant 0$ such that for all $h \in H_f^+(a^\pm)$ we have

$$\left|\varphi^0(h)\right| \leqslant C\,|h|\,.$$

Since we have for all $\xi \in a$

$$\left|H_f^+(\xi)\right| \leqslant |H_0(\xi)|\exp\left(2\pi\left|\left|f^+\right|\right|_{V^0}\right), \tag{4.4}$$

and because $|H_0|$ is flat at 0 and ∞ along $a^\pm$ and $b^\pm$, the integrals defining $\Lambda_f^\pm$ are absolutely convergent. That being said, as we want to control the magnitude of $\left|\Lambda_f^\pm\right|$ with respect to both the point $z \in V^\pm$ and the parameter λ, we need to work a little bit more.

Lemma 4.10. *The model first-integral H_0 as given in (4.1) satisfies*

$$(\forall z\ :\ \Re(z) \geqslant 0) \qquad \int_{a^+}\left|\frac{\sqrt{z}H_0(\xi)\,\mathrm{d}\xi}{\sqrt{\xi}\,(\xi - z)}\right| < \frac{3\mathfrak{m}_\mu}{2}\lambda^2.$$

Identical bounds for $\int_{a^-}$ and $\int_{b^\pm}$ also hold.

We are now ready to prove the main properties of the operator Λ_f.

Proposition 4.11. *Let $\varphi := \left(\varphi^0, \varphi^\infty\right)$ be given and assume that $(\lambda, f) \in \mathbb{R}_{>0} \times \mathcal{S}$ is adapted to it. Let $\Lambda_f^\pm$ be as in the definition and recall that σ is the involution $z \mapsto \frac{-1}{z}$.*

(1) *For all $z \in V^\pm$ we have*

$$\Lambda_f^\mp \circ \sigma(z) = -\Lambda_{f\circ\sigma}^\pm(z)\,.$$

(2) *For all $z \in V^\cap$ we have*

$$\Lambda_f^- - \Lambda_f^+ = \begin{cases}\varphi^0 \circ H_f^+ & \textit{on } V^0\\ \varphi^\infty \circ H_f^+ & \textit{on } V^\infty\end{cases}.$$

(3) *$\Lambda_f^\pm \in \mathrm{Holo_c}\left(V^\pm\right)$ and*

$$\left|\left|\Lambda_f^\pm\right|\right|_{V^\pm} \leqslant 4\mathfrak{m}_\mu\lambda^2 \times \mathrm{e}^{2\pi\left|\left|f^+\right|\right|_{V^\cap}} \times \max\left\{\left|\left|\frac{\mathrm{d}\varphi^0}{\mathrm{d}h}\right|\right|_{H_f^+(V^0)}, \left|\left|\frac{\mathrm{d}\varphi^\infty}{\mathrm{d}h}\right|\right|_{H_f^+(V^\infty)}\right\}.$$

Proof.

(1) Since $H_0 \circ \sigma = H_0$ we have $H_f^{\pm} \circ \sigma = H_{f\circ\sigma}^{\mp}$. The rest follows from applying the straightforward change of variable $u := \sigma(\xi)$, which permutes $a^{\pm} \leftrightarrow -a^{\mp}$ and $b^{\pm} \leftrightarrow -b^{\mp}$. If $z \in V^-$ we have $\sigma(z) = \frac{-1}{z} \in V^+$ and

$$\begin{aligned}\Lambda_f^+(\sigma(z)) =& \frac{1}{2\mathrm{i}\pi}\int_{a^+}\frac{\sqrt{-z\xi}\varphi^0\left(H_f^+(\xi)\right)}{-z\xi-1}\frac{\mathrm{d}\xi}{\xi}\\ &+\left(\text{same integral with }\varphi^\infty\text{ over }b^+\right)\\ =&\frac{1}{2\mathrm{i}\pi}\int_{-a^-}\frac{\sqrt{\frac{z}{u}}\varphi^0\left(H_f^+(\sigma(u))\right)}{1-\frac{z}{u}}\frac{\mathrm{d}u}{u}\\ &+\left(\text{same integral with }\varphi^\infty\text{ over }-b^-\right)\\ =&-\Lambda_{f\circ\sigma}^-(z).\end{aligned}$$

(2) is a consequence of Cauchy's formula. Assume for the sake of example that $z \in V^0$ and pick $\varepsilon > 0$ so small that

$$z \in V_\varepsilon^0 := \left\{x \in V^0 \ : \ \varepsilon < |x| < \frac{1}{\varepsilon}\right\}.$$

Hence (with the direct orientation on the boundary)

$$\int_{\partial V_\varepsilon^0}\frac{\varphi^0\left(H_f^+(\xi)\right)}{\sqrt{\xi}(\xi-z)}\mathrm{d}\xi = \frac{2\mathrm{i}\pi}{\sqrt{z}}\varphi^0\left(H_f^+(z)\right)$$

and with corresponding notations

$$\int_{\partial V_\varepsilon^\infty}\frac{\varphi^\infty\left(H_f^+(\xi)\right)}{\sqrt{\xi}(\xi-z)}\mathrm{d}\xi = 0.$$

Because $\varphi^0\left(H_f^+(\xi)\right)$ is exponentially flat near 0 and ∞ in the sector V^0 these identities hold at the limit $\varepsilon \to 0$, so that

$$\Lambda_f^-(z) - \Lambda_f^+(z) = \frac{\sqrt{z}}{2\mathrm{i}\pi}\int_{\partial V^0}\frac{\varphi^0\left(H_f^+(\xi)\right)}{\sqrt{\xi}(\xi-z)}\mathrm{d}\xi + \frac{\sqrt{z}}{2\mathrm{i}\pi}\int_{\partial V^\infty}\frac{\varphi^\infty\left(H_f^+(\xi)\right)}{\sqrt{\xi}(\xi-z)}\mathrm{d}\xi$$

yields the expected result.

(3) When $z \in V^+$ with $\Re(z) \geqslant 0$ the bound

$$\left|\Lambda_f^{\pm}(z)\right| \leqslant \max\left\{\left|\left|\frac{\mathrm{d}\varphi^0}{\mathrm{d}h}\right|\right|_{H_f^+(V^0)}, \left|\left|\frac{\mathrm{d}\varphi^\infty}{\mathrm{d}h}\right|\right|_{H_f^+(V^\infty)}\right\}\exp\left(2\pi\left|\left|f^+\right|\right|_{V^\cap}\right) \times 3\mathfrak{m}_\mu\lambda^2$$

is obtained by putting together the estimates of (4.3), (4.4) and twice Lemma 4.10 (once for a and once for b). According to 1., the same argument proves the estimate for $\left|\Lambda_f^-(z)\right|$ when $\Re(z) \leqslant 0$ and $z \in V^-$.

We must now deal with the case $z \in V^+$ and $\Re(z) < 0$. We use the following trick: since we just proved that $\left|\Lambda_f^-(z)\right|$ satisfies the expected estimate on V^-, and because $\Lambda_f^+ = \Lambda_f^- - \varphi^\sharp \circ H_f^+$ we have

$$\begin{aligned}\left|\Lambda_f^+(z)\right| &\leqslant \left|\Lambda_f^-(z)\right| + \left|\left|\frac{\mathrm{d}\varphi^\sharp}{\mathrm{d}h}\right|\right|_{H_f^+(V^\sharp)} \left|H_f^+(z)\right| \\ &\leqslant \left|\Lambda_f^-(z)\right| + \left|\left|\frac{\mathrm{d}\varphi^\sharp}{\mathrm{d}h}\right|\right|_{H_f^+(V^\sharp)} \exp\left(2\pi\left|\left|f^+\right|\right|\right) \times \frac{\mathfrak{m}_\mu}{\exp{}^1\!/\!_\lambda}\end{aligned}$$

according to Corollary 4.5. The conclusion follows from $\frac{1}{\exp{}^1\!/\!_\lambda} \leqslant 4\lambda^2\mathrm{e}^{-2}$ and $4\mathrm{e}^{-2} + 3 < 4$.

□

Definition 4.12. We call

$$\begin{aligned}\mathrm{CH}^\varphi \;:\; \mathrm{Adapt}_\lambda(\varphi) &\longrightarrow \mathcal{S} \\ f &\longmapsto \Lambda_f - \Lambda_f(0)\end{aligned}$$

the **Cauchy-Heine transform**.

We deduce the following facts from the items of Proposition 4.11.

- From 1. and thanks to the σ-action we obtain
$$0 = \mathrm{CH}^\varphi(f)^\pm(0) = \mathrm{CH}^\varphi(f)^\pm(\infty).$$
Moreover if f is a fixed-point of CH^φ then
$$f^\pm \circ \sigma = -f^\mp. \tag{4.5}$$
- From 2. we find that $\Lambda_f^+(0) = \Lambda_f^-(0)$, hence $\mathrm{CH}^\varphi(f)$ also solves the Cousin problem
$$\mathrm{CH}^\varphi(f)^- - \mathrm{CH}^\varphi(f)^+ = \varphi \circ H_f^+.$$
Moreover $\left(\Lambda_f^+, \Lambda_f^-\right)$ is 1-flat with 1-type bounded from above by that of H_f^+, that is $-\frac{6}{\lambda}$.
- From 3. we derive the estimate

$$\left|\left|\mathrm{CH}^\varphi(f)\right|\right| \leqslant 8\mathfrak{m}_\mu\lambda^2 \times \mathrm{e}^{2\pi\left|\left|f^+\right|\right|_{V^\cap}} \times \max\left\{\left|\left|\frac{\mathrm{d}\varphi^0}{\mathrm{d}h}\right|\right|_{H_f^+(V^0)}, \left|\left|\frac{\mathrm{d}\varphi^\infty}{\mathrm{d}h}\right|\right|_{H_f^+(V^\infty)}\right\}. \tag{4.6}$$

4.4. **Convergence of the fixed-point method.** From what we established in Proposition 4.11 we can derive more interesting properties of CH^φ, which will allow us to iterate it provided λ be small enough. Define for $\varphi \in h\mathbb{C}\{h\}$, and when it makes sense,

$$\left|\left|\varphi\right|\right|_\lambda := \sup_{|z| \leqslant \mathfrak{m}_\mu \exp(2\pi - {}^1\!/\!_\lambda)} \left|\varphi'(z)\right|.$$

For given φ it decreases to $|\varphi'(0)|$ as $\lambda \to 0$. For given $\lambda > 0$ it is a norm on the Banach space of bounded and holomorphic functions on the disc $\mathfrak{m}_\mu \exp(2\pi - {}^1\!/\!_\lambda)\mathbb{D}$ vanishing at 0. Moreover $|\varphi(h)| \leqslant |h| \left|\left|\varphi\right|\right|_\lambda$ for all h in the disc.

Proposition 4.13. *Let $\mathcal{B}$ stand for the closed unit ball in the Banach space $\mathcal{S}$ (as in Section 4.2). Being given $\varphi := \left(\varphi^0, \varphi^\infty\right)$ define*

$$\ell = \ell\left(\varphi\right) := \max\left\{1, \frac{1}{\mathfrak{t}_\mu}, \frac{1}{2\pi + \ln\frac{\mathfrak{m}_\mu}{\min\{\rho^0,\rho^\infty\}}}\right\}.$$

For $0 < \lambda \leqslant \ell$ let

$$\kappa_\lambda := 8\mathfrak{m}_\mu \lambda^2 \max\left\{\left|\left|\varphi^0\right|\right|_\ell, \left|\left|\varphi^\infty\right|\right|_\ell\right\}$$
$$r_\lambda := 536\kappa_\lambda$$

(which is well-defined since φ^0 and φ^∞ are holomorphic and bounded on a disc of radius at least $\mathfrak{m}_\mu \exp\left(2\pi - 1/\ell\right)$). The transform

$$CH^\varphi \ : \ \mathcal{B} \longrightarrow r_\lambda \mathcal{B}$$

is a well-defined $2\kappa_\lambda$-Lipschitz map. In particular if $\kappa_\lambda < \frac{1}{2}$ then it is a contracting self-map of $r_\lambda\mathcal{B}$.

Proof. Clearly $\lim_{\lambda\to 0}\kappa_\lambda = 0$. The fact that CH^φ ranges in $r_\lambda\mathcal{B}$ comes from (4.6), since $\exp\left(2\pi\left|\left|f^+\right|\right|_{V^\cap}\right) \leqslant \mathrm{e}^{2\pi} < 536$.

Observe next that the assumption made on λ guarantees that the condition $\mathfrak{m}_\mu \exp\left(2\pi - 1/\lambda\right) < \rho^0$ is met, hence

$$\left|\varphi^0\left(H_f^+\left(\xi\right)\right)\right| \leqslant \left|\left|\varphi^0\right|\right|_\lambda \left|H_f^+\left(\xi\right)\right| \leqslant \left|\left|\varphi^0\right|\right|_\ell \left|H_f^+\left(\xi\right)\right|$$

because

$$\left|H_f^+\left(\xi\right)\right| < \mathfrak{m}_\mu \exp\left(-1/\lambda\right)\exp\left(2\pi\left|\left|f^+\right|\right|_{V^\cap}\right) \leqslant \mathfrak{m}_\mu \exp\left(2\pi - 1/\lambda\right) \leqslant \mathfrak{m}_\mu \exp\left(2\pi - 1/\ell\right).$$

Being given $f_1,\ f_2 \in \mathcal{B}$ and $z \in V^0$ we thereby derive the bound:

$$\begin{aligned}\left|\Lambda_{f_1}^{\pm}\left(z\right) - \Lambda_{f_2}^{\pm}\left(z\right)\right| &\leqslant \frac{\left|\sqrt{z}\right|}{2\pi}\left|\left|\varphi^0\right|\right|_\ell \int_{\partial V^+}\left|\exp\left(2\mathrm{i}\pi f_1^+\left(\xi\right)\right) - \exp\left(2\mathrm{i}\pi f_2^+\left(\xi\right)\right)\right|\left|\frac{H_0\left(\xi\right)}{\sqrt{\xi}\left(\xi - z\right)}\right|\left|\mathrm{d}\xi\right| \\ &\leqslant \mathrm{e}\left|\sqrt{z}\right|\left|\left|\varphi^0\right|\right|_\ell \int_{\partial V^+}\left|f_1^+\left(\xi\right) - f_2^+\left(\xi\right)\right|\left|\frac{H_0\left(\xi\right)}{\sqrt{\xi}\left(\xi - z\right)}\right|\left|\mathrm{d}\xi\right| \\ &\leqslant \frac{\mathrm{e}}{2}\left|\left|\varphi^0\right|\right|_\ell \left|\left|f_1 - f_2\right|\right| \times 4\mathfrak{m}_\mu\lambda^2,\end{aligned}$$

the last step coming from Lemma (4.10). Hence

$$\begin{aligned}\left|\mathrm{CH}^\varphi\left(f_1\right)\left(z\right) - \mathrm{CH}^\varphi\left(f_2\right)\left(z\right)\right| &\leqslant \left|\Lambda_{f_1}^{\pm}\left(z\right) - \Lambda_{f_2}^{\pm}\left(z\right)\right| + \left|\Lambda_{f_1}^{\pm}\left(0\right) - \Lambda_{f_2}^{\pm}\left(0\right)\right| \\ &\leqslant \frac{\mathrm{e}}{2}\left|\left|\varphi^0\right|\right|_\ell \left|\left|f_1 - f_2\right|\right| \times 8\mathfrak{m}_\mu\lambda^2.\end{aligned}$$

The case of φ^∞ is completely similar. □

Corollary 4.14. *Let $\ell := \max\left\{1, \frac{1}{\mathfrak{t}_\mu}, \frac{1}{2\pi+\ln\frac{\mathfrak{m}_\mu}{\min\{\rho^0,\rho^\infty\}}}\right\}$ and take*

$$\lambda < \min\left\{\ell, \frac{1}{4\sqrt{\mathfrak{m}_\mu \max\left\{\left|\left|\varphi^0\right|\right|_\ell, \left|\left|\varphi^\infty\right|\right|_\ell\right\}}}\right\}.$$

The map $CH^\varphi|_{\mathcal{B}}$ admits a unique fixed-point $f \in \mathcal{B}$, obtained for instance by considering the CH^φ-orbit of 0. Moreover

$$\left|\left|f\right|\right| \leqslant \kappa_\lambda \exp\left(3365\kappa_\lambda\right).$$

Remark 4.15. The bound $\kappa_\lambda \exp(3365\kappa_\lambda) \to 0$ is marginally sharper as $\lambda \to 0$ than $||f|| \leqslant r_\lambda = 536\kappa_\lambda$.

Proof. Well, this is just Banach's theorem. The bound on the norm of the fixed-point f comes from the fact that

$$||f|| = ||\mathrm{CH}^\varphi(f)|| \leqslant \kappa_\lambda \exp(2\pi ||f||)$$

(Proposition 4.11) and that $||f|| \leqslant r_\lambda = 536\kappa_\lambda$. □

4.5. **Proof of the lemmas.**

4.5.1. *Proof of Lemma 4.3.* We need to bound $\mathfrak{H}$ on the half-lines $\tau = \mathrm{i}t\theta$ for $t > 0$ and for fixed $\theta \in \mathbb{S}^1$ with $\arg\theta = \pm\delta$. Define

$$\begin{aligned} M(t) &:= |\mathfrak{H}(\tau)| = \exp(2\pi\Im(\tau + \mu\log\tau)) \\ &= \exp\left(2\pi\left(t\cos\delta + \Im(\mu)\ln t + \left(\frac{\pi}{2} \pm \delta\right)\Re(\mu)\right)\right). \end{aligned}$$

An extremum is reached only if

$$t = -\frac{\Im(\mu)}{\cos\delta} > 0.$$

- If $\Im(\mu) = 0$ the function M increases from $M(0)$ to $+\infty$ as t goes from 0 to $+\infty$:
$$M(0) = \exp\left(2\pi\Re(\mu)\left(\frac{\pi}{2} \pm \delta\right)\right) > \exp\left(\pi^2\Re(\mu) - \pi^2|\Re(\mu)|\right),$$
therefore $M(0) \geqslant \exp\left(-2\pi^2|\Re(\mu)|\right) = \frac{1}{\mathfrak{m}}$.
- If $\Im(\mu) < 0$ the minimum of M is bounded from below by
$$\begin{aligned} M\left(-\frac{\Im(\mu)}{\cos\delta}\right) &\geqslant \exp\left(-2\pi\Im(\mu)\left(\ln\left|\frac{\Im(\mu)}{\cos\delta}\right| - 1\right) + \pi^2\Re(\mu) - \pi^2|\Re(\mu)|\right) \\ &\geqslant \frac{1}{\mathfrak{m}}. \end{aligned}$$
- If $\Im(\mu) > 0$ the function M increases from 0 to $+\infty$ as t runs along $\mathbb{R}_{>0}$. The unique $t > 0$ such that $M(t) = \frac{1}{\mathfrak{m}}$ satisfies
$$t\cos\delta + \Im(\mu)\ln t = \Im(\mu)\left(\ln\frac{\Im(\mu)}{\cos\delta} - 1\right) + \pi|\Re(\mu)| - \left(\frac{\pi}{2} \pm \delta\right)\Re(\mu) \leqslant \mathfrak{t}\cos\delta.$$
If $t \geqslant 1$ then $t\cos\delta + \Im(\mu)\ln t \geqslant t\cos\delta$, therefore for all $t > \mathfrak{t}$ we have $M(t) > M(\mathfrak{t}) \geqslant \frac{1}{\mathfrak{m}}$.

The case $t < 0$ is taken care of similarly.

4.5.2. *Proof of Lemma 4.4.* Assume that $z = t\theta$ with $t > 0$ and $\theta := \mathrm{i}\mathrm{e}^{\mathrm{i}\eta}$ for $\eta \in \left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$, *i.e.* $z \in V^0$. Then

$$\arg\frac{\tau}{-\mathrm{i}} = \arg\left(1 + t^2\mathrm{e}^{2\mathrm{i}\eta}\right) - \eta.$$

Because $-\frac{\pi}{2} < 2\eta < \frac{\pi}{2}$ we have

$$\left|\arg\left(1 + t^2\mathrm{e}^{2\mathrm{i}\eta}\right)\right| \leqslant 2\eta$$

and $\arg\left(1+t^2\mathrm{e}^{2\mathrm{i}\eta}\right)\to_{+\infty}2\eta$ monotonically. On the one hand we obtain, with sharp bounds,

$$-\frac{3\pi}{8}\leqslant-3\eta\leqslant\arg\frac{\tau}{-\mathrm{i}}\leqslant\eta\leqslant3\eta\leqslant\frac{3\pi}{8}.$$

On the other hand,

$$|\tau|=\frac{\left|1+t^2\mathrm{e}^{2\mathrm{i}\eta}\right|}{\lambda t}\geqslant\frac{1+\cos\left(2\eta\right)t^2}{\lambda t}\geqslant\frac{1+\frac{1}{\sqrt{2}}t^2}{\lambda t}\tag{4.7}$$

and basic calculus yields

$$(\forall t>0)\qquad\frac{1+\frac{1}{\sqrt{2}}t^2}{\lambda t}\geqslant\frac{3}{\lambda\sqrt{2\sqrt{2}}}>\frac{1}{\lambda}.$$

4.5.3. *Proof of Lemma (4.10).* For the sake of concision we only deal with the case $\int_a\frac{H_0(\xi)}{\sqrt{\xi}(\xi-z)}\mathrm{d}\xi$ where $z\in U:=\left\{z\neq0\ :\ \Re\left(z\right)\geqslant0\right\}$ and $a=\mathrm{e}^{\mathrm{i}\frac{5\pi}{8}}\mathbb{R}_{>0}$. For $\Re\left(z\right)\geqslant0$ let us define

$$d\left(z\right):=\sup\left\{\frac{\sqrt{|\xi|}}{|\xi-z|}\ :\ \xi\in a\right\}\in\left]0,+\infty\right[.$$

We have $\frac{1}{\left|\sqrt{\xi}(\xi-z)\right|}\leqslant\frac{d(z)}{|\xi|}$ so that writing $\xi=\mathrm{e}^{\mathrm{i}5\pi/8}t$ for $t>0$ yields:

$$\left|\frac{H_0\left(\xi\right)\mathrm{d}\xi}{\sqrt{\xi}\left(\xi-z\right)}\right|\leqslant\mathfrak{m}_\mu d\left(z\right)\times\frac{1}{t}\exp\frac{1+\frac{1}{\sqrt{2}}t^2}{-\lambda t}\mathrm{d}t$$

(Corollary 4.5 and bound (4.7)) while

$$\int_0^{+\infty}\frac{1}{t}\exp\left(-\frac{1}{\lambda t}-\frac{1}{\lambda\sqrt{2}}t\right)\mathrm{d}t\leqslant\max_{t>0}\left(\frac{1}{t\exp\frac{1}{\lambda t}}\right)\int_0^{+\infty}\exp\left(-\frac{1}{\lambda\sqrt{2}}t\right)\mathrm{d}t=\frac{\lambda}{\mathrm{e}}\times\lambda\sqrt{2}.$$

In order to conclude we need to bound $d\left(z\right)$. First, we claim that $|\xi-z|\geqslant|\xi-\mathrm{i}\,|z||$ for all $\xi\in a$ and $\Re\left(z\right)\geqslant0$, so that we may as well assume that $z\in\mathrm{i}\mathbb{R}_{>0}$. Next, elementary calculus provides the bound $d\left(z\right)=\frac{1}{\sqrt{2|z|}\sqrt{1-\cos\frac{\pi}{8}}}$, completing the proof since $\frac{1}{\mathrm{e}\sqrt{1-\cos\frac{\pi}{8}}}<\frac{3}{2}$.

5. Globalization Theorem

This section is devoted to the proof of the Globalization Theorem. We prove Items 1.–3. in Section 5.3 below but begin with establishing Item 4. in Section 5.1, which introduces the needed material about the action of the involution $\sigma=\frac{-1}{\mathrm{Id}}$.

Every object O involved in the paper comes as a pair of sectorial objects $O=(O^+,O^-)$, each component being meromorphic on the corresponding $V^\pm$. The action of

$$\begin{aligned}\sigma\ :\ \overline{\mathbb{C}}&\longrightarrow\overline{\mathbb{C}}\\ z&\longmapsto-\frac{1}{z}\end{aligned}$$

on such objects is defined as

$$\sigma^{\circledast}O:=\left(\sigma^*O^-,\sigma^*O^+\right)$$

where σ^* is just the usual action on holomorphic objects by change of coordinate:

on functions $\sigma^*g:=g\circ\sigma$;

on biholomorphisms $\sigma^*\phi := \sigma^{\circ-1} \circ \phi \circ \sigma$;
on vector fields $\sigma^* X := \mathrm{D}\sigma^{-1} (X \circ \sigma)$.
In what follows every object will be $\sigma^{\circledast}$**-invariant**:

$$\sigma^{\circledast} O_f = O_{\sigma^{\circledast} f},$$

as is the case for example of the sectorial first-integrals (because $H_0 \circ \sigma = H_0$ we have $\sigma^{\circledast} H_f = H_{\sigma^{\circledast} f}$), but for the Cauchy-Heine transform (Proposition 4.11 and identity (4.5)) which is $\sigma^{\circledast}$**-antivariant** since $\sigma^{\circledast}\mathrm{CH}^{\varphi}(f) = -\mathrm{CH}^{\varphi}(\sigma^{\circledast} f)$.

5.1. **Modulus at ∞.** Let $f \in \mathcal{B}$ in the unit ball of $\mathcal{S}$ give the parabolic realization Δ_f of some $\psi_f = \left(\psi_f^0, \psi_f^\infty\right)$. According to Proposition 3.3 we know that $f^- - f^+$ is a first-integral of Δ_f on the whole $V^\cap$, in particular near ∞. Therefore we can apply the converse of Proposition 3.3 to $\sigma^{\circledast} f$: there exists some parabolic germ $\Delta_{\sigma^{\circledast} f} \in \mathrm{Parab}_1$ given by the sectorial time-1 map of $X_{\sigma^{\circledast} f} = \sigma^{\circledast} X_f$ (by uniqueness of the normal form).

By definition of the Birkhof–Écalle–Voronin modulus it means that the identities

$$\begin{cases} H_f^- & = \psi_f \circ H_f^+ \\ H_{\sigma^{\circledast} f}^- & = \psi_{\sigma^{\circledast} f} \circ H_{\sigma^{\circledast} f}^+ \end{cases}$$

hold on $V^\cap$. By letting σ act on the first equality we obtain

$$\begin{aligned} H_{\sigma^{\circledast} f}^+ = H_f^- \circ \sigma = \psi_f \circ H_f^+ \circ \sigma &= \psi_f \circ H_{\sigma^{\circledast} f}^- \\ &= \psi_f \circ \psi_{\sigma^{\circledast} f} \circ H_{\sigma^{\circledast} f}^+ \end{aligned}$$

as expected.

5.2. **Quantitative bounds for the dynamics of the model Δ_0.** The first step to control precisely the dynamics of $X^\pm$ is to control that of X_0 and more precisely of $\Phi_{X_0}^\tau$ for $|\tau| \leqslant 1$. We rely on the following technical lemma.

Lemma 5.1. *Let X be a holomorphic vector field on some pointed disc $\mathcal{D} := \rho\mathbb{D}\backslash\{0\}$ and assume there exists some convenient constant $C > 0$ such that for any $0 < r < \rho$ we have:*

$$\sup\left|X\left(r\mathbb{S}^1\right)\right| \leqslant \frac{C}{r}.$$

Pick $0 < r < \rho$ and suppose moreover that $0 < \sqrt{r^2 - 2C} < \sqrt{r^2 + 2C} < \rho$. Then:

1. *for all $|\tau| \leqslant 1$ the mapping $\Delta := \Phi_X^\tau$ is holomorphic on a neighborhood of $r\mathbb{S}^1$;*
2. *$\Delta\left(r\mathbb{S}^1\right) \subset \mathcal{D}$.*

Proof. This is a simple variational argument applied to $\phi(t) := |z(t)|^2$ for $t \in [0, \tau]$, where z solves $\dot{z} = X(z)$. Indeed $\phi = z\overline{z}$ so that:

$$\dot{\phi} = 2\Re\left(X(z)\,\overline{z}\right)$$

and

$$\left|\dot{\phi}\right| \leqslant 2C.$$

Integrating the left- and right-hand sides with respect to $t \in [0, 1]$, one obtains

$$r^2 - 2C \leqslant |z(\tau)|^2 \leqslant r^2 + 2C.$$

□

We apply now this result to the model X_0.

Lemma 5.2. *Consider $\rho \leqslant \frac{1}{20}$ and assume that $\lambda < \min\left\{\frac{1}{2|\mu|}, \frac{\rho^2}{6}\right\}$.*

(1) *For each $0 < r < \rho$ and $|z \pm \mathrm{i}| = r$ we have*
$$|X_0(z)| < \frac{\lambda}{r}.$$
(2) *If $r := 2\sqrt{\lambda}$ and $|\tau| \leqslant 1$ then $\Phi^{\tau}_{X_0}$ is holomorphic on the annulus $\mathcal{A}_{\pm\mathrm{i}} := \pm\mathrm{i} + \rho\mathbb{D} \backslash \overline{r\mathbb{D}}$ and $\Phi^{\tau}_{X_0}(\mathcal{A}_{\pm\mathrm{i}}) \subset \pm\mathrm{i} + \rho\mathbb{D} \backslash \{0\}$.*

Example 5.3. For instance for $\rho := \frac{1}{20}$ we can choose the disk $D_{\pm\mathrm{i}}$ in Proposition 2.17 2. to be of radius $2\sqrt{\lambda}$ provided $\lambda < \frac{1}{2\,400}$.

Proof. The proof for an annulus centered at $-\mathrm{i}$ is identical to that of i, so we only deal with that case.

(1) The dominant part of R near i is given by $\frac{\lambda}{1+z^2}$. We use in the roughest possible fashion the triangular inequalities:
- $|z|^2 \leqslant (1+\rho)^2$;
- $\sqrt{2} - \rho < \left|1 - z^2\right| \leqslant 1 + \left(1 + \rho^2\right)$;
- $\left|1 - z^2 + \lambda\mu z\right| > \left(\sqrt{2} - \rho\right)^2 - \frac{1+\rho}{2}$, since $\lambda\,|\mu|\,|z| < \frac{1+\rho}{2}$;
- $\left|1 + z^2\right| \geqslant r\,(2 - \rho)$;

finally yielding
$$|X_0(z)| < \frac{\lambda}{r}.$$
(2) To apply Lemma 5.1, we must find r such that $0 < r^2 - 2\lambda$ and $r^2 + 2\lambda < \rho^2$. The choice $r := 2\sqrt{\lambda}$ fulfills both conditions provided $\lambda < \frac{\rho^2}{6}$.
To conclude this lemma we recall Lemma 2.15. We just proved that no point form $\mathcal{C}$ can ever reach the pole i in time τ, hence $\Phi^{\tau}_{X_0}$ must be holomorphic on $\mathcal{C}$.

□

We play the same game near the poles
$$z_{\pm} = \pm\sqrt{1 + \frac{\lambda^2\mu^2}{4}} + \frac{\lambda\mu}{2}$$
Observe that if $\lambda < \frac{1}{2|\mu|}$ then

$$|z_{\pm} \mp 1| < \frac{\lambda\,|\mu|}{2} + \frac{\lambda^2\,|\mu|^2}{8} < \lambda\frac{9\,|\mu|}{16}. \tag{5.1}$$

In particular $\frac{1}{2} < |z_{\pm}| < \frac{3}{2}$.

Lemma 5.4. *Consider $\rho \leqslant \frac{1}{20}$ and assume that $\lambda < \min\left\{\frac{1}{2|\mu|}, \frac{\rho^2}{32}\right\}$.*

(1) *For each $0 < r < \rho$ and $|z - z_{\pm}| = r$ we have*
$$|X_0(z)| < \frac{8\lambda}{r}.$$

(2) *If* $r := 4\sqrt{\lambda}$ *and* $|\tau| \leqslant 1$ *then* $\Phi^{\tau}_{X_0}$ *is holomorphic on the annulus* $\mathcal{A}_{\pm} := z_{\pm} + \rho\mathbb{D}\backslash\overline{r\mathbb{D}}$ *and* $\Phi^{\tau}_{X_0}(\mathcal{A}_{\pm}) \subset z_{\pm} + \rho\mathbb{D}\backslash\{0\}$.

Example 5.5. For instance for $\rho := \frac{1}{20}$ we can choose the disk $D_{z_{\pm}}$ in Proposition 2.17 2. to be of radius $4\sqrt{\lambda}$ provided $\lambda < \frac{1}{12\,800}$.

Proof. Again we only deal with the neighborhood of z_+, and obtain the expected bound using $|z - z_+| = r$:

- $|z^2| \leqslant \left(\frac{3}{2} + \rho\right)^2$;
- $|1 - z^2| \leqslant 1 + \left(\frac{3}{2} + \rho\right)^2$;
- (from (5.1)) $|z \pm \mathrm{i}| \geqslant \sqrt{2} - \rho - \frac{9}{32}$;
- $|1 + \lambda\mu z - z^2| = r\,|2z_+ + r| \geqslant r\,(1 - \rho)$.

$\square$

5.3. **Dynamics of $X^{\pm}$.** Here and in all the following we take $f \in \mathcal{B}$ in the unit ball of $\mathcal{S}$ and suppose

$$0 < \lambda < \min\left\{\frac{1}{2\,|\mu|}, \frac{1}{12\,800}\right\}.$$

Lemma 5.6. *Each vector field* X^- *and* X^+ *admits a unique pole in each disc* $\pm\mathrm{i} + 3\sqrt{\lambda}\mathbb{D}$, *which is simple.*

Proof. For the sake of clarity we chose a sector V^+ and let f stand for f^+, the case of f^- on V^- being completely similar. Let $\rho := 9\sqrt{\lambda} \leqslant \frac{1}{20}$ so that $0 < \lambda < \frac{\rho^2}{6}$ and we can apply Lemma 5.2: for any $0 < r < \rho$ and $|z \pm \mathrm{i}| = r$ we have

$$\frac{1}{|X_0(z)|} > \frac{r}{\lambda}.$$

Besides Cauchy's formula applied to f on the circle $\pm\mathrm{i} + \rho\mathbb{S}^1$ (which is included in $V^{\cap}$) yields for $z \in \pm\mathrm{i} + r\mathbb{S}^1$

$$|f'(z)| \leqslant \frac{1}{2\pi}\oint_{\mathrm{i}+r\mathbb{D}} \frac{|f(z)|}{|z - \mathrm{i}|^2}\,|\mathrm{d}z| \leqslant \frac{\rho}{(\rho - r)^2}.$$

Whenever $\frac{\rho}{(\rho-r)^2} < \frac{r}{\lambda}$ we can apply Rouché's theorem to the holomorphic functions $\frac{1}{X_0\cdot\mathrm{Id}}$ and $\frac{1}{X_0\cdot\mathrm{Id}} + f'$ on the disc $\pm\mathrm{i} + r\mathbb{D}$, so that $\frac{1}{X_0\cdot\mathrm{Id}} + f'$ has a single simple zero in the disc. In other words $\frac{1}{1+X_0\cdot f}X_0$ has a single (simple) pole in the disc.

The condition $\frac{\rho}{(\rho-r)^2} < \frac{r}{\lambda}$ is equivalent to $\phi(r) > \lambda$ where $\phi : r \mapsto \frac{r}{\rho}(r - \rho)^2$ vanishes at 0 and ρ. The function reaches its maximum $\frac{4\rho^2}{27} > \lambda$ at $\frac{\rho}{3} =: r$. $\square$

A similar phenomenon happens near $z_{\pm}$.

Lemma 5.7. *The vector field* $X^{\pm}$ *admits a unique pole* $p_{\pm}$ *in the disc* $\pm 1 + 5\sqrt{\lambda}\mathbb{D}$, *which is simple.*

Proof. This time we take $\rho := 15\sqrt{\lambda}$ in Lemma 5.4 and apply Cauchy's formula to the circle $z_{\pm} + r\mathbb{S}^1$ where $0 < r < \rho$ so that, for every $|z - z_{\pm}| = r$:

$$|f'(z)| \leqslant \frac{\rho}{(\rho - r)^2}$$

$$\frac{r}{8\lambda} \leqslant \frac{1}{|X_0(z)|}.$$

The condition $8\lambda < \phi(r)$ is met if $r := 5\sqrt{\lambda}$. □

This proposition ends the proof of Items 1.–3. of the Globalization Theorem.

Proposition 5.8. *Let us denote by $p_{\mathrm{i}}^{\pm}$, $p_{-\mathrm{i}}^{\pm}$ and $z_{\pm}$ the simple poles of $X^{\pm}$ provided by the previous lemmas. The following properties hold.*

(1) *$X^{\pm}$ does not have any other pole in $V^{\pm}$.*
(2) *$p_{\mathrm{i}}^{+} = p_{\mathrm{i}}^{-} =: p_{\mathrm{i}}$ and $p_{-\mathrm{i}}^{+} = p_{-\mathrm{i}}^{-} := p_{-\mathrm{i}}$. Moreover $\sigma(p_{\mathrm{i}}) = p_{-\mathrm{i}}$ and $\sigma(p_{+}) = p_{-}$.*
(3) *The vector field $X^{\pm}$ is conjugate to $\frac{1}{w}\frac{\partial}{\partial w}$ as in Lemma 2.7 on a disc centered at p and containing the two attached ramification points $\{z_p, w_p\}$, hence conjugating the monodromies of the time-1 maps.*

Proof.

(1) Recalling Proposition 2.19 3. the sectorial normalization $\Psi^{\pm}$ between X_0 and $X^{\pm}$ is holomorphic on $\mathcal{D}_{\lambda}$. If $X^{\pm}$ were to admit a pole outside $\Psi^{\pm}(\mathcal{D}_{\lambda})$ it would show in X_0. Since $\Psi^{\pm}(\partial\mathcal{D}_{\lambda})$ lies within the union of the discs given by the two previous lemmas, this cannot be the case. Indeed, according to Lemma 5.2 the circle $\pm\mathrm{i} + 2\sqrt{\lambda}\mathbb{S}^1$ is mapped by $\Psi^{\pm}$ into a disc centered at $\pm\mathrm{i}$ of radius $\rho := 3\sqrt{\lambda} > r$. The same goes accordingly near $z_{\pm}$.
(2) Recalling Lemma 2.15, which also holds locally for meromorphic vector fields, a pole p of X^{+} induces two ramification points $\{z_p, w_p\}$ in Δ, which are mapped by Δ to p. Since Δ is also the time-1 map of X^{-}, the points z_p and w_p also belong to a stable manifold of X^{-} and are sent to a pole of X^{-} by its time-1 flow. Hence p is also a pole of X^{-}.
For the very same reason that a pole of $X_f = (X^{+}, X^{-})$ is a dynamical feature, the poles of $X_{\sigma^{\circledast} f}$ and that of $\sigma^{\circledast} X_f$ coincide. But these vector fields have only poles near fixed-points $\{\pm\mathrm{i}, \pm 1\}$ of σ, hence must correspond by σ.
(3) Let p be a pole of $X^{\pm}$. For ψ to conjugate $\frac{1}{z-p}\frac{\partial}{\partial z}$ to $X^{\pm}(z) = \frac{1}{z-p}R(z)$ it needs to solve
$$X^{\pm} \cdot \psi = \frac{1}{\psi - p}$$
or, in other words
$$\psi(z) = p + 2\sqrt{\int_p^z \frac{(x-p)\,\mathrm{d}x}{R(x)}}.$$
For $R(p) \notin \{0, \infty\}$, this formula defines a holomorphic function on a simply-connected domain U as long as R is holomorphic on U. Of course U can be chosen to avoid the zeroes of R, therefore $X^{\pm}$ is conjugate to the polar model $\frac{1}{z-p}\frac{\partial}{\partial z}$ on a domain encompassing the ramification points of the model time-1 map.

□

6. Corollaries

6.1. **Parabolic Renormalization.** Let $\mathrm{Synth}_{\lambda} : \varphi \mapsto \delta$ be the synthesis map built in Section 4, that is

$$\mathrm{B\acute{E}V}(\mathrm{Id}\exp\delta) = \left(\mathrm{Id}\exp\left(4\mu\pi^2 + \varphi^0\right), \mathrm{Id}\exp\varphi^{\infty}\right)$$

and consider for given $\varphi^0 \in \mathrm{Holo}_c(\mathbb{C},0)$ the iteration $\delta_0 := 0$ and $\delta_{n+1} := \mathrm{Synth}_\lambda\left(\varphi^0, \delta_n\right)$. We wish to prove that the iteration is well-defined (that is, one can chose λ independently on n) and that it converges towards a unique fixed-point. For this we exploit the nice dynamical features of the spherical normal forms, providing us with a uniform lower bound on the radius of convergence of Δ and allowing us to control Δ'.

Let us be quantitative, as in Section 5.2 but around 0 this time.

Proposition 6.1. *Fix some* $0 < \lambda < \frac{1}{2|\mu|}$.

(1) *For all* $0 < |z| \leqslant \frac{1}{2}$ *we have*

$$\frac{2\lambda|z|^2}{5} \leqslant |X_0(z)| \leqslant \frac{10\lambda|z|^2}{3}.$$

(2) *Let* $f = (f^+, f^-) \in \mathcal{S}$.

$$\left(\forall |z| \leqslant \frac{1}{4}\right) \qquad |X_0 \cdot f|(z) \leqslant 4\lambda ||f|| .$$

(3) *Let* $\Delta := \mathrm{Id}\exp\delta$ *for* $\delta := \mathit{Synth}\left(\varphi^0, \varphi^\infty\right)$ *be some spherical normal form and assume* $\lambda < \frac{1}{8}$.

(a) *The sectorial vector fields* $X^\pm = \frac{1}{1+X_0\cdot f}X_0$ *have magnitude*

$$\left(\forall |z| \leqslant \frac{1}{4}\ ,\ \pm\Re(z) \geqslant 0\right) \quad \left|X^\pm(z)\right| \leqslant \frac{20\lambda|z|^2}{3}.$$

(b) Δ *is biholomorphic on the disc* $\frac{3}{16}\mathbb{D}$ *and*

$$\Delta\left(\frac{3}{16}\mathbb{D}\right) \subset \frac{1}{4}\mathbb{D}.$$

(c)

$$||\Delta'||_{\frac{3}{16}\mathbb{D}} \leqslant 45$$
$$||\delta||_{\frac{3}{16}\mathbb{D}} < 11$$

Proof.

(1) This is nothing but the plain triangular inequalities, taking into account that $\left|1 + \lambda\mu z - z^2\right| \geqslant 1 - r^2 - \frac{r}{2} \geqslant 1 - r$ since $\frac{1}{2} \geqslant r$.

(2) For $\Re(z) \geqslant 0$ we apply Cauchy formula to the pacman $P := \partial\left(V^+ \cap \frac{1}{2}\overline{\mathbb{D}}\right)$ and drop the index $\pm$ altogether. The length of P is bounded by $\pi + 1$, and $|x - z| \geqslant |z|\sin\frac{\pi}{8}$ for all $x \in P$ and all $|z| \leqslant \frac{1}{4}$ so that:

$$\left(\forall |z| \leqslant \frac{1}{4}\ ,\ \Re(z) \geqslant 0\right) \quad |f'(z)| \leqslant \frac{(\pi+1)||f||}{\pi\sin^2\frac{\pi}{8}|z|^2} \leqslant \frac{2||f||}{|z|^2}.$$

As a consequence

$$|X_0 \cdot f|(z) = \lambda\left|\frac{1 - z^2}{(1 + z^2)(1 + \lambda\mu z - z^2)}\right|\left|z^2 f'(z)\right| \leqslant 4\lambda ||f|| .$$

The argument is identical for $\Re(z) \leqslant 0$ on V^-.

(3) Here $f \in \mathcal{B}$.

(a) On the disc $\frac{1}{4}\mathbb{D}$ we have $|X_0 \cdot f^{\pm}| \leqslant \frac{1}{2}$ so that $\frac{1}{2} \leqslant |1 + X_0 \cdot f^{\pm}| \leqslant \frac{3}{2}$: we can bound the magnitude of the sectorial vector field $X^{\pm} = \frac{1}{1+X_0 \cdot f} X_0$ by

$$\left(\forall |z| \leqslant \frac{1}{4} \ : \ \pm\Re(z) \geqslant 0\right) \quad \frac{4\lambda |z|^2}{15} \leqslant \left|X^{\pm}(z)\right| \leqslant \frac{20\lambda |z|^2}{3}.$$

(b) Invoking the same variational argument as in the proof of Lemma 5.1, if the *a priori* bound

$$\frac{r}{1 - {}^{20\lambda r}/_{3}} \leqslant \frac{1}{4}$$

holds then the time-1 flow of $z \in r\mathbb{S}^1$ along $X^{\pm}$ is well-defined and lands in $\frac{1}{4}\mathbb{D}$. But the condition is fulfilled for $r := \frac{3}{16}$ since $\frac{20\lambda r}{3} = \frac{5\lambda}{4} < \frac{5}{32}$.

(c) Because Δ is a symmetry of $X^{\pm}$ we have $X^{\pm} \cdot \Delta = X^{\pm} \circ \Delta$, that is:

$$|\Delta'| = \frac{|X \circ \Delta|}{|X|}.$$

For $|z| = \frac{3}{16}$ we obtain

$$|\Delta'(z)| \leqslant \frac{||X^{\pm}||_{\frac{1}{4}\mathbb{D}}}{|X^{\pm}(z)|} \leqslant \frac{\frac{20\lambda}{3}\left(\frac{1}{4}\right)^2}{\frac{4\lambda}{15}\left(\frac{3}{16}\right)^2} \leqslant 45.$$

Continuing we find

$$|\delta(z)| = \left|\log \frac{\Delta(z)}{z}\right| \leqslant 2\pi + \ln \left|\frac{\Delta(z)}{z}\right| \leqslant 2\pi + \ln ||\Delta'||_{\frac{3}{16}\mathbb{D}} < 11.$$

□

Consider now Δ_1 and Δ_2 in normal form, being the respective time-1 map of $X_j^{\pm} := \frac{1}{1+X_0 \cdot f_j^{\pm}} X_0$ with $f_j \in \mathcal{B}$ for given $0 < \lambda < \min\left\{\frac{1}{2|\mu|}, \frac{1}{8}\right\}$ as in the Proposition. Being given $O = (O^-, O^+)$ we introduce the slight abuse of notation:

$$||O||_{r\mathbb{D}} := \max\left\{\sup_{|z| \leqslant r \ , \ \pm\Re(z) \geqslant 0} \left|O^{\pm}(z)\right|\right\}$$

and choose

$$0 < \rho \leqslant \frac{3}{16} \ , \ r := \frac{1}{4}.$$

For the sake of readibility we drop the superscripts $\pm$ altogether.

If z_1, z_2 is the respective trajectory of X_1, X_2 emanating from some common $z \in \rho\overline{\mathbb{D}}$, then $\phi := |z_1 - z_2|^2$ solves

$$\dot{\phi} = 2\Re\left((X_1(z_1) - X_2(z_2))\overline{z_1 - z_2}\right) \ , \ \phi(0) = 0.$$

Since $z_1(t), \ z_2(t) \in r\mathbb{D}$ for all $t \in [0,1]$ we have

$$|X_1(z_1) - X_2(z_2)| \leqslant ||X_1 - X_2||_{r\mathbb{D}} + ||X_2'||_{r\mathbb{D}} \sqrt{\phi}.$$

The Proposition yields

$$||X_1 - X_2||_{r\mathbb{D}} = \sup_{r\mathbb{D}} \frac{|X_0 \cdot (f_1 - f_2)| \, |X_0|}{|1 + X_0 \cdot f_1| \, |1 + X_0 \cdot f_2|} \leqslant \frac{10\lambda^2}{3} ||f_1 - f_2|| =: a$$

and

$$\begin{aligned}||X_2'||_{r\mathbb{D}} &= \left|\left|\frac{(X_0\cdot\mathrm{Id})'}{1+X_0\cdot f_2} - \frac{X_0\cdot^2 f_2}{(1+X_0\cdot f_2)^2}\right|\right|_{r\mathbb{D}}\\ &\leqslant 2\,||X_0'||_{r\mathbb{D}} + 4\left|\left|X_0\cdot^2 f_2\right|\right|_{\frac{1}{4}\mathbb{D}}\\ &\leqslant 2\,||X_0'||_{r\mathbb{D}} + 16\lambda\,||X_0\cdot f_2||_{\frac{1}{2}\mathbb{D}}\\ &\leqslant \frac{5}{2}\lambda + 64\lambda^2\,||f_2|| \leqslant 2.\end{aligned}$$

From a direct integration of

$$(\forall\,|t|\leqslant 1)\quad \left|\frac{\dot{\phi}}{2\sqrt{\phi}\,(a+2\sqrt{\phi})}\right| \leqslant 1$$

we finally deduce the following estimate.

Lemma 6.2. *We have*

$$||\Delta_1-\Delta_2||_{\rho\mathbb{D}} \leqslant a\frac{\mathrm{e}^2-1}{2} \leqslant 11\lambda^2\,||f_1-f_2||\,.$$

Assume now that $\mathrm{B\acute{E}V}\,(\Delta_j) = \left(\mathrm{Id}\exp\left(4\pi^2\mu+\varphi_j^0\right),\mathrm{Id}\exp\varphi_j^\infty\right)$ and pick $\rho < \min\{\rho_1^\infty,\rho_2^\infty\}$ less than the least radius of convergence of $\varphi_1^\infty,\varphi_2^\infty$, so that $\varphi_j^\infty \in \mathrm{Holo}_{\mathrm{c}}\,(\rho\mathbb{D})$, and consider

$$\ell := \min\left\{1,\mathfrak{t}_\mu,\frac{1}{2\pi+\ln\frac{\mathfrak{m}_\mu}{\min\{\rho^0,\rho/2\}}}\right\} \tag{6.1}$$

similarly as in Proposition 4.13. Invoking 4.6 we derive

$$||f_1-f_2|| \leqslant 4\mathfrak{m}_\mu\lambda^2\times\left(||\varphi_1^\infty-\varphi_2^\infty||_\ell + \mathrm{e}\left(\left|\left|\varphi^0\right|\right|_\ell + ||\varphi_2^\infty||_\ell\right)||f_1-f_2||\right),$$

that is

$$||f_1-f_2|| \leqslant 8\mathfrak{m}_\mu\lambda^2\,||\varphi_1^\infty-\varphi_2^\infty||_\ell$$

provided $\lambda>0$ be small enough to ensure $4\mathrm{e}\mathfrak{m}_\mu\lambda^2\left(\left|\left|\varphi^0\right|\right|_\ell + ||\varphi_2^\infty||_\ell\right) \leqslant \frac{1}{2}$, say. By assumption we have $\mathfrak{m}_\mu\exp\left(2\pi-\frac{1}{\lambda}\right) \leqslant \frac{\rho}{2}$ so that Cauchy's formula applied on $\rho\mathbb{S}^1$ yields

$$||\varphi||_\ell \leqslant \frac{\rho}{\left(\rho-\mathfrak{m}_\mu\exp\left(2\pi-\frac{1}{\ell}\right)\right)^2}\,||\varphi||_{\rho\mathbb{D}} \leqslant \frac{4}{\rho}\,||\varphi||_{\rho\mathbb{D}}$$

from which we deduce

$$||\Delta_1-\Delta_2||_{\rho\mathbb{D}} \leqslant 171\mathfrak{m}_\mu\lambda^4\,||\varphi_1^\infty-\varphi_2^\infty||_{\rho\mathbb{D}}\,.$$

We write $\Delta_j = \mathrm{Id}\exp\delta_j$ with $\delta_j\,(0)=0$ which, thanks to Proposition 6.1 (3)(c), satisfies $||\delta_j||_{\rho\mathbb{D}} \leqslant 11$. Because we iterate Synth_λ we lose no generality in assuming $\left|\left|\varphi_j^\infty\right|\right|_{\rho\mathbb{D}} \leqslant 11$ as well. Supposing therefore

$$\frac{171}{\rho}\mathrm{e}^{11}\mathfrak{m}_\mu\lambda^4 \leqslant \frac{1}{44}$$

we deduce for $|z|=\rho$

$$\left|\frac{\Delta_1-\Delta_2}{\Delta_2}\right|(z) \leqslant \frac{\mathrm{e}^{11}}{\rho}\,||\Delta_1-\Delta_2||_{\rho\mathbb{D}} \leqslant \frac{171}{\rho}\mathrm{e}^{11}\mathfrak{m}_\mu\lambda^4\,||\varphi_1^\infty-\varphi_2^\infty||_{\rho\mathbb{D}} \leqslant \frac{11+11}{44} = \frac{1}{2}.$$

As a matter of consequence we have

$$\begin{aligned}||\delta_1-\delta_2||_{\rho\mathbb{D}} = \left|\left|\log\left(1+\frac{\Delta_1-\Delta_2}{\Delta_2}\right)\right|\right|_{\rho\mathbb{S}^1} &\leqslant 2\left|\left|\frac{\Delta_1-\Delta_2}{\Delta_2}\right|\right|_{\rho\mathbb{S}^1}\\ &\leqslant \frac{1}{22}\left|\left|\varphi_1^\infty-\varphi_2^\infty\right|\right|_{\rho\mathbb{D}}.\end{aligned}$$

Now we can chose the right constants in advance.

Corollary 6.3. *Pick* $\varphi^0 \in \mathrm{Holo}(\mathbb{C},0)$ *and* $\rho := \frac{3}{16}$*; let* $\mathcal{B}_{11}$ *be the ball* $\left\{\varphi \in \mathrm{Holo}_c(\rho\mathbb{D})\ :\ ||\varphi||_{\rho\mathbb{D}} \leqslant 11\right\}$. *Define*

$$\widehat{\ell} = \widehat{\ell}\left(\varphi^0\right) := \min\left\{1, \mathfrak{t}_\mu, \frac{10^{-2}}{\sqrt[4]{\mathfrak{m}_\mu}}, \frac{1}{2\pi + \ln\frac{\mathfrak{m}_\mu}{\min\{\rho^0, 3/32\}}}\right\}.$$

For any

$$0 < \lambda \leqslant \widehat{\lambda} := \min\left\{\ell, \frac{1}{8\mathrm{e}\sqrt{\mathfrak{m}_\mu}\sqrt{||\varphi^0||_{\widehat{\ell}}+9}}\right\}$$

the partial map Synth$_\lambda\left(\varphi^0, \bullet\right)$

$$\begin{aligned}\mathcal{B}_{11} &\longrightarrow \mathcal{B}_{11}\\ \varphi^\infty &\longmapsto \delta = \mathit{Synth}_\lambda\left(\varphi^0,\varphi^\infty\right)\end{aligned}$$

is a well-defined $\frac{1}{22}$*-lipschitz map. Its unique fixed-point* δ_* *provides the sought fixed-point* $\Delta_* := \mathrm{Id}\exp\delta_*$ *of the parabolic renormalization.*

6.2. **Real Synthesis.** Let $\mu \in \mathbb{R}$ and $\psi = \left(\psi^0, \psi^\infty\right)$ be given such that condition ($\square$) holds:

$$(\square)\qquad (\forall h \in (\mathbb{C},0))\qquad \overline{\psi^0\left(\overline{h}\right)} = \frac{\mathrm{e}^{4\pi^2\mu}}{\psi^\infty\left(\frac{1}{h}\right)}.$$

Expressed for the data $\varphi = \left(\varphi^0, \varphi^\infty\right)$ it becomes

$$(\widetilde{\square})\qquad (\forall h \in (\mathbb{C},0))\qquad \overline{\varphi^0\left(\overline{h}\right)} = -\varphi^\infty\left(\frac{1}{h}\right).$$

Recall that $\Delta = \Phi_X^1$ where $X = \frac{1}{1+X_0\cdot f}X_0$ and f is obtained as the limit of the iteration $\left(f_n\right)_n$ defined by:

$$\begin{aligned}f_0 &:= (0,0)\\ f_{n+1} &:= \mathrm{CH}^\varphi\left(f_n\right) = \Lambda_{f_n} - \Lambda_{f_n}(0)\end{aligned}$$

as in Definition 4.12. We wish to prove that f is a real function by induction on n. This is clearly true for $n := 0$ and it is sufficient to prove that Λ_f is real.

Assume that $\overline{f_n(z)} = f_n(\overline{z})$ for every $z \in V$ (this identity is well-defined because V is invariant by the complex conjugation). Because $\overline{a} = b$ we only need to check that

$$(\forall \xi \in b)\quad \overline{\varphi^0\left(H_n\left(\overline{\xi}\right)\right)} = -\varphi^\infty\left(H_n(\xi)\right),$$

where we have set

$$H_n(\xi) := H_{f_n}^+(\xi) = H_0(\xi)\exp\left(2\mathrm{i}\pi f_n^+(\xi)\right),$$

which in turn holds thanks to $\left(\widetilde{\square}\right)$ whenever $\overline{H_n\left(\overline{\xi}\right)} = \frac{1}{H_n(\xi)}$. By the induction hypothesis we can assert

$$\overline{H_n\left(\overline{\xi}\right)} = \overline{H_0\left(\overline{\xi}\right)} \exp\left(-2\mathrm{i}\pi f_n^+(\xi)\right) = \frac{1}{H_n(\xi)}$$

since, according to (4.1):

$$\overline{H_0\left(\overline{\xi}\right)} = \exp\left(2\mathrm{i}\pi\frac{1-\xi^2}{\lambda\xi} + 2\mathrm{i}\pi\mu\log\frac{1-\xi^2}{\lambda\xi}\right) = \frac{1}{H_0(\xi)}.$$

This completes the proof of the Real Synthesis Corollary.

References

[Bak62] I. N. Baker, *Permutable entire functions*, Math. Z., vol. 79: pp. 243–249, doi:10.1007/BF01193120, 1962

[BD10] B. Branner and K. Dias, *Classification of complex polynomial vector fields in one complex variable*, J. Difference Equ. Appl., vol. 16, #5-6: pp. 463–517, doi:10.1080/10236190903251746, 2010

[Bir39] G. D. Birkhoff, *Déformations analytiques et fonctions auto-équivalentes*, Ann. Inst. H. Poincaré, vol. 9: pp. 51–122, 1939

[DES05] A. Douady, F. Estrada and P. Sentenac, *Champs de vecteurs polynomiaux sur* $\mathbb{C}$, Preprint 2005

[Éca75] J. Écalle, *Théorie itérative: introduction à la théorie des invariants holomorphes*, J. Math. Pures Appl. (9), vol. 54: pp. 183–258, 1975

[Éca85] J. Écalle, Les fonctions résurgentes. Tome III, *Publications Mathématiques d'Orsay [Mathematical Publications of Orsay]*, vol. 85, Université de Paris-Sud, Département de Mathématiques, Orsay, l'équation du pont et la classification analytique des objects locaux. [The bridge equation and analytic classification of local objects], 1985

[Eca05] J. Ecalle, *Twisted resurgence monomials and canonical-spherical synthesis of local objects*, *in* Analyzable functions and applications, *Contemp. Math.*, vol. 373, pp. 207–315, Amer. Math. Soc., Providence, RI, doi:10.1090/conm/373/06921, 2005

[Eps93] A. Epstein, *Towers of Finite Type Complex Analytic Maps*, Ph.D. thesis, 1993

[Lor04] F. Loray, *Versal deformation of the analytic saddle-node*, Astérisque, , #297: pp. 167–187, analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes. II, 2004

[LR16] M. Loday-Richaud, Divergent series, summability and resurgence. II, *Lecture Notes in Mathematics*, vol. 2154, Springer, [Cham], doi:10.1007/978-3-319-29075-1, simple and multiple summability, With prefaces by Jean-Pierre Ramis, Éric Delabaere, Claude Mitschi and David Sauzin, 2016

[RT0p] C. Rousseau and L. Teyssier, *Analytic normal forms and inverse problems for unfoldings of 2-dimensional saddle-nodes with analytic center manifold*, Annales Scientifiques de l'ENS, 2020 (100p)

[Sma81] S. Smale, *The fundamental theorem of algebra and complexity theory*, Bull. Amer. Math. Soc. (N.S.), vol. 4, #1: pp. 1–36, doi:10.1090/S0273-0979-1981-14858-8, 1981

[ST15] R. Schäfke and L. Teyssier, *Analytic normal forms for convergent saddle-node vector fields*, Ann. Inst. Fourier, vol. 65, #3: pp. 933–974, 2015

[STW88] M. Shub, D. Tischler and R. F. Williams, *The Newtonian graph of a complex polynomial*, SIAM J. Math. Anal., vol. 19, #1: pp. 246–256, doi:10.1137/0519018, 1988

[Vor81] S. Voronin, *Analytic classification of germs of conformal mappings* $(\mathbf{C},0) \to (\mathbf{C},0)$, Functional Anal. Appl., vol. 15, #1: pp. 1–13, 1981

Laboratoire IRMA (UMR 7501), Université de Strasbourg — CNRS, France

Email address: teyssier@math.unistra.fr